Factorization and Unitarity of Helicity Amplitudes

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Outline

All fundamental processes are reversible

Feynman

- Trees: Collinear-Limit $\Leftrightarrow$ Recurrence Relation
  - Spinor Formalism
  - MHV Amplitudes
  - CSW diagrams
  - BCFW Recurrence Relation
- Loop: Cutting Loops $\Leftrightarrow$ Sewing Trees
  - Unitarity & Cut-Constructibility
  - General Algorithm for Cuts in 4-dim: multiple-cuts
  - Recurrence for Rational Terms
  - D-dimensional Cuts
  - Unitarity-motivated momentum decomposition

Spinor Formalism

Xu, Zhang, Chang

Berends, Kleiss, De Causmaeker

Gastmans, Wu

Gunion, Kunzst

- on-shell massless Spinors

$$|i\rangle \equiv |k_i^+\rangle \equiv u_+(k_i) = v_-(k_i) \ , \qquad [i| \equiv \langle k_i^+| \equiv \bar{u}_+(k_i) = \bar{v}_-(k_i) \ ,$$

- $k^2 = 0$: $\quad k_{a\dot{a}} \equiv k_\mu \sigma^\mu_{a\dot{a}} = \lambda^k_a \tilde{\lambda}^k_{\dot{a}} \qquad \text{or} \qquad \not{k} = |k\rangle[k| + |k]\langle k|$

- Spinor Inner Products

$$\langle i\, j\rangle \equiv \langle i^-|j^+\rangle = \varepsilon_{ab}\, \lambda_i^a \lambda_j^b = \sqrt{|s_{ij}|}\; e^{i\Phi_{ij}} \ , \qquad [i\, j] \equiv \langle i^+|j^-\rangle = \varepsilon_{\dot{a}\dot{b}}\, \tilde{\lambda}_i^{\dot{a}}\, \tilde{\lambda}_j^{\dot{b}} = -\langle i\, j\rangle^* \ ,$$

with $s_{ij} = (k_i + k_j)^2 = 2k_i \cdot k_j = \langle i\, j\rangle[j\, i]$.

- Polarization Vector

$$\varepsilon^+_\mu(k;q) = \frac{\langle q|\gamma_\mu|k]}{\sqrt{2}\langle q\, k\rangle} \ , \qquad \varepsilon^-_\mu(k;q) = \frac{[q|\gamma_\mu|k\rangle}{\sqrt{2}[k\, q]} \ ,$$

with $\varepsilon^2 = 0 \ , \quad k_\mu \cdot \varepsilon^\pm_\mu(k;q) = 0 \ , \quad \varepsilon^+ \cdot \varepsilon^- = -1$.

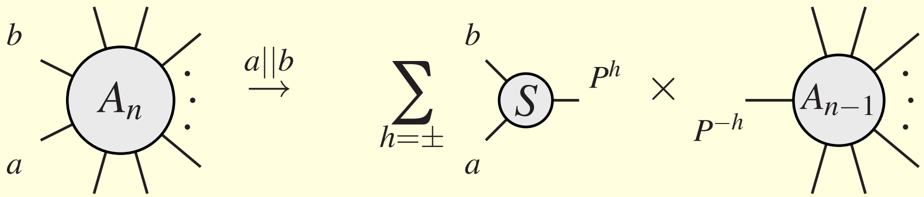
Changes in ref. mom. q are equivalent to gauge trasformations.

Factorization of Tree Amplitudes

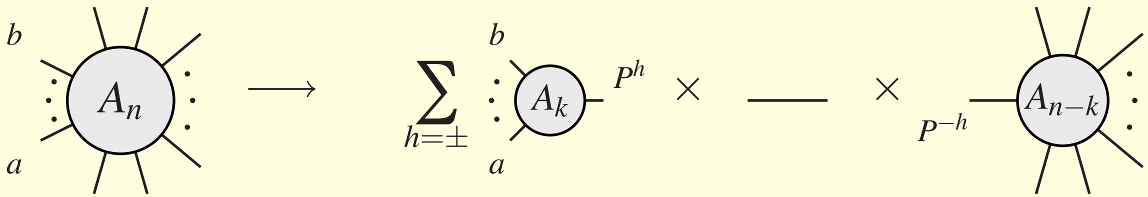
Parke & Taylor

Berends & Giele

- Two-Particles Collinearity



- Multi-Particles Collinearity: $(p_a + \ldots + p_b)^2 \to 0$



Gluon Amplitudes in Twistor Space

Witten [hep-th/0312155]

- Twistor Space Penrose (1967): $(Z_1, Z_2, Z_3, Z_4) = (\lambda^1, \lambda^2, \mu^{\dot 1}, \mu^{\dot 2}), \quad \mu_{\dot a} = -i \frac{\partial}{\partial \tilde\lambda^{\dot a}}$

as a Fourier transform with respect to the **anti-holomorphic** spinors.

- n-gluon Amplitudes

$$A_n^{\rm MHV}(1^-, 2^-, 3^+, \ldots, n^+) = \frac{\langle 1\, 2\rangle^4}{\langle 1\, 2\rangle\langle 2\, 3\rangle \cdots \langle n\, 1\rangle} \delta^4\left(\sum_{k=1}^n p_k\right)$$

$$= \frac{\langle 1\, 2\rangle^4}{\langle 1\, 2\rangle\langle 2\, 3\rangle \cdots \langle n\, 1\rangle} \int d^4x \, \exp\left(i \sum_{k=1}^n \langle k|x|k]\right)$$

Holomorphic in $\langle i\, j\rangle$-product !!!

In Twistor Space, as a consequence of the holomorphy,

$$\tilde A^{MHV} = \prod_k \int [d k\, k] \, e^{i[\mu_k k]} \, A^{MHV} = \frac{\langle 1\, 2\rangle^4}{\langle 1\, 2\rangle\langle 2\, 3\rangle \cdots \langle n\, 1\rangle} \int d^4x \prod_k \int [d k\, k] \, e^{i[\Omega_k k]}$$

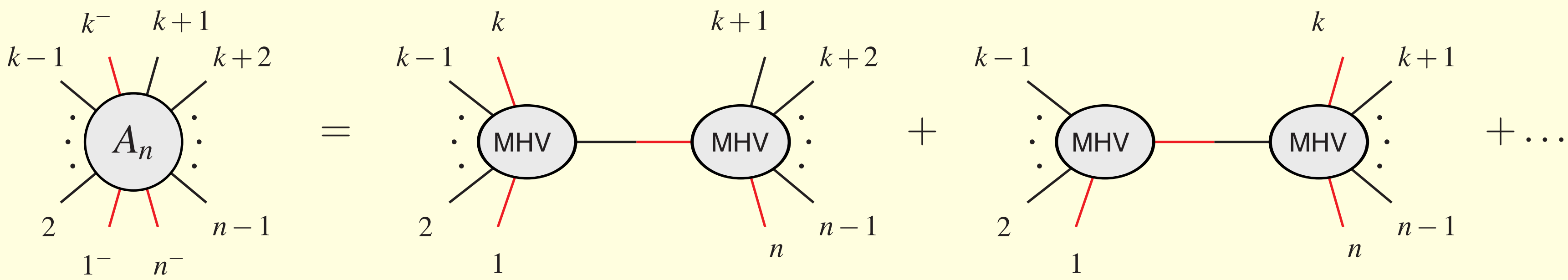
$$= \frac{\langle 1\, 2\rangle^4}{\langle 1\, 2\rangle\langle 2\, 3\rangle \cdots \langle n\, 1\rangle} \int d^4x \prod_k \delta^2([\Omega_k|) \,, \qquad [\Omega_k| = [\mu_k| + \langle k|\not x$$

MHV amplitudes supported on *lines* in Twistor Space corresponding to *points* in Minkowsky Space.

MHV-rules

All the non-MHV n-gluon tree amplitudes are expressed as sum of tree graphs whose vertices are MHV amplitudes continued off-shell, and connected by scalar propagators

Cachazo, Svrček, Witten (2004) .



- CSW off-shell continuation (Massless Projection) Bena, Bern, Kosower

$$P_\mu = P^\flat_\mu + \frac{P^2}{2P\cdot\eta}\,\eta_\mu \quad \Rightarrow \quad \langle i\,P\rangle \to \langle i\,P^\flat\rangle$$

with $\eta^2 = 0$ an arbitrary reference momentum and $(P^\flat)^2 = 0$.

- (some) Applications:

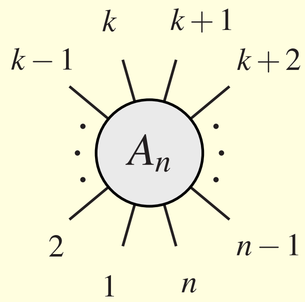
- fermion-gluon coupling Georgiou & Khoze
- massive Higgs Dixon, Glover & Khoze
- Vector Boson Currents Bern, Forde, Kosower & PM
- massless QED Ozeren & Stirling
- multi-collinear Splitting Birthwright, Glover, Khoze & Marquard

BCFW Recurrence Relation

Britto, Cachazo, Feng; [hep-th/0412308]

& Witten; [hep-th/0501052]

Consider $A(1,2,\ldots,n)$, and pick up any **two special legs**, say 1 and n.



- Analytic continuation, $A \rightarrow A(z)$:

$$
\begin{aligned}
p_1^\mu \quad &\rightarrow \quad p_1^\mu(z) \equiv p_1^\mu + z\langle 1|\gamma^\mu|n] \\
p_n^\mu \quad &\rightarrow \quad p_n^\mu(z) \equiv p_n^\mu - z\langle 1|\gamma^\mu|n]
\end{aligned}
$$

If $n \in \{i,\ldots,j\}$ && $1 \notin \{i,\ldots,j\}$

$$\Rightarrow \quad P_{ij}^2 \equiv (p_i + \ldots + p_j)^2 \rightarrow P_{ij}^2(z) = P_{ij}^2 - z\langle 1|\not{P}_{ij}|n]$$

$\therefore$ the propagator develops a **simple pole** @ $z = z_{ij} \equiv \frac{P_{ij}^2}{\langle 1|P_{ij}|n]}$.

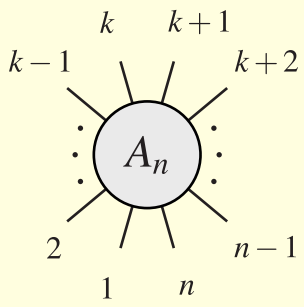
- **Cauchy's Residue Theorem**

$$\frac{1}{2\pi i}\oint \frac{dz}{z}A(z) = C_\infty = A(0) + \sum_\alpha \mathrm{Res}|_{z=z_\alpha}\frac{A(z)}{z}$$

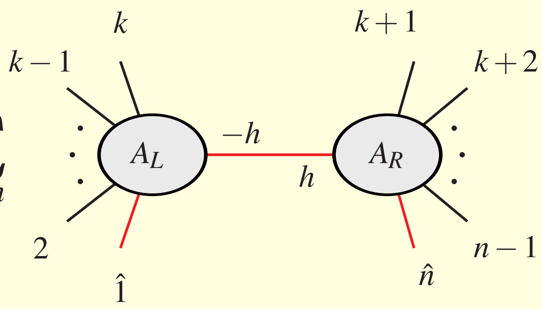
To get back the physical amplitude, $A(0)$, use the residue theorem

Investigation of tree level Feynman diagrams shows that there exist shifts of momenta yielding the surface term to vanish $C_\infty = 0$!

- **BCFW on-shell Recurrence Relation**



$$= A(0) = -\sum_\alpha \mathrm{Res}|_{z=z_\alpha}\frac{A(z)}{z} = \sum_{k,h}$$



On-Shell Complex Momenta enable the *inversion* of the Collinear Limit!

- SUSY massless fermions Luo & Wen
- MHV vs BCFW Risager
- Gravity Bedford, Brandhuber, Spence & Travaglini; Cachazo & Svrĉek; Bjerrum-Bohr, Dunbar, Ita, Perkins & Risager; Benincasa & Cachazo;
- Feynman vs BFCW Draggiotis, Kleiss, Lazopoulos & Papadopoulos;
- Largest Time Eqn & BFCW Vaman & Yao;
- massive scalars and fermions Badger, Glover, Khoze & Svrĉek; Forde & Kosower; Ferrario, Rodrigo & Talavera;
- massive Higgs Badger, Dixon, Glover & Khoze

Massive Particles

- Massive Propagators Badger, Glover, Khoze, Svrček

$$P_{ij}^2 - m^2 \to P_{ij}^2(z) - m^2 = P_{ij}^2 - m^2 - z\langle 1|P_{ij}|n]$$

$\therefore$ the propagator develops a **simple pole** @ $z = z_{ij} \equiv \frac{P_{ij}^2 - m^2}{\langle 1|P_{ij}|n]}$.

- Polarization of External Massive Fermions Schwinn, Weinzierl

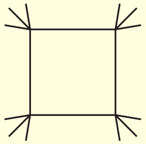
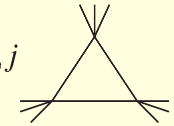
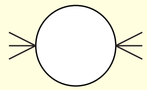
$$\not{P}^\flat = \not{P} - \frac{m^2}{2P.\eta}\not{\eta} \,, \qquad (P^\flat)^2 = 0 \,;$$

$$|P\rangle \equiv \frac{(\not{P} + m)}{[P^\flat \eta]}|\eta] = |P^\flat\rangle + \frac{m}{[P^\flat \eta]}|\eta]$$

For massive particles, the reference momentum η is associated to the spin-quantization axis. Helicity amplitudes, in this case, depend on the choice of η.

One Loop Amplitudes

P-V Tensor Reduction

$A = \sum_i c_{4,i}$  $+ \sum_j c_{3,j}$  $+ \sum_k c_{2,k}$  $+$ rational

Since the D-regularised scalar functions associated to **boxes** $(I_4^{(4m)}, I_4^{(3m)}, I_4^{(2m,\mathrm{e})}, I_4^{(2m,\mathrm{h})}, I_4^{(1m)}, I_4^{(0m)})$, **triangles** $(I_3^{(3m)}, I_3^{(2m)}, I_3^{(1m)})$ and **bubbles** (I_2) are analytically known

't Hooft & Veltman (1979)

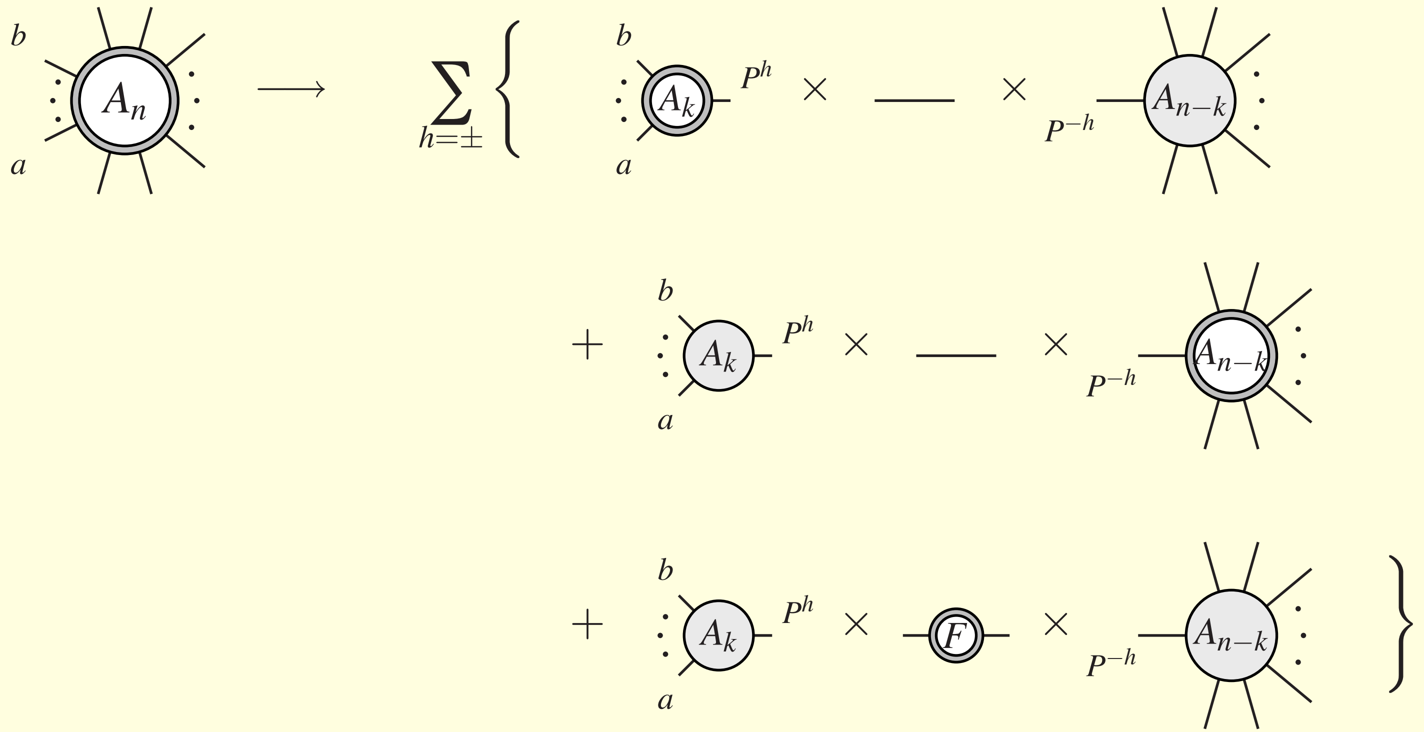
Bern, Dixon & Kosower (1993)

- A is known, once the coefficients c_4, c_3, c_2 and the rational term are known: they all are rational functions of spinor products $\langle i\, j\rangle$, $[i\, j]$

Factorization of One-Loop Amplitudes

Bern & Chalmers (1995)

- Multi-Particles Collinearity: $(p_a + \ldots + p_b)^2 \to 0$



- naîve Recurrence: doesn't work! Bern, Dixon, Kosower (2005)

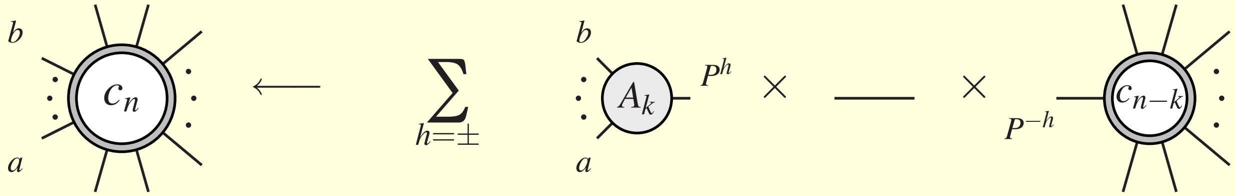
because of unreal poles in complex-momentum space from *1L-Splitting Functions*: $\frac{\langle a b \rangle}{[a b]}$

- but ...

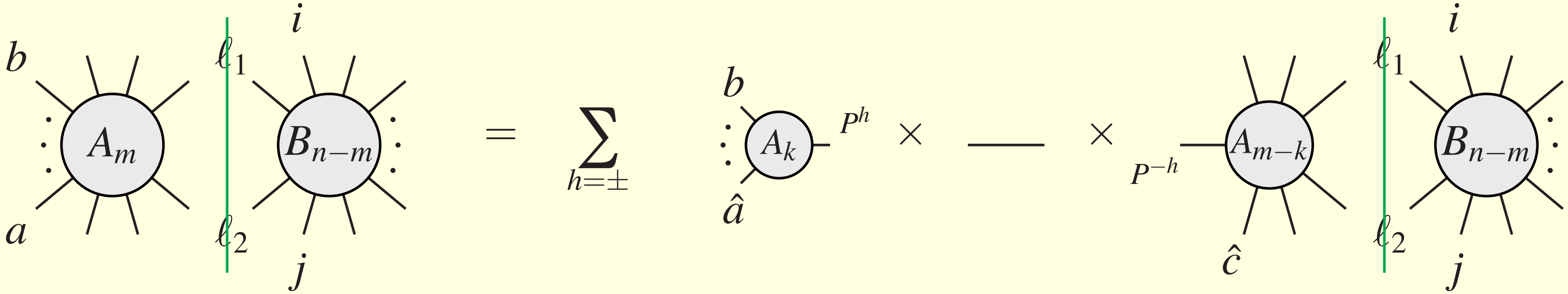
Recurrence for Coefficients

Bern, Bjerrum-Bohr, Dunbar, Ita (2005)

- BFCW-type:



- $P_{i,\dots,j}$–channel:

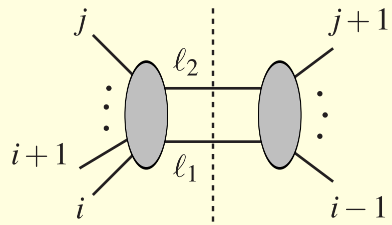


which works only for special helicity-configurations, ex. $(-,-,-,\dots,+,+,+)$

Unitarity & Cut-Constructibility

- Discontinuity accross the Cut

Cut Integral in the P_{ij}^2-channel



$$C_{i\ldots j} = \Delta(A_n^{\text{1-loop}}) = \int d^4\Phi \, A^{\text{tree}}(\ell_1, i, \ldots, j, \ell_2) A^{\text{tree}}(-\ell_2, j+1, \ldots, i-1, -\ell_1)$$

with

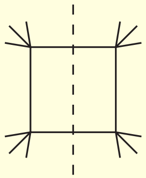
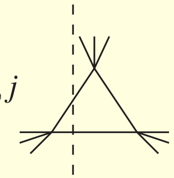
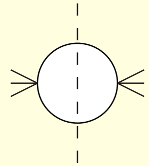
$$d^4\Phi = d^4\ell_1 \; d^4\ell_2 \; \delta^{(4)}(\ell_1 + \ell_2 - P_{ij}) \; \delta^{(+)}(\ell_1^2) \; \delta^{(+)}(\ell_2^2)$$

- loop-Reconstruction

Bern, Dixon, Dunbar & Kosower
Anastasiou & Melnikov
Brandhuber, Mc Namara, Spence & Travaglini

- channel-by-channel reconstruction of the loop-intgeral: $\delta^{(+)}(p^2) \leftrightarrow \frac{1}{(p^2 - i0)}$

- loop-tools integrations: PV-tensor reduction, or integration-by-parts identitities

$$C_{i\ldots j} = \Delta(A_n^{\text{1-loop}}) = \sum_i c_{4,i}$$  $$+ \sum_j c_{3,j}$$  $$+ \sum_k c_{2,k}$$ 

- The Cut carries information about the coefficients.

- In 4-dim we loose any information about the rational term

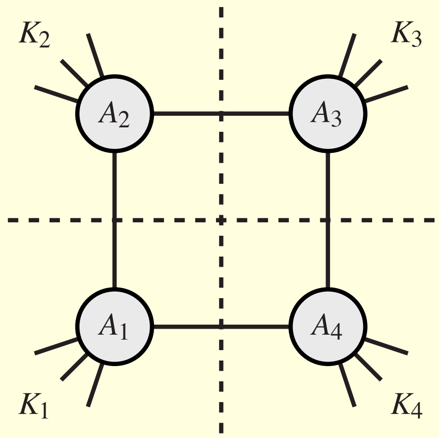
- coefficients show up entangled in a given cut: how do we disentangle them?

The polylogarithmic structure of boxes, 3m-triangles, and bubbles is different. Therefore their multiple cuts have specific signature which enable us to distinguish unequivocally among them.

Quadruple Cuts

Boxes

- Multiple Cuts   Bern, Dixon, Dunbar, Kosower (1994)



The discontinuity across the leading singularity, *via* quadruple cuts, is **unique**, and corresponds to the **coefficient** of the master **box**   Britto, Cachazo, Feng (2004)

$$c_{4,i} \quad \propto \quad A_1^{\text{tree}} A_2^{\text{tree}} A_3^{\text{tree}} A_4^{\text{tree}}$$

with a frozen loop momentum: $\ell^\mu = \alpha K_1^\mu + \beta K_2^\mu + \gamma K_3^\mu + \delta \varepsilon^\mu_{\nu\rho\sigma} K_1^\nu K_2^\rho K_3^\sigma$

Double Cuts

Triangles & Bubbles

$$C_{i\ldots j} = \Delta(A_n^{\text{1-loop}}) = \int d^4\Phi \, A^{\text{tree}}(\ell_1, i, \ldots, j, \ell_2) A^{\text{tree}}(-\ell_2, j+1, \ldots, i-1, -\ell_1)$$

with $d^4\Phi = d^4\ell_1 \, d^4\ell_2 \, \delta^{(4)}(\ell_1 + \ell_2 - P_{ij}) \, \delta^{(+)}(\ell_1^2) \, \delta^{(+)}(\ell_2^2)$

- Twistor-motivated Integration Measure — Cacahazo, Svrček & Witten (2004)

Use the $\delta^{(4)}$ integral to reduce just to a single loop momentum variable ℓ such that:

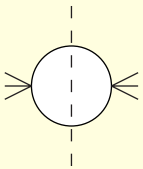
$$\boldsymbol{\ell} = |\ell\rangle[\ell| \equiv t|\lambda\rangle[\lambda|$$

$$\Rightarrow \int d^4\Phi = \int d^4\ell \, \delta^{(+)}(\ell^2) \, \delta^{(+)}((\ell - P_{ij})^2) = \int \frac{\langle\lambda \, d\lambda\rangle[\lambda \, d\lambda]}{\langle\lambda|P_{ij}|\lambda]} \int_0^\infty t \, dt \delta^{(+)}\left(t - \frac{P_{ij}^2}{\langle\lambda|P_{ij}|\lambda]}\right)$$

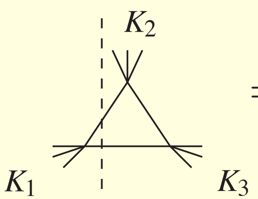
Double-Cuts $\oplus$ Spinor-Integration

Britto, Buchbinder, Cachazo & Feng [hep-ph/0503132]

Britto, Feng & PM [hep-ph/0602187]



$$= \int d^4\ell \, \delta^{(+)}(\ell^2) \, \delta^{(+)}((\ell - K)^2) = K^2 \int \frac{\langle \lambda \, d\lambda \rangle [\lambda \, d\lambda]}{\langle \lambda | K | \lambda]^2} = 1 \; ;$$



$$= \int d^4\ell \, \delta^{(+)}(\ell^2) \, \frac{\delta^{(+)}((\ell - K_1)^2)}{(\ell + K_3)^2} = \int \frac{\langle \lambda \, d\lambda \rangle [\lambda \, d\lambda]}{\langle \lambda | K_1 | \lambda] \langle \lambda | Q | \lambda]} = \int_0^1 dx \int \frac{\langle \lambda \, d\lambda \rangle [\lambda \, d\lambda]}{\langle \lambda | R | \lambda]^2} = \int_0^1 dx \frac{1}{R^2}$$

$$\not{Q} = (K_3^2 / K_1^2) \not{K}_1 + \not{K}_3 \, , \quad \not{R} = (1-x) \not{K}_1 + x \not{Q} \; \Rightarrow R^2 \text{ quadratic in } x$$

- The discontinuity of a bubble is **rational** !!!
- The discontinuity a 3m-Triangle is a $\ln$**(irrational argument)** !!!

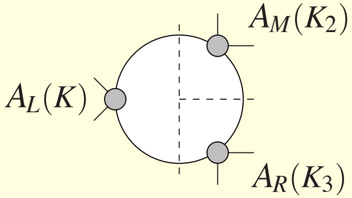
and if needed ...

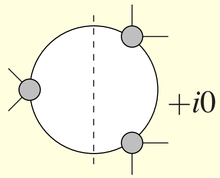
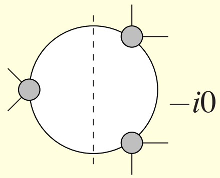
- The double cut detect box-coefficient as well. One can show that the discontinuity of a 1m-,2m-,3m-box is a $\ln$**(rational argument)** – but boxes are known from 4-ple cuts.

Triple Cuts

PM [hep-th/0611091]

Triangles



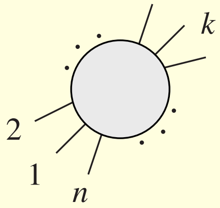
$= \frac{1}{(2\pi i)} \Big\{$  $-$  $\Big\}$

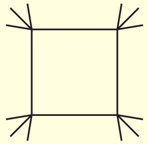
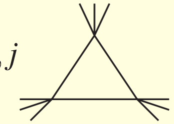
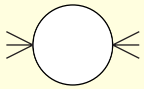
$$= \quad \cdots = \frac{1}{(2\pi i)} \int dx \left\{ \frac{1}{R^2 + i0} - \frac{1}{R^2 - i0} \right\} = \int dx \, \delta(R^2)$$

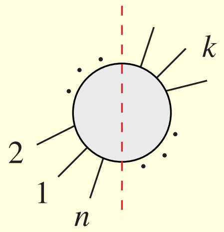
with

$$R^2 \quad = \quad ax^2 + 2bx + c \, , \qquad x_{1,2} = \frac{-b \pm \sqrt{\Delta}}{a} \, .$$

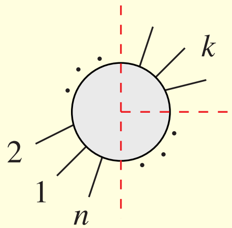
Cut-Constructible Part of One-Loop Amplitudes

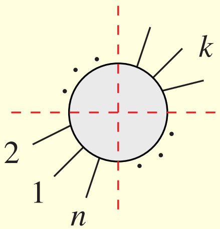
$A =$ 

$= \sum_i c_{4,i}$  $+ \sum_j c_{3,j}$  $+ \sum_k c_{2,k}$ 

$c_{2,i} = \Big[$ 

$\Big]_{\propto \text{ rational}}$

$c_{3,i} =$ 

$c_{4,i} =$ 

On-Shell Complex Momenta enable the *fulfillment* of the cut-constrains!

Master Formulae

Schouten identity to reduce $|\lambda]$

$$\frac{[\lambda\, a]}{[\lambda\, b]\; [\lambda\, c]} = \frac{[b\, a]}{[b\, c]} \frac{1}{[\lambda\, b]} + \frac{[c\, b]}{[c\, b]} \frac{1}{[\lambda\, c]} \tag{1}$$

Integration-by-Parts in $|\lambda]$

$$[\lambda\; d\lambda] \frac{[\eta\, \lambda]^n}{\langle\lambda|P|\lambda]^{n+2}} = \frac{[d\lambda\; \partial_{|\lambda]}]}{(n+1)} \frac{[\eta\, \lambda]^{n+1}}{\langle\lambda|P|\lambda]^{n+1} \langle\lambda|P|\eta]} \; . \tag{2}$$

Cauchy's Residue Theorem in $|\lambda\rangle$.

Residues in Feynman parameters, at the *zeroes* of the Standard Quadratic Function.

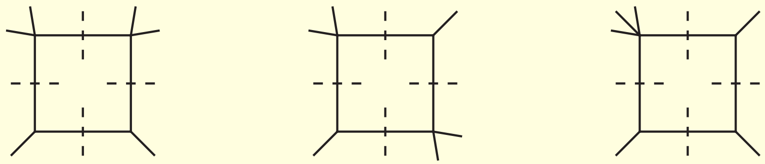
These *zeroes* are the signature of the Master Integrals.

6-gluon Amplitude

- Numerical Result: Ellis, Giele, Zanderighi (2006)
- Analytical Result:

Amplitude	$N = 4$	$N = 1$	$N = 0\|_{\mathrm{CC}}$	$N = 0\|_{\mathrm{rat}}$
$(--++++)$	BDDK'94	BDDK'94	BDDK'94	BDK'05, KF'05
$(-+-+++)$	BDDK'94	BDDK'94	BBST'04	BBDFK'06, XYZ'06
$(-++-++)$	BDDK'94	BDDK'94	BBST'04	BBDFK'06, XYZ'06
$(---+++)$	BDDK'94	BBDD'04	BBDI'05, BFM'06	BBDFK'06
$(--+-++)$	BDDK'94	BBCF'05, BBDP'05	BFM'06	XYZ'06
$(-+-+-+)$	BDDK'94	BBCF'05, BBDP'05	BFM'06	XYZ'06

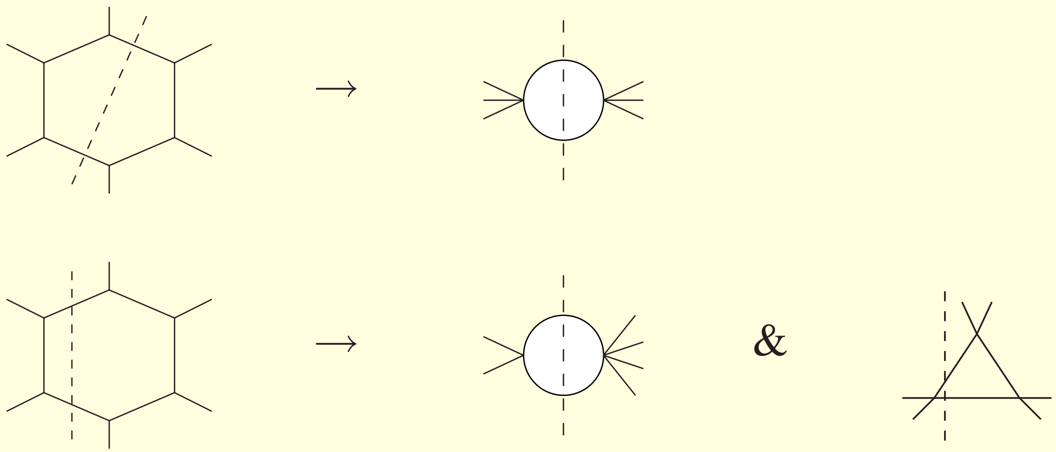
Quadruple Cuts



Bidder, Bjerrum-Bohr, Dunbar & Perkins (2005)

Double Cuts

Britto, Feng & PM (2006)



6-photon Amplitude

Mahlon (1996)

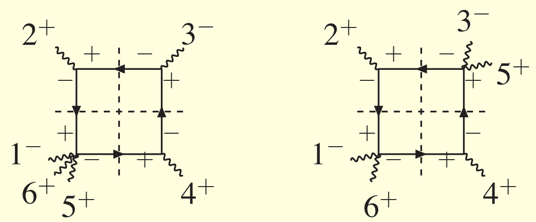
Nagy & Soper (2006)

Binoth, Guillet & Heinrich (2006)

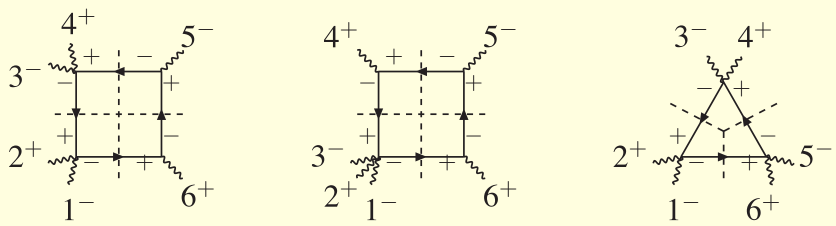
Binoth, Gehrmann, Heinrich & PM [hep-ph/0703311]

Ossola, Papadopoulous & Pittau (2007); Forde (2007)

- $(1^-, 2^+, 3^-, 4^+, 5^+, 6^+)$



- $(1^-, 2^+, 3^-, 4^+, 5^-, 6^+)$



n-gluon⊕Higgs Amplitudes

- Heavy-top limit

- H + 4 partons Ellis, Giele, Zanderighi (2005)
- H + 5 partons Campbell, Ellis, Zanderighi (2006)

- H + n-gluons

$$\phi = \frac{1}{2}(H + iA)$$

$$G^{\mu\nu}_{SD} = \frac{1}{2}(G^{\mu\nu} + \tilde{G}^{\mu\nu}) , \quad G^{\mu\nu}_{ASD} = \frac{1}{2}(G^{\mu\nu} - \tilde{G}^{\mu\nu}) , \quad \tilde{G}^{\mu\nu} = \frac{i}{2}\varepsilon_{\mu\nu\rho\sigma}G^{\rho\sigma}$$

$$L_{\rm int} \propto H \, {\rm tr} G_{\mu\nu}G^{\mu\nu} + iA \, {\rm tr}\tilde{G}_{\mu\nu}\tilde{G}^{\mu\nu} \; = \; \phi \, {\rm tr} G_{SD,\mu\nu}G^{\mu\nu}_{SD} + \; \phi^\dagger \, {\rm tr}\tilde{G}_{ASD,\mu\nu}\tilde{G}^{\mu\nu}_{ASD} \, ,$$

- A(ϕ + n-gluons) $\longrightarrow$ A(n-gluons) w/out momentum conservation Dixon, Glover & Kohze

- ϕ-nite Berger, Del Duca, Dixon (2006)
- ϕ-MHV amplitudes (nearest neighbour) Badger, Glover, Risager (2007)
- ϕ-MHV amplitudes (generic configuration) Glover, Williams, PM (coming soon)

Rational Part of One-Loop Amplitudes

Berger, Bern, Dixon, Forde & Kosower

- Cut Completion:

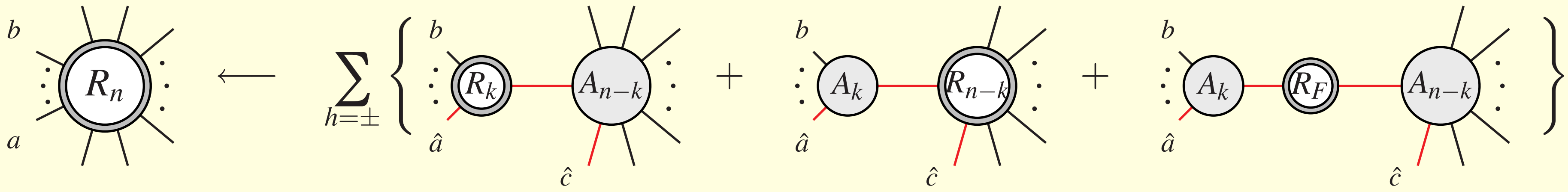
$$A_n = C_n + R_n = \left[C_n + \hat{CR}_n\right] + \left[R_n - \hat{CR}_n\right] = \hat{C}_n + \hat{R}_n$$

- BCFW Analytic Continuation: $A_n \longrightarrow A_n(z) = \hat{C}_n(z) + \hat{R}_n(z)$

- Cauchy-Theorem with branch-cuts $\oplus$ boundary terms:

$$A_n(0) = A_n^\infty + \hat{C}_n(0) - \hat{C}_n^\infty - \sum_{\text{poles } \alpha} \text{Res}_{z=z_\alpha} \frac{R_n(z)}{z} + \sum_{\text{poles } \alpha} \text{Res}_{z=z_\alpha} \frac{\hat{CR}_n(z)}{z}$$

- on-shell RR for Rational Term:



- Additional Shift for boundary terms, to capture the large-z behaviour

D-dimension Unitarity-Cut

Van Neerven; Mahlon;

Bern, Dixon, Kosower, Morgan

Brandhuber, McNamara, Spence, Travaglini

- Decomposition of D-loop-measure:

$$\int d^D L = \Omega(\varepsilon) \int_0^\infty d\mu \, (\mu^2)^{-1-\varepsilon} \int d^4 L \, , \qquad \varepsilon = (4-D)/2 \, .$$

- D-dimensional Cut

$$C_{i\ldots j} = \Delta(A_n^{\text{1-loop}}) = \int d^D\Phi \, A^{\text{tree}}(L_1, i, \ldots, j, L_2) A^{\text{tree}}(-L_2, j+1, \ldots, i-1, -L_1)$$

with,

$$\begin{aligned}
\int d^D\Phi &= \int d^D L_1 \, d^D L_2 \, \delta^{(D)}(L_1 + L_2 - P_{ij}) \, \delta^{(+)}(L_1^2) \, \delta^{(+)}(L_2^2) \\
&= \int d^D L_1 \; \underbrace{\delta^{(+)}(L_1^2) \, \delta^{(+)}((L_1 - P_{ij})^2)}_{\text{massless in D-dim}} \\
&= \Omega(\varepsilon) \int_0^\infty d\mu \, (\mu^2)^{-1-\varepsilon} \int d^4 L_1 \; \underbrace{\delta^{(+)}(L_1^2 - \mu^2) \, \delta^{(+)}((L_1 - P_{ij})^2 - \mu^2)}_{\text{massive in 4-dim}}
\end{aligned}$$

D-Unitarity-Cut & Spinor Integration

Anastasiou, Britto, Feng, Kunst, & PM (2006)

PM (2006); Britto & Feng (2006)

- Massive-cut *vs* massless-cut:

shift: $$L_1 = \ell_1 + z P_{ij} \,, \qquad L_1^2 = \mu^2 \,, \quad \ell_1^2 = 0$$

$$\Rightarrow \quad \int d^4 L_1 \; \underbrace{\delta(L_1^2 - \mu^2)\, \delta((L_1 - P_{ij})^2 - \mu^2)}_{\text{massive in 4-dim}}$$

$$= \int dz\, (1-2z)\, P_{ij}^2\, \delta(z(1-z)P_{ij}^2 - \mu^2) \int d^4\ell_1 \; \underbrace{\delta(\ell_1^2)\, \delta((1-2z)P_{ij}^2 - 2\ell_1 \cdot P_{ij})}_{\text{massless in 4-dim}}$$

- D-dimensional Cut: $\quad u \equiv 4\mu^2 / P_{ij}^2, \qquad |\ell_1\rangle[\ell_1| \equiv t|\lambda\rangle[\lambda|$

$$\int d^D\Phi \quad = \quad \Omega(\epsilon, P_{ij}) \int_0^1 du\, u^{-1-\epsilon} \int dz\, \delta(z - (1 - \sqrt{1-u})/2) \;\times$$

$$\int \frac{\langle\lambda\, d\lambda\rangle[\lambda\, d\lambda]}{\langle\lambda|P_{ij}|\lambda]} \int t\, dt\, \delta\!\left(t - \frac{(1-2z)P_{ij}^2}{\langle\lambda|P_{ij}|\lambda]}\right)$$

Unitarity-motivated Momentum Decomposition

De L'Aguila, Pittau (1996); Ossola, Papadopulos, Pittau (2006)

Forde (2007); Ellis, Giele, Kunszt (2007)

- Loop-Momentum Parametrization:

$$\ell^\mu = \sum_{i=1}^{4} \alpha_i \, v_i^\mu$$

1. v_i are: external momenta, polarization vectors, or van Neerven-Vermaseren vectors;

2. $\alpha_j \; (1 \leq j \leq 4)$ frozen by cut-conditions;

3. unconstrained α_j parametrize the residual integrations for triangle-, bubble- and tadpole-coefficients:

   (a) OPP & EGK: integrand decomposition (partial fractioning) $\oplus$ spurious terms treatment

   (b) Forde: spinor formalism $\oplus$ pinch contribution of unconstrained parameters

Analytic Tools for One-Loop Amplitudes

Unitarity-based methods

Bern, Dixon, Dunbar & Kosower (1993)

Brandhuber, Mc Namara, Spence, & Travaglini (2004/5) Quigley & Roszali (2004)

Britto, Buchbinder, Cachazo, Svrček & Witten (2004/5) Britto, Feng & PM (2006)

Anastasiou, Britto, Feng, Kunszt & PM (2006)

Britto & Feng (2006); Forde (2007)

▷ terms with discontinuities $\Leftarrow$ $\text{input}: \; 4-\text{dim Cuts}$

▷ idem ⊕ rational terms $\Leftarrow$ $\text{input}: D-\text{dim Cuts}$

on-shell Recurrence Relations

Britto, Cachazo, Feng & Witten (2004) Bern, Dixon & Kosower (2005)

Bern, Bjerrum-Bohr, Dunbar & Ita (2005)

Berger, Bern, Dixon, Forde & Kosower (2006)

▷ rational terms $\Leftarrow \begin{cases} \text{input1}: \text{rational term @ less number of legs} \\ \text{input2}: \text{cut term @ same number of legs} \end{cases}$

▷ terms with discontinuities $\Leftarrow$ $\text{input}: \text{cut term @ less number of legs w/in the same class of polylog}$

Improved Tensor Reduction

Xiao, Yang & Zhu (2006)

Ossola, Papadopulos & Pittau (2006)

Binoth, Guillet & Heinrich (2006)

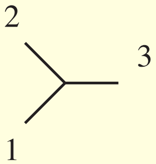
▷ rational terms $\Leftarrow$ input: standard loop-integrals

Outlook & ...

- One-Loop processes [pick-up your favourite one from the Les-Houches whishlist]
- 5-point One-Loop Bhabha
- MHV-rules for One-Loop
- Multi-loop [multiple-cuts, iterative structure, ...]
- Gravity amplitudes [N=8 SuGra UV-behaviour]
- S@M (Spinor @ MATHEMATICA) Maître & PM (to be released)

...Summary

- on-shell 3-point amplitude: $k_i^2 = 0$



$$0 = k_1^2 = (k_2 + k_3)^2 = 2k_2 \cdot k_3 = \langle 2\,3 \rangle [3\,2] \begin{cases} \langle 2\,3 \rangle \neq 0 \\ \\ |3] \,//\, |2] \end{cases} \quad (k_3 \text{ on} - \text{shell \& complex})$$

The imaginary number is a fine and wonderful recourse of the divine spirit, almost an amphibian between being and non-being. [...] there is something fishy about [...] imaginaries, but one can calculate with them because their form is correct. Leibniz