NNLO corrections to the charm quark mass dependent matrix elements in $\bar{B} \to X_s\gamma$

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Radcor October 2007

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Motivations

- $\bar{B} \to X_s\gamma$ most precise short-distance information currently available for $\Delta B = 1$ FCNC

$$\boxed{\mathcal{B}(\bar{B} \to X_s\gamma)^{\rm exp}_{E_\gamma > 1.6\,{\rm GeV}} = (3.55 \pm 0.26) \times 10^{-4}}$$

[HFAG2006]

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  - less sensitive to non-perturbative effects
    dominant ones: $\mathcal{O}(\Lambda^2/m_b^2)$, $\mathcal{O}(\Lambda^2/m_c^2)$, $\mathcal{O}(\alpha_s\Lambda/m_b)$

$$\Longrightarrow \quad \Gamma(\bar{B} \to X_s\gamma) \approx \Gamma(b \to X_s^{parton}\gamma) = \Gamma(b \to s\gamma) + \Gamma(b \to s\gamma g) + \ldots$$

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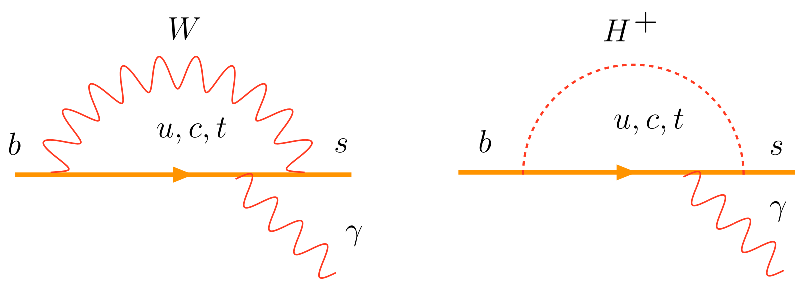
  - loop induced in SM and highly sensitive to new physics which is not suppressed by factors of $\boldsymbol{\alpha}$ as compared to SM contributions



Motivations

- decay mode measured by several independent experiments

  - $\mathcal{B}(\bar{B} \to X_s\gamma)^{\mathrm{th,NLO}}_{E_\gamma > 1.6\,\mathrm{GeV}} = (3.57 \pm 0.30) \times 10^{-4}$
    [Misiak et al 2001,Buras et al 2002]

  - $\mathcal{B}(\bar{B} \to X_s\gamma)^{\mathrm{exp}} = (3.55 \pm 0.26) \times 10^{-4}$
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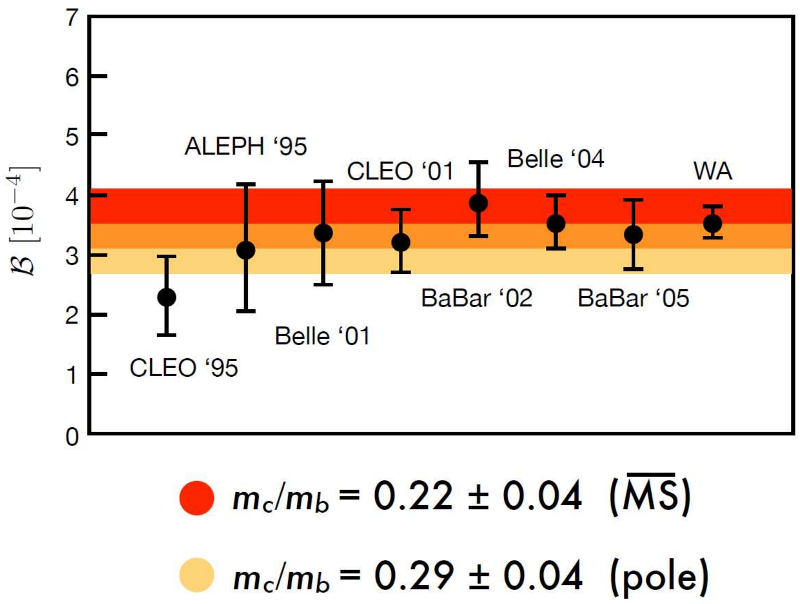
  Super-B factory: 5% uncertainty possible
  (more statistics, lower E_γ)



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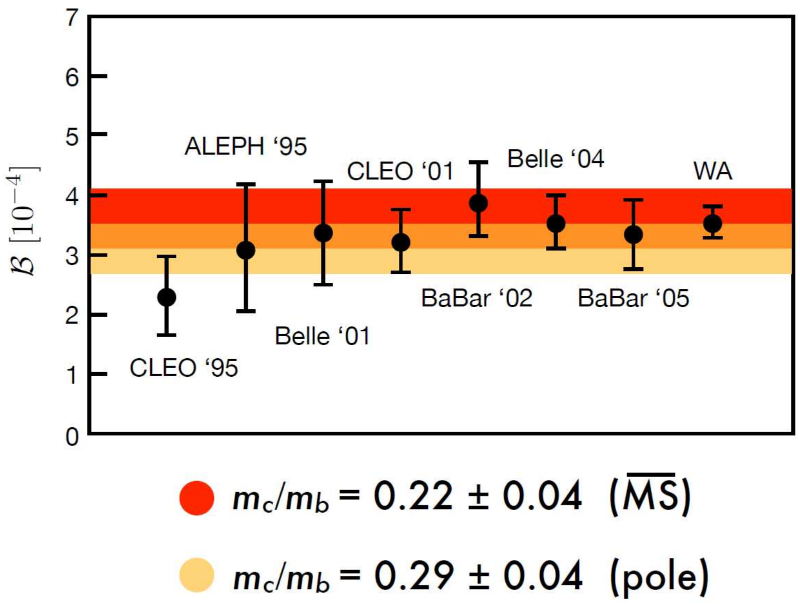

$\Longrightarrow$ strong constraints on new physics require better theoretical precision

Motivations

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[HFAG 2006]

- Contributions to the theory prediction

$$\mathcal{B}(\bar{B} \to X_s\gamma)_{E_\gamma > 1.6\,{\rm GeV}} = \mathcal{B}(\bar{B} \to X_c e\bar{\nu})_{\rm exp} \left[\frac{\Gamma(b\to s\gamma)}{\Gamma(b\to ce\bar{\nu})}\right]_{\rm LO\ EW} f\left(\frac{\alpha_s(M_W)}{\alpha_s(m_b)}\right)\times$$

$$\times \underbrace{\Big\{1 + \underset{\substack{\textbf{NLO}\\ \sim 25\%}}{\mathcal{O}(\alpha_s)} + \underset{\substack{\textbf{NNLO}\\ \sim 7\%}}{\mathcal{O}(\alpha_s^2)} + \underset{\sim 4\%}{\mathcal{O}(\alpha_{\rm em})}}_{\text{perturbative corrections}} + \underbrace{\underset{\sim 1\%}{\mathcal{O}\left(\frac{\Lambda^2}{m_b^2}\right)} + \underset{\sim 3\%}{\mathcal{O}\left(\frac{\Lambda^2}{m_c^2}\right)} + \underset{<\sim 5\%}{\mathcal{O}\left(\frac{\Lambda}{m_b}\alpha_s\right)}\Big\}}_{\text{non-perturbative corrections}}$$

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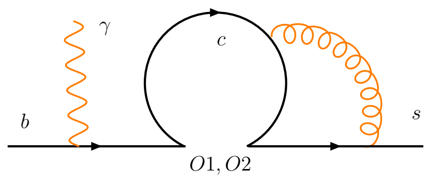
expected NNLO corrections to $\mathcal{B}$ ($\sim 7\%$) are of the same size as the experimental error

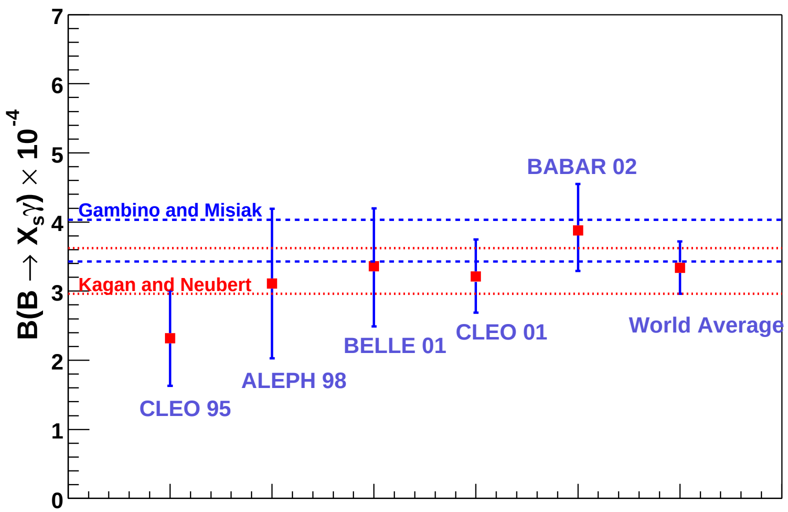
Motivations

- Charm quark mass definition ambiguity
  - dependence of $\mathcal{B}(\bar{B} \to X_s\gamma)^{theo}$ on m_c enters through the $\langle s\gamma|O_{1,2}|b\rangle$ which start contributing at $\mathcal{O}(\alpha_s)$



[Aubertetal 02]

  - $m_c^{pole}/m_b^{pole} = 0.29 \pm 0.02$
    $\mathcal{B}(\bar{B} \to X_s\gamma)^{theo} = (3.32 \pm 0.30) \times 10^{-4}$
  - $\overline{m}_c(m_b/2)/m_b^{pole} = 0.22 \pm 0.04$
    $\mathcal{B}(\bar{B} \to X_s\gamma)^{theo} = (3.70 \pm 0.30) \times 10^{-4}$



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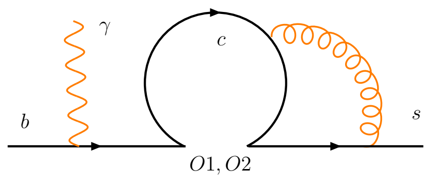

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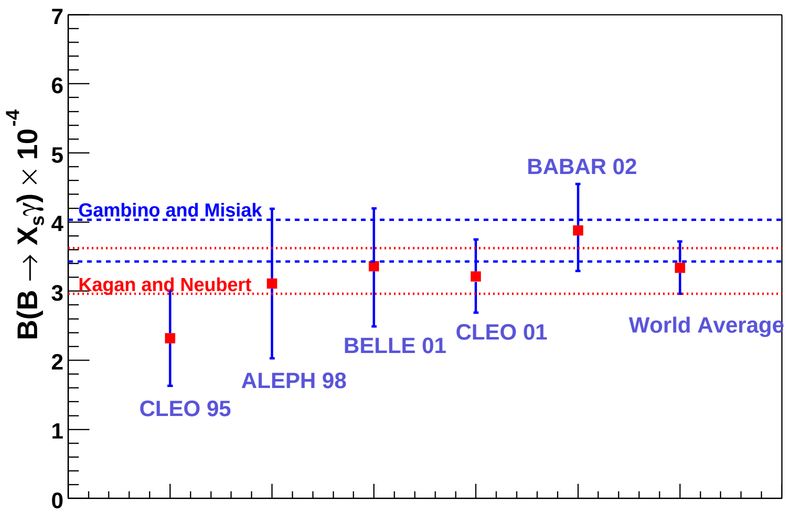

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- difference between using $\overline{\boldsymbol{m}}_{\boldsymbol{c}}(\mu)$ and $\boldsymbol{m_c^{pole}}$ is a NNLO effect in the branching ratio
  $\Longrightarrow$ resolving the ambiguity requires going to the NNLO level

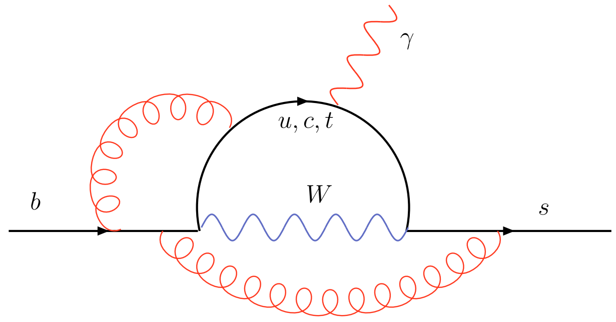
Theoretical framework

- diagrams involve scales with large hierarchy

  $M_W, M_t \gg m_b \gg m_s \Longrightarrow$ large $\log\left(\frac{M_W^2}{m_b^2}\right)$

  $\longrightarrow$ resummation of $\alpha_s \log\left(\frac{M_W^2}{m_b^2}\right)$ is necessary

  using RG techniques



- start by introducing an effective theory without the heavy fields

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_{\text{QCD}\times\text{QED}}(u, d, s, c, b) + \frac{4G_F}{\sqrt{2}} V_{ts}^* V_{tb} \sum_i C_i(\mu)\, O_i(\mu)$$

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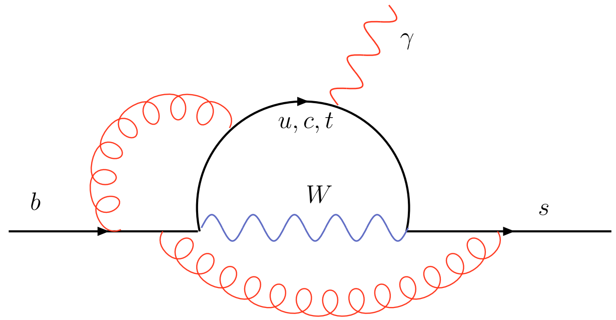

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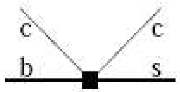
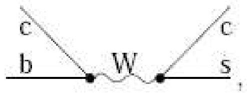
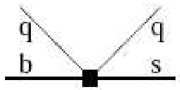
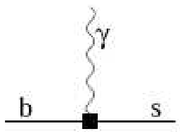
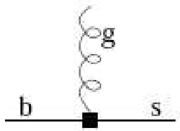
$$\mathcal{L}_{\rm eff} = \mathcal{L}_{\rm QCD\times QED}(u,d,s,c,b) + \frac{4G_F}{\sqrt{2}} V_{ts}^* V_{tb} \sum_i C_i(\mu)\, O_i(\mu)$$

$O_{1,2} =$  $= (\bar{s}\Gamma_i c)(\bar{c}\Gamma_i' b)$, from , $|C_i(m_b)| \sim 1$

$O_{3,4,5,6} =$  $= (\bar{s}\Gamma_i b)\sum_q(\bar{q}\Gamma_i' q)$, $|C_i(m_b)| < 0.07$

$O_7 =$  $= \frac{e m_b}{16\pi^2}\, \bar{s}_L \sigma^{\mu\nu} b_R F_{\mu\nu}$, $C_7(m_b) \simeq -0.3$

$O_8 =$  $= \frac{g m_b}{16\pi^2}\, \bar{s}_L \sigma^{\mu\nu} T^a b_R G^a_{\mu\nu}$, $C_8(m_b) \simeq -0.15$

Theoretical framework

Calculation done in three steps:

- Matching find the Wilson coefficients $C_i(\mu)$ by comparing the full and the effective theory at the mass scale $\mu \approx M_W$
$\Longrightarrow$ no large logarithms and only vacuum diagrams

- Mixing compute the anomalous dimensions of the operators and solve the renormalization group equations to go down with the Wilson coefficients to $\mu \approx m_b$

$$\frac{d}{d\mu} C_j(\mu) = C_i(\mu)\gamma_{ij}(\mu)$$

- Matrix elements calculate the matrix elements of all the operators at $\mu \approx m_b \Longrightarrow$ no large logarithms as no heavy masses are present

Current state of the art for NNLO corrections

1. Matching
   - 2-loop matching for (O_1, . . . ,O_6) [Bobeth,Misiak,Urban 00]
   - 3-loop matching for O_7 and O_8 [Misiak,Steinhauser 04]
2. Mixing
   - 3-loop: $(O_1, \ldots, O_6)$ and $(O_7,\, O_8)$ sectors [Gorbahn,Haisch 05] [Gorbahn,Haisch,Misiak 05]
   - 4-loop $(O_1, \ldots, O_6) \longrightarrow (O_7,\, O_8)$ [Czakon,Haisch,Misiak 06]

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3. Matrix elements
   - O_1, O_2, O_7, O_8 large β_0 [Bieri,Greub,Steinhauser 03]
   - O_7 [Blokland,Czarnecki,Misiak,Slusarczyk,Tkachov 05] [Asatrian,Hovhannisyan,Poghosyan,Ewerth,Greub,Hurth 06]
   - O_7, photon spectrum [Melnikov,Mitov 05] [Asatrian,Ewerth,Ferroglia,Gambino,Greub 06]
   - O_1, O_2 leading term for $m_c \gg m_b$ [Misiak,Steinhauser 06]

The NNLO estimated Branching Ratio

$$\boxed{\mathcal{B}(\bar{B} \to X_s\gamma)^{\rm theo}_{E_\gamma > 1.6\,{\rm GeV}} = (3.15 \pm 0.23) \times 10^{-4}}$$

[Misiak et al 06] [Misiak,Steinhauser 06]

- Decomposition of Uncertainty
  - non-perturbative 5% $\mathcal{O}(\alpha_s \Lambda/m_b)$
  - parametric 3% $\alpha_s(M_Z)$, $\mathcal{B}^{exp}_{SL}$, $m_c \ldots$
  - m_c interpolation 3% ($O_{1,2}$ matrix elements)
  - higher order 3% (μ_b, μ_c, μ_0 dependence)

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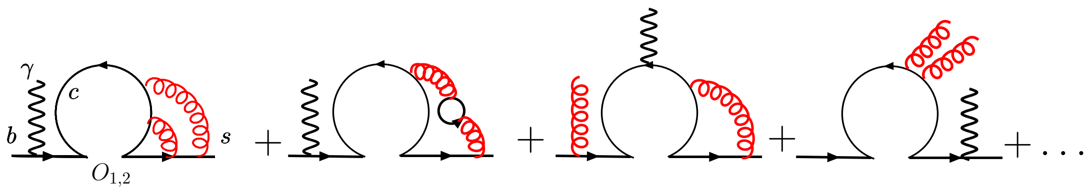
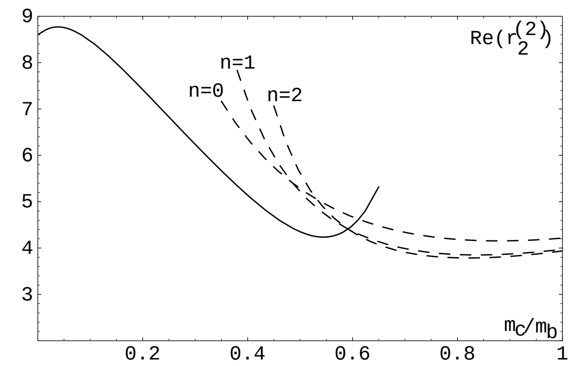
- source of the interpolation uncertainty is the missing $\mathcal{O}\left(\alpha_s^2\right)$ correction to $\langle s\gamma|O_{1,2}|b\rangle$



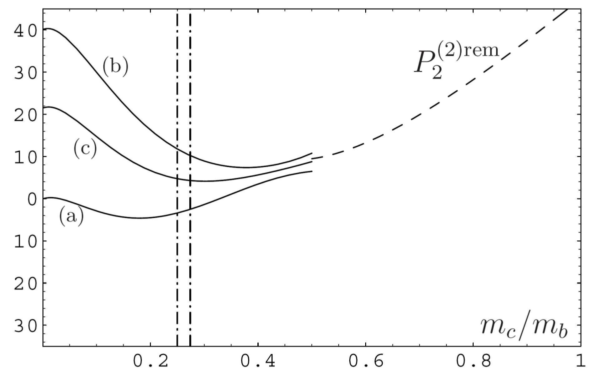
More about the interpolation uncertainty

- $\mathcal{O}\left(\alpha_s^2\right)$ perturbative contribution to $\mathcal{B}(\bar{B} \to X_s\gamma)$: $$P_2^{(2)} = \sum_{i,j=1}^{8} C_i^{(0)} C_j^{(0)} \left(n_f\, A_{ij} + B_{ij}\right)$$
- using large β_0 approx. $$P_2^{(2)} = \sum_{i,j=1}^{8} C_i^{(0)} C_j^{(0)} \left(\frac{-3}{2}\beta_0\, A_{ij} + B'_{ij}\right) = P_2^{(2),\beta_0} + P_2^{(2),rem}$$
  - $P_2^{(2),\beta_0}$ known for $\langle s\gamma|O_{1,2,7,8}|b\rangle$

- expansions in limits $m_c/m_b \to 0$ and $m_c \gg m_b$ match nicely for Re$\langle s\gamma|O_2|b\rangle^{\beta_0}$
- good approximation already for $n = 0$
- no large $c\bar{c}$ threshold effects at $m_c = m_b/2$



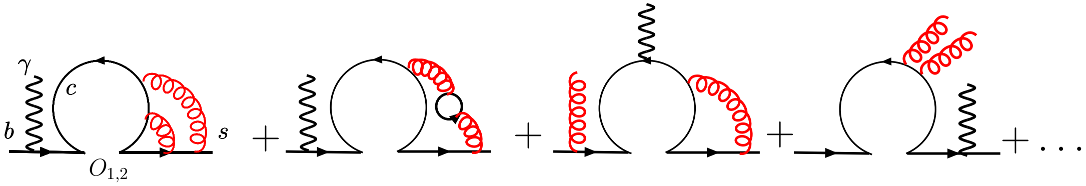
- calculate the leading term of large m_c expansion for $P_2^{(2),rem}$ and interpolate to physical m_c
- making assumptions for $P_2^{(2),rem}$ at $m_c = 0$ is the source of the interpolation uncertainty



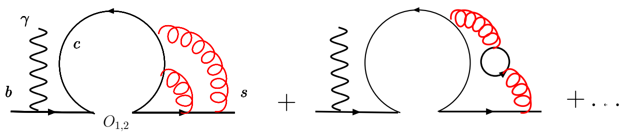
Reducing the overall uncertainty of $\mathcal{B}(\bar{B} \to X_s\gamma)^{\text{theo,NNLO}}_{E_\gamma > 1.6\,\text{GeV}}$

- removing the interpolation uncertainty

$\Longrightarrow$ need a complete calculation of $\langle s\gamma|O_{1,2}|b\rangle$ at $m_c \neq 0$



$\longrightarrow$ working on the virtual part [R. B, Czakon, Schutzmeier]



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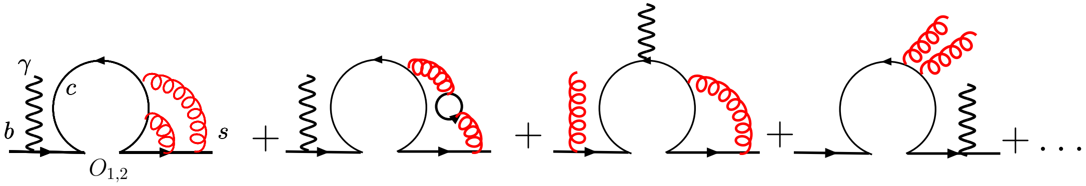

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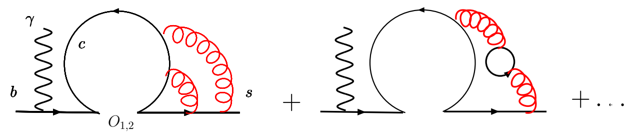

- reducing the interpolation uncertainty

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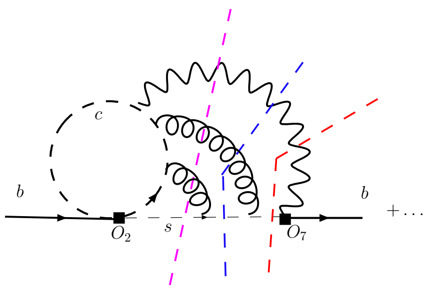
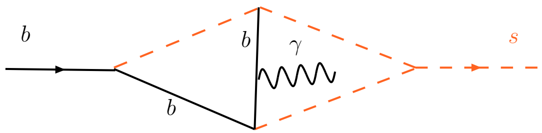


in progress [R. B, Czakon, Schutzmeier]

Removing the interpolation uncertainty: virtual part

- Around $\mathbf{400}$ 3-loop vertex diagrams are involved in the on-shell calculation of $\langle s\gamma|O_{1,2}|b\rangle^{virt}$, with two different scales: m_b and m_c
  - amplitudes reduced to linear combinations of master integrals using integration by parts identities and Laporta's algorithm
  - About $\mathbf{500}$ masters are involved in the bare 3-loop calculation **!**
  - complicated 2-loop diagrams needed for the renormalization: around $\mathbf{50}$ masters out of which 4 are non-planar vertex graphs

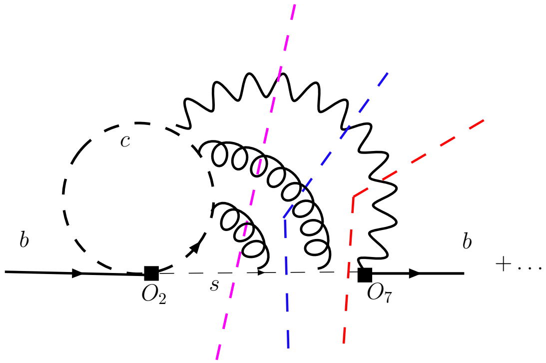
    e.g.



    - getting a result for this master involved solving 62 other integrals
  - masters are being calculated with a mixed approach: Mellin-Barnes and differential equations solved numerically
  - reduction to masters and their calculation are not yet complete, work in progress...

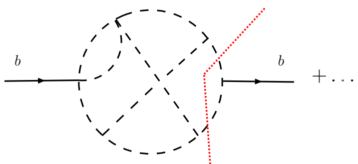
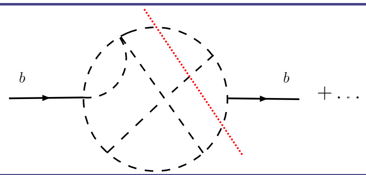
Would it be less complicated just to reduce this uncertainty?

- calculating $\mathcal{O}\left(\alpha_s^2\right)$ correction to $\langle s\gamma|O_{1,2}|b\rangle$ at $m_c = 0$ helps significantly in reducing the interpolation uncertainty

$\Longrightarrow$ 

up to 5-particles cuts: $\gamma s, \gamma sg, \gamma sgg, \gamma sq\bar{q}, \gamma sgq\bar{q}$

- 524 four-loop self-energy master integrals subdivided into two groups
  - with b-quark internal lines: doable with differential eqts derived off-shell and solved numerically $\longrightarrow$ boundaries are the problem ...
  - massless masters: no straightforward way to get all of them

2 particles cut	
3 particles cut	

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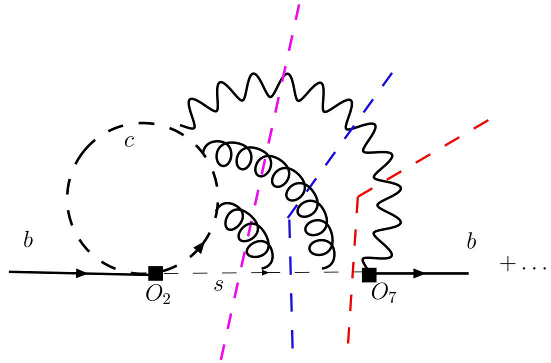
$\Longrightarrow$ 

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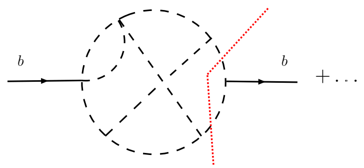
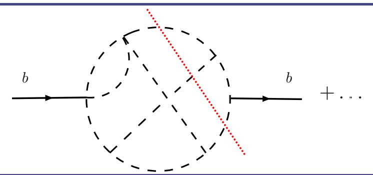
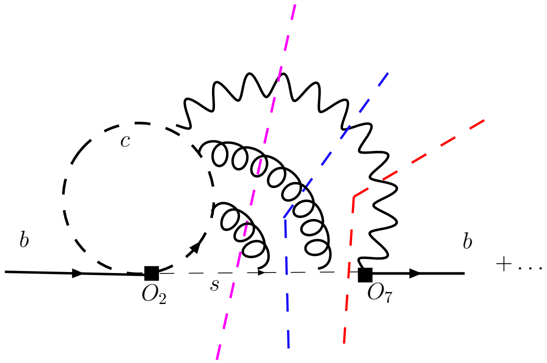
2 particles cut	
3 particles cut	

- derive an MB representation for the loop part
- do the phase space integral
- do the MB integral

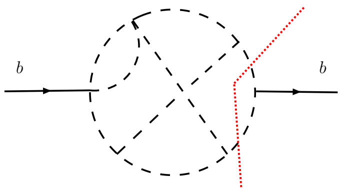
Would it be less complicated just to reduce this uncertainty?

- calculating $\mathcal{O}\left(\alpha_s^2\right)$ correction to $\langle s\gamma|O_{1,2}|b\rangle$ at $m_c = 0$ helps significantly in reducing the interpolation uncertainty

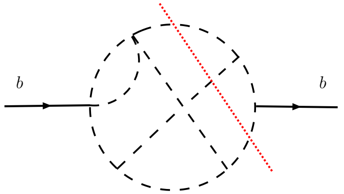
$\Longrightarrow$ 

up to 5-particles cuts: $\gamma s, \gamma sg, \gamma sgg, \gamma sq\bar{q}, \gamma sgq\bar{q}$

- 524 four-loop self-energy master integrals subdivided into two groups
  - with b-quark internal lines: doable with differential eqts derived off-shell and solved numerically $\longrightarrow$ boundaries are the problem ...
  - massless masters: no straightforward way to get all of them



$$= \frac{1.571\,i}{\varepsilon^5} - \frac{14.804 - 6.283\,i}{\varepsilon^4} - \frac{59.218 + 69.002\,i}{\varepsilon^3} + \frac{211.983 - 382.96\,i}{\varepsilon^2} + \ldots$$



$$= -\frac{3.142\,i}{\varepsilon^5} + \frac{19.74 - 12.567\,i}{\varepsilon^4} + \frac{78.957 + 86.326\,i}{\varepsilon^3} - \frac{282.645 - 596.976\,i}{\varepsilon^2} + \ldots$$

Would it be less complicated just to reduce this uncertainty?

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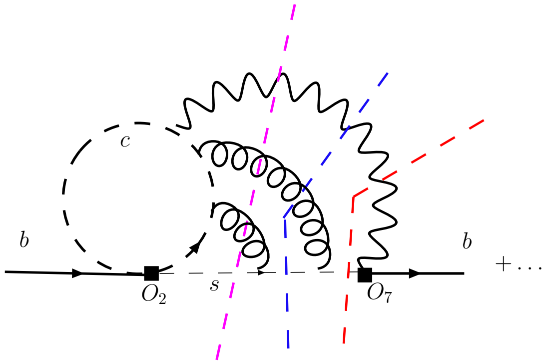
$\Longrightarrow$ 

up to 5-particles cuts: $\gamma s, \gamma sg, \gamma sgg, \gamma sq\bar{q}, \gamma sgq\bar{q}$

- 524 four-loop self-energy master integrals divided into two groups
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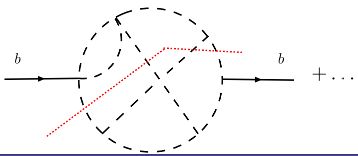
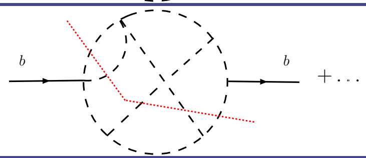
4 particles cut	
5 particles cut	

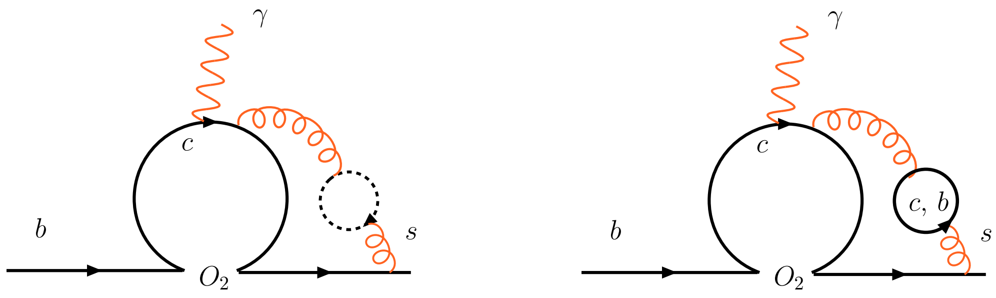
- still under investigation ...

$\mathcal{O}\left(\alpha_s^2 n_f\right)$ correction to $\langle s\gamma|O_{1,2}|b\rangle$

- analyzing the color structure of O_1 and O_2 shows that at $\mathcal{O}\left(\alpha_s^2\, n_f\right)$

$$\langle s\gamma|O_1|b\rangle = -\frac{1}{2\,N_c}\langle s\gamma|O_2|b\rangle$$

- Two types of diagrams are involved in the $\boldsymbol{\mathcal{O}\left(\alpha_s^2 n_f\right)}$ correction to $\boldsymbol{\langle s\gamma|O_2|b\rangle}$



1. massless quark loop insertion
   - 36 diagrams expressed through 18 master integrals
   - the masters are calculated using Mellin Barnes method in two ways. In both of them an MB representation is derived automatically for each master integral then analytically continued using the MB package [MB : Czakon 05] , [MBrepresentation : Chachamis, Czakon 06] .

MB for masters with massless c and b loop insertions

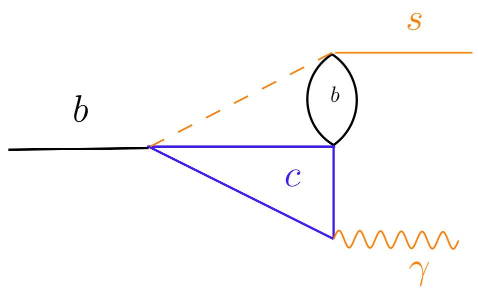
- first way: a numerical integration of the MB representations is performed for specific values of z using the `MB` package (exact dependence on $z = m_c^2/m_b^2$ is therefore kept)
- second way:
  - perform an expansion in $z = m_c^2/m_b^2$ by closing contours
  - coefficients of the expansion are given by at most a 1-dimensional MB integral expressed as a sum over residues
  - sum these infinite series using `XSummer` [Moch & Uwer 05]
- The results of both methods are consistent with each other. our z-expanded result (second way) confirms the one of [Bieri, Greub, Steinhauser 03]

MB for masters with massless c and b loop insertions

2. massive quark loop insertion

- b-loop: 142 integrals reduced to 47 masters
- c-loop: 181 integrals reduced to 38 masters
- MB alone was not enough to calculate all the master integrals

  due to poor convergence, eg.



- for these cases we used differential equations numerically

differential eqts for the massive case

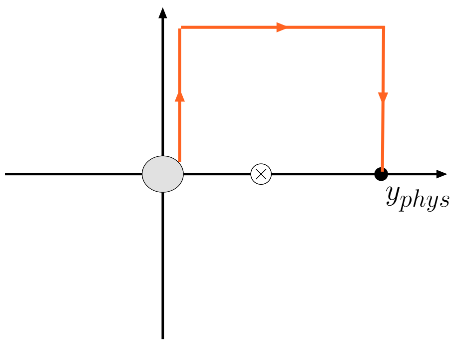
- Our masters V_i are functions of ϵ and $z = m_c^2/m_b^2$ or its inverse $y = z^{-1}$
  $\Longrightarrow$ a system of differential eqts in y can be derived:

$$\frac{d}{dy} V_i(y,\epsilon) = A_{ij}(y,\epsilon) V_j(y,\epsilon)$$

- expand the masters in ϵ and y for $\epsilon,\ y \to 0$ using the ansatz:

$$V_i(y,\epsilon) = \sum_{nmk} c_{inmk} \epsilon^n y^m \log^k y, \quad {\scriptstyle n,m=-3,-2,\ldots;\ \ k=0,\ldots,3+n\,\Theta(n)}$$

- c_{inmk} are calculated recursively up to higher powers of y but not all of them can be determined from the differential eqt
- use boundary conditions from large mass expansion of vertex diagrams for $m_c \gg m_b$
  $\Longrightarrow$ all the masters are provided with high precision for $y \ll 1$.
- use this series as a starting point for the numerical integration that ends in the physical region $y \gg 1$.
- points of numerical instability on the real axis are avoided by shifting the integration path to the complex plane.



Results: massless approximation for $\langle s\gamma|O_2|b\rangle_{\mathcal{O}(\alpha_s^2 n_f)}$

$$\langle s\gamma|O_2|b\rangle_{\mathcal{O}(\alpha_s^2 n_f)} = \left(\frac{\alpha_s}{4\pi}\right)^2 m_b\, n_f\, \langle s\gamma|O_2|b\rangle_{n_f}^{(2),M}\ \bar{u}_s\, R\, \not{\epsilon}\, \not{q}\, u_b\,, \qquad M=\{0,m_c,m_b\}$$

- result in the massless approximation expanded in $z = m_c^2/m_b^2$ up to z^3 ($L = \ln z$)

$$\langle s\gamma|O_2|b\rangle_{n_f}^{(2),0} = \left(t_2^{(2)} \ln^2(m_b/\mu) + l_2^{(2)} \ln(m_b/\mu) + r_2^{(2)}\right),$$

- it confirms the result of [Bieri, Greub, Steinhauser 03]

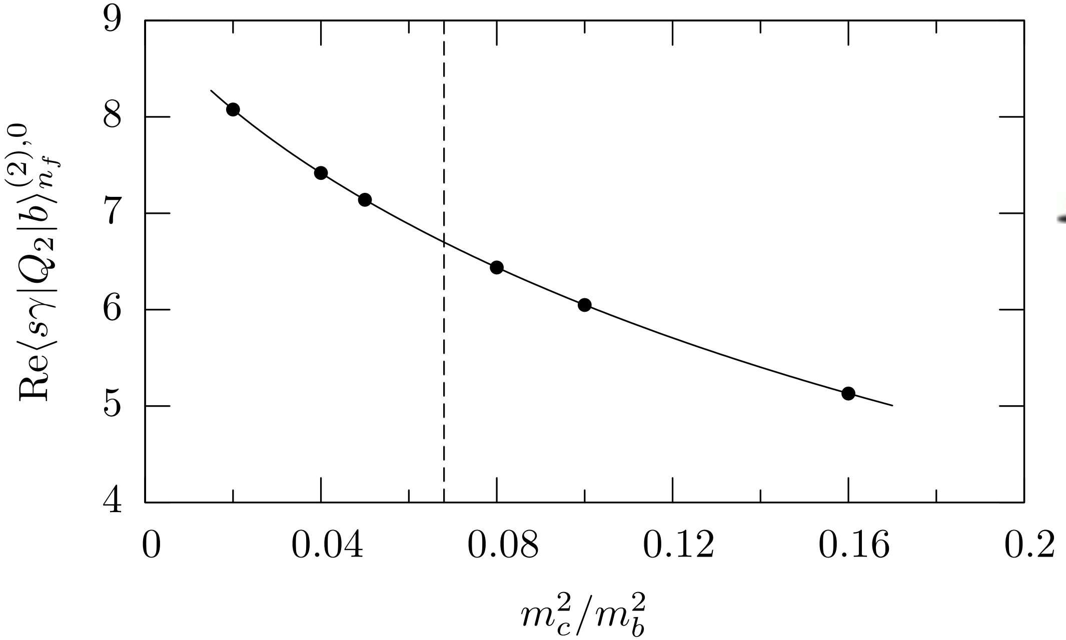
$$\begin{aligned}
t_2^{(2)} &= \frac{800}{243}\,,\\
\mathrm{Re}\left(l_2^{(2)}\right) &= \frac{16}{243}\Big(-145 + \left(288 - 30\pi^2 - 216\zeta(3) + 216L - 54\pi^2 L + 18L^2\right.\\
&\left. +6L^3\right) z + 24\pi^2 z^{3/2} + 6\left(18 + 2\pi^2 + 12L - 6\pi^2 L + L^3\right) z^2\\
&-\left(9 + 14\pi^2 - 182L + 126L^2\right) z^3\Big) + \mathcal{O}(z^4)\,,\\
\mathrm{Im}\left(l_2^{(2)}\right) &= \frac{16\pi}{243}\Big(-22 + \left(180 - 12\pi^2 + 36L + 36L^2\right) z\\
&-\left(12\pi^2 - 36L^2\right) z^2 + \left(112 - 48L\right) z^3\Big) + \mathcal{O}(z^4)\,,
\end{aligned}$$

$$\begin{aligned}
\mathrm{Re}\left(r_2^{(2)}\right) &= \frac{67454}{6561} - \frac{124\pi^2}{729} - \frac{4}{1215}\left(11280 - 1520\pi^2 - 171\pi^4 - 5760\zeta(3)\right.\\
&\left. +6840L - 1440\pi^2 L - 2520\zeta(3)L + 120L^2 + 100L^3 - 30L^4\right) z\\
&-\frac{64\pi^2}{243}\left(43 - 12\ln(2) - 3L\right) z^{3/2} - \frac{2}{1215}\left(11475 - 380\pi^2 + 96\pi^4\right.\\
&+7200\zeta(3) - 1110L - 1560\pi^2 L + 1440\zeta(3)L + 990L^2 + 260L^3\\
&\left. -60L^4\right) z^2 + \frac{2240\pi^2}{243} z^{5/2} - \frac{2}{2187}\left(62471 - 2424\pi^2 - 33264\zeta(3)\right.\\
&\left. -19494L - 504\pi^2 L - 5184L^2 + 2160L^3\right) z^3 + \mathcal{O}(z^{7/2})\,,\\
\mathrm{Im}\left(r_2^{(2)}\right) &= \frac{4\pi}{729}\Big(495 - 12\left(375 - 19\pi^2 + 36\zeta(3) + 84L + 48L^2 - 6L^3\right) z\\
&+6\left(207 + 38\pi^2 - 72\zeta(3) - 126L - 78L^2 + 12L^3\right) z^2\\
&+8\left(67 - 12\pi^2 - 48L\right) z^3\Big) + \mathcal{O}(z^4)\,,
\end{aligned}$$

Results: massless approximation for $\langle s\gamma|O_2|b\rangle_{\mathcal{O}(\alpha_s^2 n_f)}$

- numerical evaluation in terms of multi-fold MB integrals:
  result provided as a fitting formula as a function of z

$$\begin{aligned}
\mathrm{Re}\langle s\gamma|O_2|b\rangle^{(2),0}_{n_f} = \\
&+ \quad 9.080 - 0.7624\, z - 5.069\, z^2 + 12.61\, z\, \ln z \\
&+ \quad (-9.679 + 5.157\, z + 1.726\, z^2 - 16.18\, z\, \ln z)\, \ln(m_b/\mu) \\
&+ \quad \frac{800}{243}\, \ln^2(m_b/\mu)
\end{aligned}$$

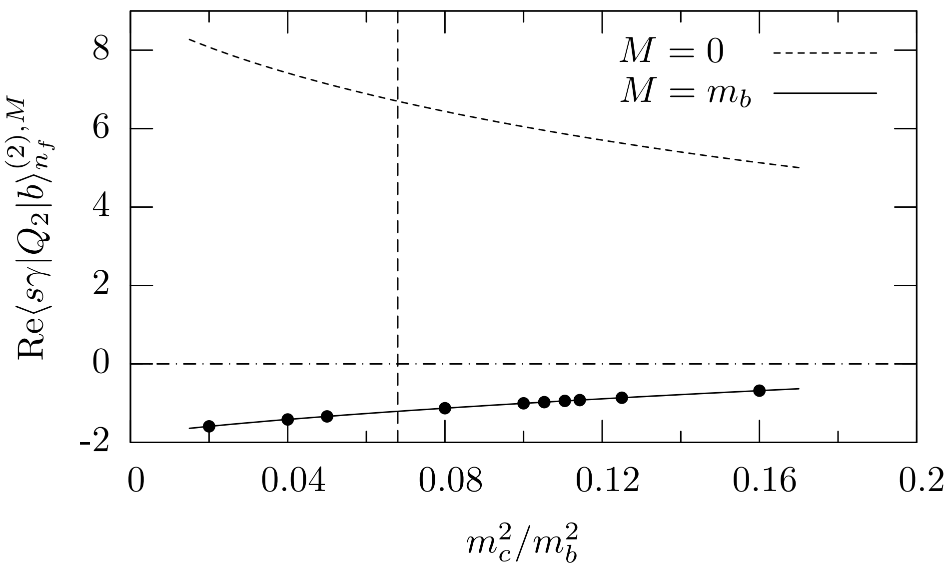


- fit function reproduces the exact values with a relative precision of 10^{-4}

Results: $\langle s\gamma|O_2|b\rangle_{\mathcal{O}(\alpha_s^2 n_f)}$ using massive b

- matrix element for O_2 using diagrams with a massive b-quark loop insertion

$$
\begin{aligned}
\text{Re}\langle s\gamma|O_2|b\rangle^{(2),m_b}_{n_f} = \\
- \quad & 1.836 + 2.608\, z + 0.8271\, z^2 - 2.441\, z\, \ln z \\
+ \quad & (-9.595 + 5.157\, z + 1.726\, z^2 - 16.18\, z\, \ln z)\ln(m_b/\mu) \\
+ \quad & \frac{800}{243}\ln^2(m_b/\mu)
\end{aligned}
$$

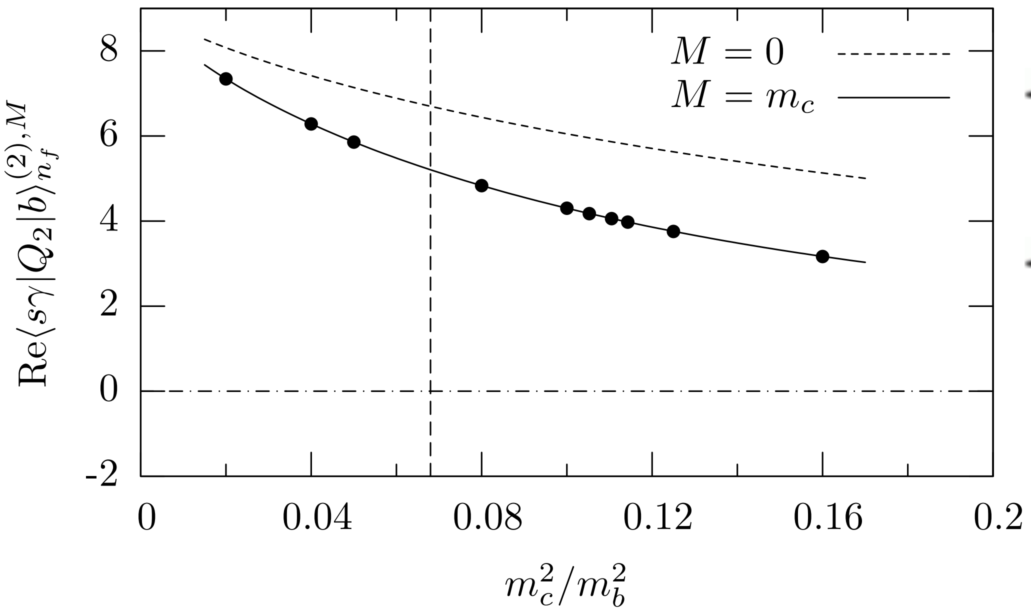


- fit function reproduces the exact values with a relative precision of 10^{-4}
- massless approximation overestimates the massive b result by a factor of 6 and has the opposite sign !

Results: $\langle s\gamma|O_2|b\rangle_{\mathcal{O}(\alpha_s^2 n_f)}$ using massive c

- matrix element for O_2 using diagrams with a massive c-quark loop insertion

$$
\begin{aligned}
\mathrm{Re}\langle s\gamma|O_2|b\rangle_{n_f}^{(2),m_c} &= 9.099 + 13.20\,z - 19.68\,z^2 + 25.71\,z\,\ln z \\
&+ (-9.679 + 13.62\,z - 13.94\,z^2 - 12.98\,z\,\ln z)\ln(m_b/\mu) \\
&+ \frac{800}{243}\ln^2(m_b/\mu)
\end{aligned}
$$



- fit function reproduces the exact values with a relative precision of 10^{-4}
- less pronounced differences for the c-quark $\longrightarrow$ moderate negative corrections wrt. massless approximation

Results: $\mathcal{O}(\alpha_s^2 n_f)$ for $m_c \gg m_b$

- we confirm the $\mathcal{O}(\alpha_s^2 n_f)$ contribution to Re$\langle s\gamma|O_2|b\rangle_{n_f}^{(2)}$ for $m_c \gg m_b$ in the massless approximation [Misiak, Steinhauser06]
- new results for massive b and c and for $\mu = m_b$:

$$\begin{aligned}
\langle s\gamma|Q_2|b\rangle_{n_f}^{(2),m_b} &= 4.25648 + 0.503085 \ln z + 0.888889 \ln^2 z \\
&+ \frac{1}{z}\left(-0.725053 - 1.80916 \ln z + 0.0938272 \ln^2 z\right) \\
&+ \frac{1}{z^2}\left(-1.39486 - 0.968501 \ln z - 0.147443 \ln^2 z\right) + \mathcal{O}\left(\frac{1}{z^3}\right),
\end{aligned}$$

$$\begin{aligned}
\langle s\gamma|Q_2|b\rangle_{n_f}^{(2),m_c} &= 1.67932 + 0.526749 \ln z + 0.823045 \ln^2 z \\
&+ \frac{1}{z}\left(0.20839 + 0.11775 \ln z + 0.128395 \ln^2 z\right) \\
&+ \frac{1}{z^2}\left(-0.0360638 - 0.0470166 \ln z + 0.0324515 \ln^2 z\right) + \mathcal{O}\left(\frac{1}{z^3}\right).
\end{aligned}$$

Summary

- Removing the interpolation uncertainty or even just reducing it requires an involved $O(\alpha_s^2)$ calculation of the $\langle s\gamma|O_{1,2}|b\rangle$

  $\longrightarrow$ work in progress $[\mathtt{R.\ B, Czakon, Schutzmeier}]$

- current NNLO theoretical estimate of $\mathcal{B}(\bar{B} \to X_s\gamma)$ used $\mathcal{O}\left(\alpha_s^2\, n_f\right)$ result for $\langle s\gamma|O_{1,2}|b\rangle$ calculated for $n_f = 5$ massless quarks

  $\longrightarrow$ our independent calculation done in two different ways confirms the existing result of $[\mathtt{Bieri, Greub, Steinhauser\ 03}]$

- Validity of the massless approximation has been explicitly checked by keeping full mass dependence of b and c quarks in the fermionic loop inserted into the gluon propagator

  $\longrightarrow$ sizable contribution from b-loop

  $\longrightarrow$ estimated impact of the mass corrections on $\mathcal{B}(\bar{B} \to X_s\gamma)$[a] $\approx +$ 1-2% depending on μ_b

---

[a] thanks to M. Misiak