Bhabha Scattering at NNLO

Andrea Ferroglia

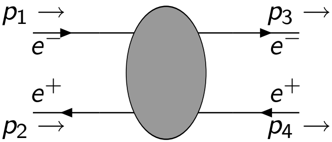
Universität Zürich

GGI - Firenze, 13 September '07

Outline

Bhabha Scattering and Luminosity-I

$$e^+ e^- \longrightarrow e^+ e^-$$



$$s \equiv -P^2 = -(p_1 + p_2)^2 = 4E^2 > 4m^2 \quad t \equiv -Q^2 = -(p_1 - p_3)^2 = -4(E^2 - m^2)\sin^2\frac{\theta}{2} < 0$$

- Effective tool for the Luminosity measurement @ e^+e^- colliders

$$\sigma_{\text{exp}} \equiv \frac{N}{L} \qquad L = \frac{N}{\sigma_{\text{bh-th}}}$$

Bhabha Scattering and Luminosity-II

- In the region employed for L measurements the Bhabha scattering cross section is large and QED dominated
- SABH is employed at LEP and ILC, while LABH is employed at colliders operating at $\sqrt{s} = 1 - 10\text{GeV}$
- Due to beam-beam interactions, at ILC the colliding energy $\sqrt{s}$ shows a continuous spectrum: the LABH can also be used to determine the luminosity spectrum

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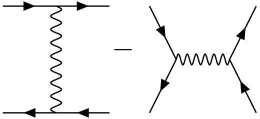
> The accuracy of the theoretical evaluation of the Bhabha scattering cross section directly affects the luminosity determination

$\Longrightarrow$ calculation of radiative corrections

Warning

$$\mathcal{M} = $$ 

- In this talk we consider the QED process only
- We consider differential cross-sections summed over the spins of the final state particles and averaged over the spin of the initial ones

$$\frac{d\sigma_0(s,t)}{d\Omega} = \frac{\alpha^2}{s}\left\{\frac{1}{s^2}\left[st + \frac{s^2}{2} + (t-2m^2)^2\right] + \frac{1}{t^2}\left[st + \frac{t^2}{2} + (s-2m^2)^2\right] + \frac{1}{st}\left[(s+t)^2 - 4m^4\right]\right\}$$

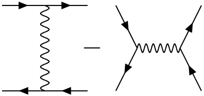
Virtual Corrections to the Cross Section -I

$$\frac{d\sigma(s,t)}{d\Omega} = \frac{d\sigma_0(s,t)}{d\Omega} + \left(\frac{\alpha}{\pi}\right)\frac{d\sigma_1(s,t)}{d\Omega} + \left(\frac{\alpha}{\pi}\right)^2\frac{d\sigma_2(s,t)}{d\Omega} + \mathcal{O}\left((\alpha/\pi)^3\right)$$

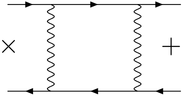
The $\mathcal{O}(\alpha^3)$ virtual corrections (one-loop $\times$ tree-level) are well known (in the full SM), no problem in keeping $m_e \neq 0$

M. Consoli (1979),
M. Böhm, A. Denner, and W. Hollik (1988),
M. Greco (1988),...

$$\left(\frac{\alpha}{\pi}\right)\frac{d\sigma_1^V(s,t)}{d\Omega} = \frac{s}{16}\sum_{\text{spin}}\Bigg\{\Bigg($$



$$\Bigg)^* \times$$



$$+\ \text{c.c.} + \cdots\Bigg\}$$

Order α^4 corrections:

- Contributions from two-loop $\times$ tree-level and one-loop $\times$ one-loop
- Can be divided in three sets,
  i) with a closed electron loop,
  ii) closed heavy(er) flavor loop, and
  iii) photonic (without fermion loops)

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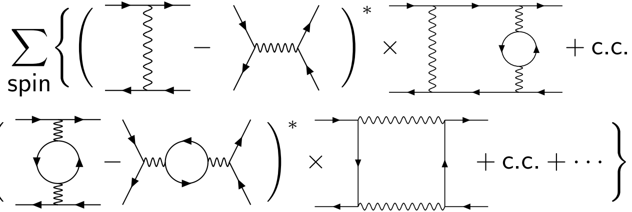
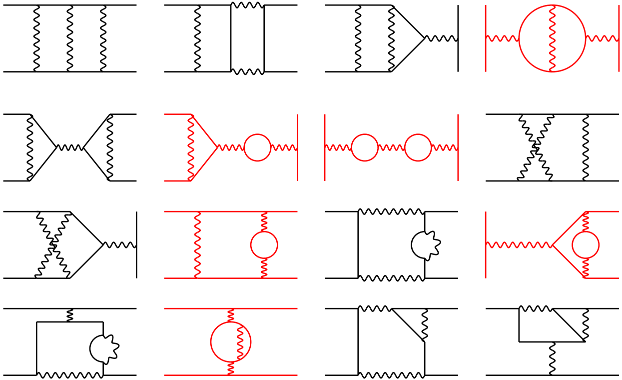


2-Loop QED diagrams

(suppressing fermionic arrows)

- Fermion loop corrections, Photonic corrections
- Boxes: known exactly for $m_e \neq 0$, known only in the for $m_e = 0$



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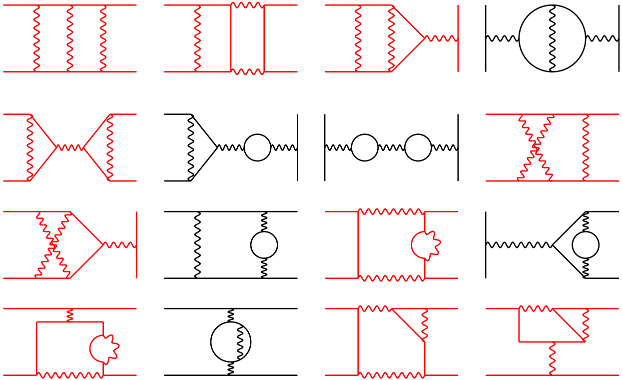

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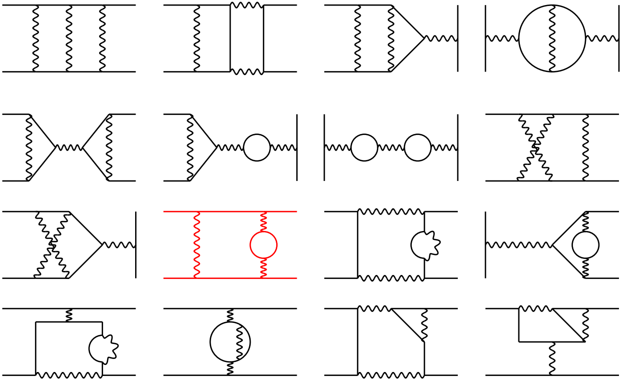

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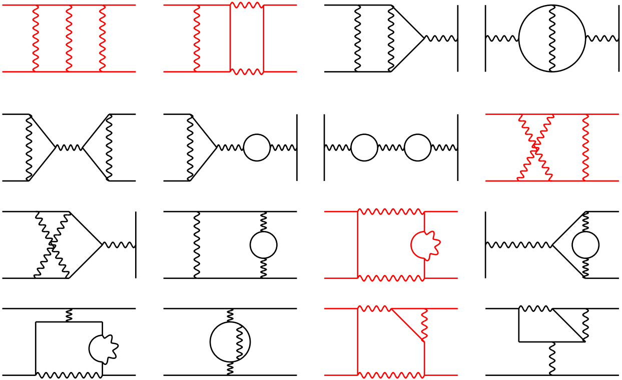

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  Z. Bern *et al.* ('00)

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R. Bonciani *et al.* ('04-'05)

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A. Penin ('05)

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S. Actis *et al.* ('07)

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T. Becher and K. Melnikov ('07)

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- Actually, starting from the massless cross section, it is possible to reconstruct both photonic and fermion loop corrections in the limit $s, t, u \gg m_f^2 \gg m_e^2$.
- By exploiting the collinear structure of the virtual corrections involving an "heavy flavor" fermion, it is possible to calculate the exact dependence of the cross section on m_f.

R. Bonciani, A. F., and A. Penin (hopefully soon)

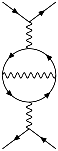
α^4 Corrections With a Closed Electron Loop

All the two-loop graphs including a closed electron loop can be calculated also keeping $m_e \neq 0$ and without relying on any approximation or expansion

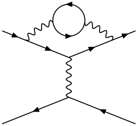
- The relevant integrals can be reduced to combination of a relatively small set of Master Integrals employing the Laporta algorithm
- The MIs (including the ones for the box) can be evaluated employing the differential equation method

R. Bonciani *et al.*('03-'04)

$\mathcal{O}(\alpha^4(N_F = 1))$ Corrections - Virtual Diagrams



$\otimes$ Born Amplitude



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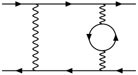

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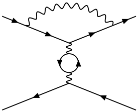

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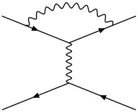
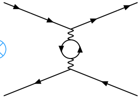
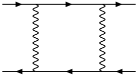
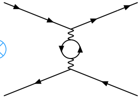
 $\otimes$ 

 $\otimes$ 

$\mathcal{O}(\alpha^4(N_F = 1))$ Virtual Corrections

In the $\mathcal{O}(\alpha^4(N_F = 1))$ corrections to the CS

- both UV and IR divergences are regularized within the DIM REG scheme
- the UV renormalization is carried out in the on-shell scheme
- the graphs are at first calculated in the non physical region $s < 0$ and then analytically continued to the physical region $s > 4m_e^2$
- the cross section can be expressed in terms of HPLs and 2dHPLs with arguments

$$x = \frac{\sqrt{s} - \sqrt{s - 4m_e^2}}{\sqrt{s} + \sqrt{s - 4m_e^2}} \quad y = \frac{\sqrt{4m_e^2 - t} - \sqrt{-t}}{\sqrt{-t} + \sqrt{4m_e^2 - t}} \quad z = \frac{\sqrt{4m_e^2 - u} - \sqrt{-u}}{\sqrt{-u} + \sqrt{4m_e^2 - u}}$$

$$\frac{(1-z)^2}{z} = \frac{1}{x}\,(x - y)\,(x - 1/y)$$

$\mathcal{O}(\alpha^4(N_F = 1))$ Corrections - Soft Photon-I

After UV renormalization, the virtual CS still includes poles in $D - 4$, of IR origin, that can be eliminated by adding the contribution of the soft photon emission diagrams

In order to cancel the IR divergent terms in the virtual cross section at $\mathcal{O}(\alpha^3)$ and $\mathcal{O}(\alpha^4(N_F = 1))$ it is sufficient to consider the contribution of the single photon emission graphs

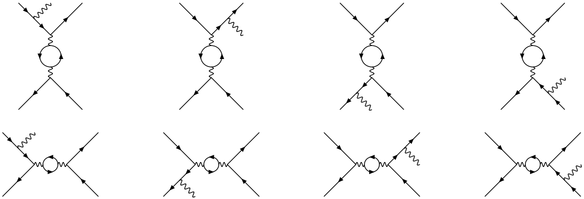
$$e^-(p_1) + e^+(p_2) \longrightarrow e^-(p_3) + e^+(p_4) + \gamma(k) \qquad k_0 < \omega$$

$\mathcal{O}(\alpha^4(N_F = 1))$ Corrections - Soft Photon-II

The soft emission cross section at $\mathcal{O}(\alpha^4(N_F = 1))$ is

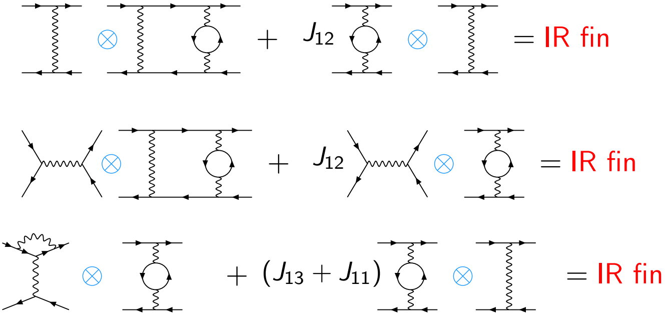
$$\frac{d\sigma_2^S(s,t,m^2)}{d\Omega} = \frac{d\sigma_1^D(s,t,m^2)}{d\Omega} \sum_{i,j=1}^{4} J_{ij}$$

where σ_1^D is the interference of the tree level soft emission with



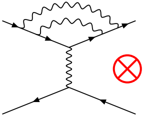
Remember: σ_1^D is finite (after UV renormalization)

It is possible to understand how the cancellation of the IR poles works from a diagrammatic point of view:



With the same techniques employed in obtaining the $\mathcal{O}(\alpha^4(N_F = 1))$ non-approximated differential CS, it is possible to calculate the photonic virtual corrections (and related soft photon emission) to the CS at order $\mathcal{O}(\alpha^4)$, except for the ones arising from the the two loop photonic boxes

R. Bonciani, A. F. ('05)



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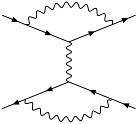

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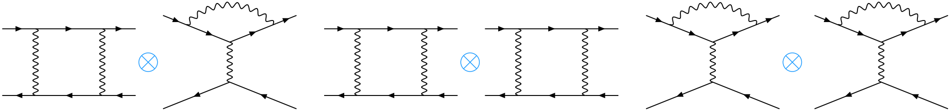
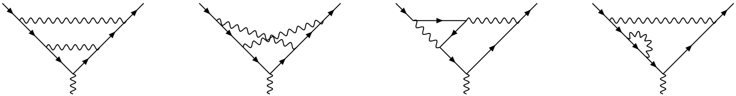


α^4 Photonic Corrections-II

- The two-loop irreducible photonic vertex corrections are gauge independent

R. Bonciani et al. ('03)



- In order to cancel the IR poles it is necessary to add also the contribution of the double photon (soft) emission

$$\frac{d\sigma_2^{(S)}(s,t,m^2)}{d\Omega} = \frac{1}{2}\frac{d\sigma_0^{D}(s,t,m^2)}{d\Omega}\left(\sum_{i,j=1}^{4} J_{ij}\right)^2 + \frac{d\sigma_1^{(V,D)}(s,t,m^2)}{d\Omega}\left(\sum_{i,j=1}^{4} J_{ij}\right)$$

- The one- and two-loop Dirac form factors in the t-channel are sufficient to determine completely the small angle cross section

$$\frac{d\sigma_2}{d\sigma_0} = 6\left({F_1}^{(1l)}(t)\right)^2 + 4{F_1}^{(2l)}(t)$$

α^4 Photonic Corrections-m_e^2/s Expansion

- Building on the BDG result and on works by A. B. Arbuzov *et al.*, B. Tausk, N. Glover, and J. J. van der Bij ('01) obtained the terms proportional to $L = \ln m_e^2/s$ of the full (virtual + soft) photonic CS (i. e. graphs including a closed electron loop have been neglected)
- Recently A. Penin obtained also the constant terms of the photonic CS in the m_e^2/s expansion

Therefore, in the expansion

$$\frac{d\sigma_2}{d\sigma_0} = \delta_2^{(2)} \ln^2\left(\frac{s}{m_e^2}\right) + \delta_2^{(1)} \ln\left(\frac{s}{m_e^2}\right) + \delta_2^{(0)} + \mathcal{O}\left(\frac{m_e^2}{s}\right)$$

$\delta_2^{(2)}$, $\delta_2^{(1)}$, and $\delta_2^{(0)}$ are known

- Several partial cross-checks of this results were possible by comparing it with the $m_e^2/s \to 0$ limit of the exact result for the photonic vertex and one-loop by one-loop corrections

Penin's technique (in a Nutshell)

- Consider the amplitude of the two loop virtual corrections to the cross-section in which collinear and IR divergencies are regularized by m_e and λ: $\mathcal{A}^{(2)}(m_e, \lambda)$
- Build an auxiliary amplitude $\overline{\mathcal{A}}^{(2)}(m_e, \lambda)$ with the same IR singularities of the $\mathcal{A}^{(2)}(m_e, \lambda)$ but sufficiently simple to be evaluated in the small mass expansion
- The quantity $\delta\mathcal{A}^{(2)} = \mathcal{A}^{(2)} - \overline{\mathcal{A}}^{(2)}$ has a finite limit when m_e and λ tend to zero
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$\Longrightarrow$ The method cannot be applied to the $\alpha^4(N_F = 1)$ corrections

Mass from Massless-I

As can be seen from Penin's result, when neglecting positive powers of the electron mass, the problem is to change the regularization scheme for the collinear singularities:

Is it possible to calculate graphs employing DIM REG to regulate both IR and collinear singularities and then translate a posteriori the collinear poles into collinear logs?

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For a generic QED/QCD process, with no closed fermion loops

$$\mathcal{M}^{(m\neq 0)} = \prod_{i\in\{\text{all legs}\}} Z_i^{\frac{1}{2}}(m,\varepsilon)\mathcal{M}^{(m=0)}$$

where Z is defined through the Dirac form factor

$$F^{(m\neq 0)}(Q^2) = Z(m,\varepsilon)\, F^{(m=0)}(Q^2) + \mathcal{O}(m^2/Q^2)$$

A. Mitov and S. Moch ('06)

Mass from Massless-II

in QED, using SCET, it was possible to find a factorization formula that relates massive and massless amplitudes also in presence of fermion loops, as long as $s, |t|, |u| \gg m_f^2 \gg m_e^2$

$$F^{(m \neq 0)}(Q^2) = Z(m, \varepsilon)\, S(Q^2, m, \varepsilon)\, F^{(m=0)}(Q^2) + \mathcal{O}(m^2/Q^2)$$

T. Becher and K. Melnikov ('07)

$$\begin{aligned}
S(Q^2, m, \varepsilon) &= 1 + a_0^2 m^{-4\varepsilon} \ln\left(\frac{Q^2}{m^2}\right)\left(-\frac{1}{12} + \frac{5}{36\varepsilon} - \frac{7}{27} - \frac{\pi^2}{72} + \mathcal{O}(\varepsilon)\right) \\
Z(m, \varepsilon) &= 1 + a_0 m_e^{-2\varepsilon}\left[\frac{1}{2\varepsilon^2} + \frac{1}{4\varepsilon} + \frac{\pi^2}{24} + 1 + \varepsilon\left(2 + \frac{\pi^2}{48} - \frac{\zeta(3)}{6}\right)\right. \\
&\quad \left. + \varepsilon^2\left(4 - \frac{\zeta(3)}{12} + \frac{\pi^4}{320} + \frac{\pi^2}{12}\right) + \mathcal{O}(\varepsilon^3)\right] + \mathcal{O}(a_0^2)
\end{aligned}$$

Mass from Massless-III

with this technique Becher and Melnikov could calculate all the NNLO corrections in the limit $s, |t|, |u| \gg m_f^2 \gg m_e^2$

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{s}\left(\frac{1-r+r^2}{r}\right)\left[1+\frac{\alpha}{\pi}\delta_1+\left(\frac{\alpha}{\pi}\right)^2\delta_2\right]$$

$(r = 1/2(1-\cos\theta))$

$$\delta_2 = \delta_2^{\text{photonic}} + \delta_2^{\text{electron loop}} + \delta_2^{\text{heavy flavor loop}}$$

- photonic corrections in agreement with A. Penin ('05)
- electron loop corrections in agreement with R. Bonciani *et al* ('04)
- "heavy flavor" loop corrections in agreement with S. Actis *et al* ('07)

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- "heavy flavor" loop corrections in agreement with S. Actis *et al* ('07)

Mass from Massless-III

with this technique Becher and Melnikov could calculate all the NNLO corrections in the limit $s, |t|, |u| \gg m_f^2 \gg m_e^2$

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{s}\left(\frac{1-r+r^2}{r}\right)\left[1+\frac{\alpha}{\pi}\delta_1+\left(\frac{\alpha}{\pi}\right)^2\delta_2\right]$$

$(r = 1/2(1-\cos\theta))$

$$\delta_2 = \delta_2^{\text{photonic}} + \delta_2^{\text{electron loop}} + \delta_2^{\text{heavy flavor loop}}$$

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BEYOND $s \gg m_f^2$

In any realistic case the approximation $s, |t|, |u| \gg m_e^2$ is more than enough
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for example

- τ loop at KLOE, where $\sqrt{s} = 1\,\mathrm{GeV} < m_\tau$
- top quark loop at ILC, where $\sqrt{s} \approx 500\,\mathrm{GeV}$ and $m_t^2/t, m_t^2/u < 1$ just in the angular region $40^o < \theta < 140^o$

Beyond $s \gg m_f^2$

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It would be nice to calculate the NNLO corrections including an heavy fermion loop retaining the exact dependence on m_f

$$s, |t|, |u|, m_f^2 \gg m_e^2$$

this is a non trivial problem involving four-scale two-loop boxes . . .

$\Longrightarrow$

Cancellation of the Collinear Poles

What is the collinear structure of these corrections?

$$\delta_2 = \delta_2^C(s, t, m_f^2) \ln\left(\frac{s}{m_e^2}\right) + \delta_2^R(s, t, m_f^2) + \mathcal{O}\left(\frac{m_e^2}{s}\right)$$

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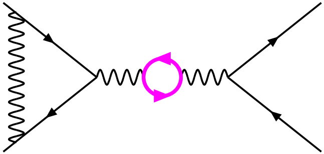
It is possible to show that the collinear logarithm arises from trivial reducible graphs only



The Calculation of the Boxes

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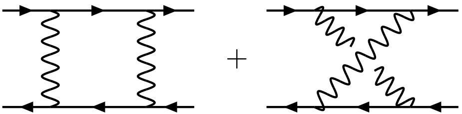 $=$ Free of collinear poles

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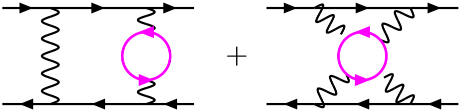


$= $ Free of collinear poles

$$u = -s - t$$

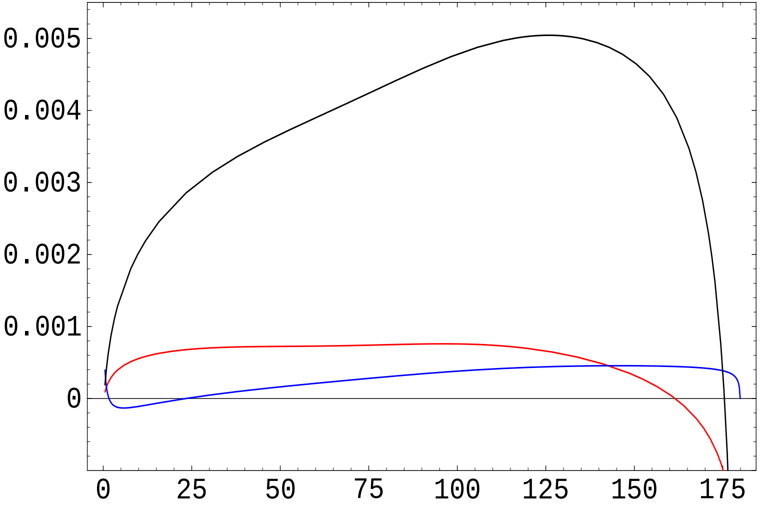
After UV renormalization, the only remaining poles are the IR (soft) ones

Technical Details

R. Bonciani, A. F., and A. Penin (soon)

- It was possible to calculate the boxes for $m_e = 0$ and obtain the cross section for generic $s, |t|, |u|, m_f^2 \gg m_e^2$
- We employed IBPs and Differential Eq. Method
- The result can be expressed in terms of HPL and a few GHPLs of a new class. The latter can be expressed in closed form in terms of polylogs
- by expanding the exact result it was possible to recover the result of Actis *al* and Becher Melnikov
- With the exact dependence of the cross section on m_f we can get numbers for the τ loop at intermediate energies and top loop at ILC energies

Results - Expansion



$\sqrt{s} = 1\,\text{GeV}$, $\omega_{IR} = \sqrt{s}/2$ black → photonic, red → electron, blu → muon

Summary & Conclusions

- A precise knowledge of the Bhabha scattering cross section (both at small and large angle) is crucial in order to determine the luminosity at ILC
- In the last few years the NNLO QED corrections for $m_e \neq 0$ were extensively studied
- The calculation of NNLO QED radiative corrections required the use of a number of powerful tools for the calculation of multi-loop diagrams: IBPs & Laporta-Remiddi Algorithm, Differential Equation Method, Mellin-Barnes techniques, study of the factorization properties, etc.
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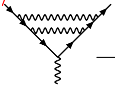
The result obtained need to be critically compared/interfaced with the existing Monte Carlo generators (talk by C. Carloni Calame)

Auxiliary Slides

Consider the virtual corrections to a given physical quantity

- Using projector techniques or summing over spins get rid of the Dirac/Lorentz structure of the Feynman diagrams numerators



$$\rightarrow \overline{v}(p_2)\left[F_1(Q^2)\gamma^\mu + \frac{1}{2m}F_2(Q^2)\sigma^{\mu\nu}Q_\nu\right]u(p_1)$$

- The "form factors" we want to calculate are linear combinations of a (huge) number of scalar integrals
- The scalar integrals are related via Integration By Parts (and other) identities
- Building the IBPs for growing powers of the propagators and scalar products the number of equations grows faster that the number of unknown: one finds a system of equations which is apparently over-constrained
- Solving the system of IBPs (in a problem with a small number of scales) one finds that only a few of the scalar integrals above (if any) are independent: the MIs.

The Laporta-Remiddi Algorithm

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$$\int \mathfrak{D}^D k_1 \mathcal{D}^D k_2 \frac{S_1^{n_1} \cdots S_q^{n_q}}{\mathcal{D}_1^{m_1} \cdots \mathcal{D}_t^{m_t}}$$

$S \rightarrow$ scalar products $k_i \cdot p_j$

$\mathcal{D} \rightarrow$ propagators $(\sum k + \sum p)^2 + M^2$

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$$\int \mathfrak{D}^D k_1 \mathcal{D}^D k_2 \frac{\partial}{\partial k_i^\mu} \left[v^\mu \frac{S_1^{n_1} \cdots S_q^{n_q}}{\mathcal{D}_1^{m_1} \cdots \mathcal{D}_t^{m_t}} \right] = 0 \quad v^\mu = k_1, k_2, p_1, \cdots, p_n$$

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The Differential Equation Method

For each Master Integral belonging to a given topology $F_i^{(t)}$

- One can take the derivative of a given integrals with respect to the external momenta p_i

$$p_j^\mu \frac{\partial}{\partial p_i^\mu} F_i^{(t)} = \int \mathfrak{D}^D k_1 \mathcal{D}^D k_2 \, p_j^\mu \frac{\partial}{\partial p_i^\mu} \frac{S_1^{n_1} \cdots S_q^{n_q}}{\mathcal{D}_1^{m_1} \cdots \mathcal{D}_t^{m_t}}$$

- The integral are regularized, therefore we can apply the derivative to the integrand in the r. h. s. and use the IBPs to rewrite it as a linear combination of the MIs
- On the left hand side one can rewrite the derivatives with respect to the external momenta as functions of the derivatives with respect to the Mandelstam invariants of the problem
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$$p_j^\mu \frac{\partial}{\partial p_k^\mu} F_i^{(t)} = p_j^\mu \sum_\xi \frac{\partial s_\xi}{\partial p_k^\mu} \frac{\partial}{\partial s_\xi} F_i^{(t)}$$

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Harmonic Polylogarithms (HPL)

E. Remiddi, J. Vermaseren (1999); E. Remiddi, T. Gehrmann (2001)

Functions of the variable x and a set of indices $\vec{a}$ with weight w;
each index can assume values $1, 0, -1$

$$H(\mathbf{a}; x)$$

Definitions: $w = 1$

$$\begin{aligned} H(1;x) &= \int_0^x \frac{dt}{1-t} = -\ln(1-x) \\ H(0;x) &= \ln x \\ H(-1;x) &= \int_0^x \frac{dt}{1+t} = \ln(1+x) \end{aligned}$$

$$\frac{d}{dx} H(a;x) = f(a;x) \quad f(1;x) = \frac{1}{1-x} \quad f(0;x) = \frac{1}{x} \quad f(-1;x) = \frac{1}{1+x}$$

HPLs: Definitions

Definitions: $w > 1$

$$\text{if } \vec{a} = 0, 0, \ldots, 0 \ (w \text{ times}) \quad H(\vec{0}_w; x) = \frac{1}{w!} \ln^w x$$

$$\text{else} \quad H(i, \vec{a}; x) = \int_0^x dt f(i; t) H(\vec{a}; t)$$

$$\text{consequences:} \quad \frac{d}{dx} H(i, \vec{a}; x) = f(i; x) H(\vec{a}; t) \quad H(\vec{a} \notin \vec{0}; 0) = 0$$

a few examples @ $w = 2$

$$H(0, 1; x) = \int_0^x dt f(0; t) H(1; t) = -\int_0^x dt \frac{1}{t} \ln(1 - t) = \mathrm{Li}_2(x)$$

$$H(1, 0; x) = \int_0^x dt f(1; t) H(0; t) = \int_0^x dt \frac{1}{1 - t} \ln t$$

$$= -\ln x \ln(1 - x) + \mathrm{Li}_2(x)$$

HPLs as a Generalization of the Nielsen's PolyLogs

The HPLs include the Nielsen's PolyLogs

$$S_{n,p(x)} = \frac{(-1)^{n+p-1}}{(n+p)!p!} \int_0^1 \frac{dt}{t} \ln^{n-1} t \ln^p(1-xt) \quad \mathrm{Li}_n(x) = S_{n-1,1}(x)$$

$$\begin{aligned} \mathrm{Li}_n(x) &= H(\vec{0}_{n-1}, 1; x) \\ S_{n,p}(x) &= H(\vec{0}_n, \vec{1}_p; x) \end{aligned}$$

but the HPLs are a larger set of functions: from $w = 4$ one finds things as

$$H(-1,0,0,1;x) = \int_0^x \frac{dt}{1+t} \mathrm{Li}_3(x) \notin \sum \text{Nielsen's PolyLogs}$$

The HPLs Algebra

- Shuffle Algebra:

$$H(\vec{p};x)H(\vec{q};x) = \sum_{\vec{r}=\vec{p}\uplus\vec{q}} H(\vec{r};x)$$

some examples

$$\begin{aligned} H(a;x)H(b;x) &= H(a,b;x) + H(b,a;x) \\ H(a;x)H(b,c;x) &= H(a,b,c;x) + H(b,a,c;x) + H(b,c,a;x) \end{aligned}$$

- Product Ids:

$$\begin{aligned} H(m_1,\ldots,m_q;x) &= H(m_1;x)H(m_2,\ldots,m_q;x) \\ &- H(m_2,m_1;x)H(m_3,\ldots,m_q;x) \\ &+ \cdots + (-1)^{q+1}H(m_q,\ldots,m_1;x) \end{aligned}$$

2-dimensional Harmonic Polylogarithms (2dHPL)

E. Remiddi, T. Gehrmann (2000)

As for the HPLs, they are obtained by repeated integration over a new set of factors depending on a second variable.

$$f(-y;x) = \frac{1}{x+y} \quad f(-1/y;x) = \frac{1}{x+1/y}$$

$$G(i,\vec{a};x) = \int_0^x dt f(i;t) G(\vec{a};t)$$

a few examples:

$$G(-y;x) = \int_0^x \frac{dz}{z+y} = \ln\left(1+\frac{x}{y}\right) \quad G(-1/y;x) = \int_0^x \frac{dz}{z+1/y} = \ln(1+xy)$$

$$G(-y,0;x) = \ln x \ln\left(1+\frac{x}{y}\right) + \mathrm{Li}_2\left(-\frac{x}{y}\right)$$

2-dimensional Harmonic Polylogarithms (2dHPL)-II

The 2dHPLs share the properties of the HPLs

Up to $w = 3$ (our case) the 2dHPLs can be expressed in terms of $\ln$, Li_2, Li_3, $S_{1,2}$

The analytic properties of both HPLs & 2dHPLs are know

Codes for their numerical evaluation are available

E. Remiddi, T. Gehrmann (2001-2002)