Hot and Dense QCD on the lattice
Frithjof Karsch, BNL

Introduction:

$T$, $gT$, $g^2T$, ...
screening and the running coupling

Bulk thermodynamics

$T_c$ and the equation of state in (2+1)-flavor QCD
with an almost realistic quark mass spectrum

Thermodynamics at non-zero baryon number density

hadronic fluctuations

isentropic equation of state

Conclusions
Critical behavior in hot and dense matter: QCD phase diagram

crossover vs. phase transition

- Quark-gluon plasma (deconfined, $\chi$-symmetric)
- Hadron gas (confined, $\chi$-SB)
- Color superconductor

T ~170 MeV

~ few times nuclear matter density

F. Karsch, apeNEXT, Florence 2007 – p.2/32
Critical behavior in hot and dense matter: QCD phase diagram

- Continuous/rapid (crossover) transition
- Quark-gluon plasma: deconfined, $\chi$-symmetric
- Hadron gas: confined, $\chi$-SB
- Color superconductor

- Continuous transition for small chemical potential and small quark masses

$T \approx 170$ MeV

$\mu_0$ few times nuclear matter density $\mu$
Critical behavior in hot and dense matter: QCD phase diagram

- QCD phase diagram
  - Continuous/rapid (crossover) transition
  - Quark-gluon plasma: deconfined, $\chi$-symmetric
  - Hadron gas: confined, $\chi$-SB
  - Color superconductor
  - Critical point

- $T_c(\mu)$ under investigation
  - 2nd order phase transition; Ising universality class
  - Location of CCP uncertain: volume and quark mass dependence

- Continuous transition for small chemical potential and small quark masses

F. Karsch, apeNEXT, Florence 2007 – p.2/32
Critical behavior in hot and dense matter: QCD phase diagram

- Continuous/rapid (crossover) transition
- Quark-gluon plasma
  - Deconfined, $\chi$-symmetric
- Hadron gas
  - Confined, $\chi$-SB
- Color superconductor
  - Chiral critical point
    - Second order phase transition; Ising universality class $T_c(\mu)$ under investigation
  - Location of CCP uncertain: volume and quark mass dependence
    - Improving accuracy on $T_c$, $\epsilon_c$, $\epsilon(p)$ and the phase boundary is mandatory to make contact to HIC phenomenology
Non-perturbative QGP

- Perturbation theory provides a hierarchy of length scales:
  \[ T \gg gT \gg g^2T \implies \text{guiding principle for effective theories, resummation, dimensional reduction...} \]

- Early lattice results show that \( g^2(T) > 1 \) even at \( T \sim 5T_c \)
  
  G. Boyd et al, NP B469 (1996) 419: SU(3) thermodynamics..

  ...one has to conclude that the temperature dependent running coupling has to be large, \( g^2(T) \sim 2 \) even at \( T \sim 5T_c \)

- The Debye screening mass is large close to \( T_c \)

- The spatial string tension does not vanish above \( T_c \)

  \[ \sqrt{\sigma_s} \neq 0 \implies \text{the QGP is ”non-perturbative” up to very high } T \]
Screening of heavy quark free energies
– remnant of confinement above $T_c$ –


- singlet free energy

$T \simeq T_c$ : screening for $r \gtrsim 0.5\text{fm}$

$F_1(r, T) \sim \frac{\alpha(T)}{r} e^{-\mu(T)r} + \text{const.}$

$F_1(r, T)$ follows linear rise of $V_{\bar{q}q}(r, T = 0) = -\frac{4\alpha(r, T = 0)}{3r} + \sigma r$

for $T \lesssim 1.5T_c$, $r \lesssim 0.3 \text{ fm}$
Singlet free energy and asymptotic freedom


Singlet free energy defines a running coupling:

$$\alpha_{\text{eff}} = \frac{3r^2}{4} \frac{dF_1(r, T)}{dr}$$

(in Coulomb gauge)
Singlet free energy and asymptotic freedom


Singlet free energy defines a running coupling:

$$\alpha_{\text{eff}} = \frac{3r^2}{4} \frac{dF_1(r, T)}{dr}$$

(in Coulomb gauge)

large distance: constant Coulomb term (string model)

short distance: running coupling $\alpha(r)$ from ($T = 0$), 3-loop

short distance physics $\Leftrightarrow$ vacuum physics

T-dependence starts in non-perturbative regime for $T < 3T_c$
Singlet free energy and asymptotic freedom


- Singlet free energy defines a running coupling:

\[ \alpha_{\text{eff}} = \frac{3r^2}{4} \frac{dF_1(r, T)}{dr} \]

(in Coulomb gauge)

- Large distance: constant Coulomb term (string model)

- Short distance: running coupling \( \alpha(r) \) from \( T = 0 \), 3-loop


- Short distance physics \( \Leftrightarrow \) vacuum physics

\[ \alpha \equiv \pi/16 \]

T-dependence starts in non-perturbative regime for \( T \lesssim 3T_c \)

\[ \alpha_{qq}(r, T) \]

rise due to confinement \( \alpha_{\text{eff}} \sim \sigma r^2 \)

F. Karsch, apeNEXT, Florence 2007 – p.5/32
Non-perturbative Debye screening

- leading order perturbation theory: \( m_D = g(T)T \sqrt{1 + \frac{n_f}{6}} \)

- \( T_c < T \lesssim 10T_c \): non-perturbative effects are well represented by an "A-factor": \( m_D \equiv Ag(T)T, A \approx 1.5 \)

- perturbative limit is reached very slowly (logarithms at work!!)

\[
\begin{align*}
\text{SU(3)} & \\
m/gT & \\
T/\Lambda_{\text{MS}} & \\
\end{align*}
\]

\( g(T) \approx 1.5 \Leftrightarrow \alpha(T) \approx 0.18 \)

O.Kaczmarek, F.Zantow, PRD 71 (2005) 114510

K.Kajantie et al, PRL 79 (1997) 3130
The spatial string tension

Non-perturbative, vanishes in high-T perturbation theory:

$$\sqrt{\sigma_s} = -\lim_{R_x, R_y \to \infty} \ln \frac{W(R_x, R_y)}{R_x R_y}$$

$$\frac{\sqrt{\sigma_s}}{g^2(T)T} = c_M f_M(g(T)), \quad c_M = 0.553(1)$$

$c_M$: 3-d SU(3), LGT
$g_M \equiv g^2 f_M$: dim. red. pert. th.

$g^2(T) \simeq 2 \iff \alpha(T) \simeq 0.16$

dimensional reduction works for $T \gtrsim 2T_c$

- $c_M$ (almost) flavor independent
- $g^2(T)$ shows 2-loop running

$c = 0.566(13) \ [SU(3)]$
$c = 0.594(39) \ [QCD]$
**μ = 0: Equation of State and \( T_c \)**

- **QCD EoS**
  - \( \varepsilon = 3p \) deviations from ideal gas behavior
  - \( T_c \leq T \sim 3T_c \) and even at high \( T \)
  - Improved staggered fermions but still on rather coarse lattices: \( N_\tau = 4 \), i.e. \( a^{-1} \simeq 0.8 \text{ GeV} \) with moderately light quarks

- **Transition temperature**
  - \( T_c = (173 \pm 8 \pm \text{sys}) \text{ MeV} \)
  - Weak quark mass and flavor dependence

EoS and $T_c$

**Goal:** QCD thermodynamics with realistic quark masses and controlled extrapolation to the continuum limit $T_c$, EoS, $\mu_q > 0$, ...

- use an **improved staggered fermion action** that removes $\mathcal{O}(a^2)$ errors in bulk thermodynamic quantities and reduces flavor symmetry breaking inherent to the staggered formulation

**RBC-Bielefeld choice:** p4-action + 3-link smearing (p4fat3)

**MILC:** Naik-action + (3,5,7)-link smearing (asqtad);

**Wuppertal:** standard staggered + exponentiated 3-link smearing (stout)
EoS and $T_c$

Goal: QCD thermodynamics with realistic quark masses and controlled extrapolation to the continuum limit $T_c$, EoS, $\mu_q > 0$, ...

- use an improved staggered fermion action that removes $O(\alpha^2)$ errors in bulk thermodynamic quantities and reduces flavor symmetry breaking inherent to the staggered formulation

RBC-Bielefeld choice: p4-action + 3-link smearing (p4fat3)

- p4-action: smooth high-T behavior for bulk thermodynamics on lattice with temporal extent $N_\tau$

\[ p(N_\tau)/T^4 = p_{SB}^{cont}/T^4 + O(N_\tau^{-4}) \]

- p4&Naik: similarly small cut-off dependence of renormalized Polyakov loops and quark number susceptibilities
Thermodynamics on QCDOC and apeNEXT

**US/RBRC QCDOC**
20,000,000,000,000 ops/sec

**BI – apeNEXT**
5,000,000,000,000 ops/sec

- critical temperature
- equation of state
- finite density QCD
EoS and $T_c$

**Goal:** QCD thermodynamics with realistic quark masses and controlled extrapolation to the continuum limit $T_c$, EoS, $\mu_q > 0$, ...

- use an improved staggered fermion action that removes $O(a^2)$ errors in bulk thermodynamic quantities and reduces flavor symmetry breaking inherent to the staggered formulation
- RBC-Bielefeld choice: p4-action + 3-link smearing (p4fat3)
- use the newly developed RHMC algorithm to remove 'step-size errors' in the numerical simulation
- perform simulations with (3-4) different light quark masses corresponding to $150 \text{ MeV} \lesssim m_\pi \lesssim 500 \text{ MeV}$ at 2 different values of the lattice cut-off controlled by the spatial lattice size $N_\tau = 4, 6$ to perform the chiral and continuum extrapolation

previous results with p4-action:
2-flavor QCD: $N_\tau = 4$, $m_\pi \simeq 770 \text{ MeV}$
Transition temperature

crossover rather than phase transition:
need to determine location of the transition from various susceptibilities:
(disconnected part of the) light and strange quark chiral susceptibility; Polyakov loop and quark number susceptibility,...

thermodynamic limit:
need to control finite volume effects;

continuum limit:
need to analyze cut-off dependence in $T > 0$ and $T = 0$ calculations;

- large statistics; several ten thousand trajectories
- find little volume dependence of location of transition point
- overall scale setting using $T = 0$ potential parameter;
  find weak cut-off dependence
Chiral susceptibility, $N_f = 4, 6$

- Weak volume dependence
- Peak location consistent with that of Polyakov loop susceptibility and maximum of quartic fluctuation of quark number density
Chiral and L susceptibility, $\mathcal{N}_\tau = 4$
Chiral and L susceptibility, $N_T = 4$

![Graph showing chiral susceptibility with error band and data points for different $\beta$ values.]

Data sample for smallest quark mass on $16^3 \times 4$ lattice

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>no. of conf.</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.2900</td>
<td>38960</td>
</tr>
<tr>
<td>3.3000</td>
<td>40570</td>
</tr>
<tr>
<td>3.3050</td>
<td>32950</td>
</tr>
<tr>
<td>3.3100</td>
<td>42300</td>
</tr>
<tr>
<td>3.3200</td>
<td>39050</td>
</tr>
</tbody>
</table>

2.5% error band $\Leftrightarrow$ 5 MeV

200,000/10 trajectories enter Ferrenberg-Swendsen sample
Chiral and L susceptibility, $\mathcal{N}_\tau = 4$

2.5% error band $\Leftrightarrow$ 5 MeV

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>no. of conf.</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.2900</td>
<td>38960</td>
</tr>
<tr>
<td>3.3000</td>
<td>40570</td>
</tr>
<tr>
<td>3.3050</td>
<td>32950</td>
</tr>
<tr>
<td>3.3100</td>
<td>42300</td>
</tr>
<tr>
<td>3.3200</td>
<td>39050</td>
</tr>
</tbody>
</table>
Ambiguities in locating the crossover point

differences of pseudo-critical couplings locating peaks in light ($\beta_l$), strange ($\beta_s$) and Polyakov loop ($\beta_L$) susceptibilities

2.5% ($N_\tau = 4$) or 4% ($N_\tau = 6$) error band $\Leftrightarrow$ 5 or 8 MeV

differences in the location of pseudo-critical couplings are taken into account as systematic error
$T = 0$ scale setting using the heavy quark potential

use $r_0$ or string tension to set the scale for $T_c = 1/N_\tau a(\beta_c)$

$$V(r) = -\frac{\alpha}{r} + \sigma r, \quad r^2 \frac{dV(r)}{dr}\bigg|_{r=r_0} = 1.65$$

no significant cut-off dependence when cut-off varies by a factor 4

i.e. from the transition region on $N_\tau = 4$ lattices to that on $N_\tau = 16$ lattices !!

we use $r_0 = 0.469(7)$ fm determined from quarkonium spectroscopy

⇒ $T_c r_0$, $T_c / \sqrt{\sigma}$

extrapolation to chiral and continuum limit

$$(r_0 T_c)_N = (r_0 T_c)_{\text{cont.}} + b (m_{PS} r_0)^d + c / N_{\tau}^2$$

(d=1.08 (O(4), 2nd ord.), d=2 (1st ord.))

⇒ $r_0 T_c = 0.456(7)^{+3}_{-1}$, $T_c / \sqrt{\sigma} = 0.408(7)^{+3}_{-1}$ at phys. point

⇒ $T_c = 192(7)(4)$ MeV

(1st error: stat. error on $\beta_c$ and $r_0$; 2nd error: $N_{\tau}^{-2}$ extrapolation)
Transition temperature

staggered fermions $N_\tau = 4, 6$

RBC-Bielefeld (p4fat3 (p4))
Transition temperature

staggered fermions $N_\tau = 4, 6$

- RBC-Bielefeld (p4fat3 (p4)) vs. MILC (asqtad (Naik)) and Wuppertal (stout (stand. staggered))

- asqtad results for $N_\tau = 4$ and 6 agree with p4 results within statistical errors; (C.Bernard et al., PR D71, 034504 (2005))

- results obtained with stout action for $N_\tau = 4$ and 6 are about 15% lower; $eta_c$ from $N_\tau = 8, 10$ covers $(151 - 176)$ MeV; (Y. Aoki et al., hep-lat/0609068)

asqtad data for $T_c r_1$ rescaled with $r_0/r_1 = 1.4795$

asqtad: continuum extrapolation:

quoted $T_c$ from $m_q/m_s \leq 1$

and fit in $m_\pi/m_\rho$ yields $T_c = 169(12)(4)$ MeV

using $m_q/m_s \leq 0.4$

and fit in $m_\pi r_0$ yields $T_c = 173(13)(4)$ MeV
Transition temperature

staggered fermions $N_\tau = 4, 6$ and Wilson fermions $N_\tau = 6 - 10$

- RBC-Bielefeld (p4fat3 (p4)) vs. MILC (asqtad (Naik)) and Wuppertal (stout (stand. staggered))

- $T_c$ from Wilson/Clover fermions so far only for $m_{ps}r_0 > 1.5$; consistent with staggered results

- Wilson for $N_\tau \geq 6$ show no significant cutoff effects
  (V.G. Bornyakov et al., hep-lat/0509122)

scale setting uncertainties:

staggered: $r_0 = 0.469(7)$ fm
(MILC + heavy quark spec. )

Clover: $r_0 = 0.516(21)$ fm
(CP-PACS+JLQCD, light quark spec.)
extrapolations to phys. point

- RBC-Bielefeld (p4fat3 (p4)) vs. Wuppertal (stout (stand. staggered))
- results for $N_\tau = 4, 6$ differ by 15% but show similar cut-off dependence
- stout results for different observables no longer consistent with each other for $N_\tau = 8, 10$

overall scale set with $r_0 = 0.469$ fm
Calculating the EoS on lines of constant physics (LCP)

The pressure

\[
\left. \frac{p}{T^4} \right|_{\beta_0} = N_f^4 \int_{\beta_0}^{\beta} d\beta' \left[ \frac{1}{N_\sigma^3 N_t} (\langle S_g \rangle_0 - \langle S_g \rangle_T) - \left( 2 \langle \bar{\psi} \psi \rangle_{l0} - \langle \bar{\psi} \psi \rangle_{lT} \right) + \frac{\hat{m}_s}{\hat{m}_l} \left( \langle \bar{\psi} \psi \rangle_{s0} - \langle \bar{\psi} \psi \rangle_{sT} \right) \left( \frac{\partial \hat{m}_l}{\partial \beta'} \right) \hat{m}_s/\hat{m}_l \right]
\]

The interaction measure for \( N_f = 2 + 1 \)

\[
\frac{\epsilon - 3p}{T^4} = T \frac{d}{dT} \left( \frac{p}{T^4} \right) = \left( a \frac{d\beta}{da} \right)_{\text{LCP}} \frac{\partial p/T^4}{\partial \beta} = \left( \frac{\epsilon - 3p}{T^4} \right)_{\text{gluon}} + \left( \frac{\epsilon - 3p}{T^4} \right)_{\text{fermion}} + \left( \frac{\epsilon - 3p}{T^4} \right) \hat{m}_s/\hat{m}_l
\]
Using an RG-inspired 2-loop $\beta$-function underestimates $(\epsilon - 3p)/T^4$ in the transition region and stretches the temperature interval in the low temperature regime artificially, i.e. makes the transition region look broader than it is.

\begin{itemize}
  \item Using an RG-inspired 2-loop $\beta$-function underestimates $(\epsilon - 3p)/T^4$ in the transition region and stretches the temperature interval in the low temperature regime artificially, i.e. makes the transition region look broader than it is.
  \item Overall good agreement
\end{itemize}

Note:
$T$-scale is not dependent on $T_c$ determination

RBC-Bielefeld, preliminary

asqtad data:
C. Bernard et al., hep-lat/0611031
Energy density and pressure

\[ N_\tau = 4, 6 \]

- **RBC-Bielefeld vs. MILC:** the RBC-Bi energy/entropy density on \( N_\tau = 4 \) lattices rises more steeply;
direct consequence of the use of a non-perturbative \( \beta \)-function directly deduced from calculated \( r_0/a \) values

- overall good agreement for \( N_\tau = 4, 6 \),
  **Note:** \( T \)-scale does not depend on \( T_c \) determination!!

\[ T = (192 \pm 11) \text{ MeV} \]

RBC-Bielefeld, preliminary

Pressure increased slightly with smaller quark mass
Lattice EoS: energy density $\Leftrightarrow$ temperature
$\Rightarrow$ conditions for heavy $q\bar{q}$ bound states

LGT: $T_c \simeq 190\,\text{MeV}$

$T = T_c$: $\epsilon_c/T_c^4 \simeq 6 \Rightarrow \epsilon_c \simeq 1\,\text{GeV}/\text{fm}^3$

$T \geq 1.5T_c$: $\epsilon/T^4 \simeq (13 - 14)$

$T = 1.5T_c$: $\epsilon \simeq 11\,\text{GeV}/\text{fm}^3$

$T = 2.0T_c$: $\epsilon \simeq 35\,\text{GeV}/\text{fm}^3$

$\downarrow$

observable consequences:

$J/\psi$ suppression

RHIC

$R_{Au} \simeq 7\,\text{fm}$;

$\tau_0 \simeq 1\,\text{fm}$

$\langle E_T \rangle \simeq 1\,\text{GeV}$

$dN/dy \simeq 1000$

$\downarrow$

$\epsilon_{Bj} \simeq 7\,\text{GeV}/\text{fm}^3$

maybe: $\tau_0 \simeq 0.5\,\text{fm}$

$\downarrow$

$\epsilon_{Bj} \simeq 14\,\text{GeV}/\text{fm}^3$
Lattice EoS: energy density $\Leftrightarrow$ temperature
$\Rightarrow$ conditions for heavy $q\bar{q}$ bound states

LGT: $T_c \simeq 190$ MeV

$T = T_c$: $\epsilon_c / T_c^4 \simeq 6 \Rightarrow \epsilon_c \simeq 1$ GeV/fm$^3$

$T \geq 1.5T_c$: $\epsilon / T^4 \simeq (13 - 14)$

$T = 1.5T_c$: $\epsilon \simeq 11$ GeV/fm$^3$

$T = 2.0T_c$: $\epsilon \simeq 35$ GeV/fm$^3$

$\downarrow$

$\chi_c, \psi'$ suppression at RHIC

direct $J/\psi$ suppression unlikely

$\downarrow$

$S(J/\psi) \simeq 0.6 + 0.4S(\chi_c)$

(assume $S(\chi_c) \simeq S(\psi')$)

RHIC

$R_{Au} \simeq 7$ fm;

$\tau_0 \simeq 1$ fm

$\langle E_T \rangle \simeq 1$ GeV

$dN/dy \simeq 1000$

$\downarrow$

$\epsilon_{Bj} \simeq 7$ GeV/fm$^3$

maybe: $\tau_0 \simeq 0.5$ fm

$\downarrow$

$\epsilon_{Bj} \simeq 14$ GeV/fm$^3$
Bulk thermodynamics with non-vanishing chemical potential

\[ Z(V, T, \mu) = \int \mathcal{D}A \mathcal{D}\psi \mathcal{D}\bar{\psi} \ e^{-S_E(V, T, \mu)} \]
\[ = \int \mathcal{D}A \ [\text{det} \ M(\mu)]^f \ e^{-S_G(V, T)} \]

↑complex fermion determinant;
Bulk thermodynamics with non-vanishing chemical potential

\[ Z(V, T, \mu) = \int \mathcal{D}A \mathcal{D}\psi \mathcal{D}\bar{\psi} \, e^{-S_E(V, T, \mu)} \]

\[ = \int \mathcal{D}A \, [\text{det} M(\mu)]^f \, e^{-S_G(V, T)} \]

↑ complex fermion determinant;

ways to circumvent this problem:

- **Reweighting**: works well on small lattices; requires exact evaluation of \( \text{det} M \)
  

- **Taylor expansion** around \( \mu = 0 \): works well for small \( \mu \);
  
  

- **Imaginary chemical potential**: works well for small \( \mu \); requires analytic continuation
  
  
Energy and Entropy density for $\mu_q > 0$

S. Ejiri, F. Karsch, E. Laermann and C. Schmidt, hep-lat/0512040

Thermodynamics: (NB: continuum $\hat{m} \equiv m_q$
lattice $\hat{m} \equiv m_q a$, implicit $T$-dependence)

- pressure
  \[ \frac{p}{T^4} \equiv \frac{1}{V T^3} \ln Z(T, \mu_q) = \sum_{n=0}^{\infty} c_n(T, \hat{m}) \left( \frac{\mu_q}{T} \right)^n \]

- energy density from "interaction measure"
  \[ \frac{\epsilon - 3p}{T^4} = \sum_{n=0}^{\infty} c'_n(T, \hat{m}) \left( \frac{\mu_q}{T} \right)^n, \quad c'_n(T, \hat{m}) \equiv T \frac{dc_n(T, \hat{m})}{dT} \]

- entropy density
  \[ \frac{s}{T^3} \equiv \frac{\epsilon + p - \mu_q n_q}{T^4} = \sum_{n=0}^{\infty} ((4 - n)c_n(T, \hat{m}) + c'_n(T, \hat{m})) \left( \frac{\mu_q}{T} \right)^n \]
Bulk thermodynamics for small \( \mu_q/T \) on \( 16^3 \times 4 \) lattices

Taylor expansion of pressure up to \( \mathcal{O}((\mu_q/T)^6) \)

\[
\frac{p}{T^4} = \frac{1}{VT^3} \ln Z = \sum_{n=0}^{\infty} c_n(T) \left( \frac{\mu_q}{T} \right)^n \simeq c_0 + c_2 \left( \frac{\mu_q}{T} \right)^2 + c_4 \left( \frac{\mu_q}{T} \right)^4 + c_6 \left( \frac{\mu_q}{T} \right)^6
\]

quark number density

\[
\frac{n_q}{T^3} = 2c_2 \frac{\mu_q}{T} + 4c_4 \left( \frac{\mu_q}{T} \right)^3 + 6c_6 \left( \frac{\mu_q}{T} \right)^5
\]

quark number susceptibility

\[
\frac{\chi_q}{T^2} = 2c_2 + 12c_4 \left( \frac{\mu_q}{T} \right)^2 + 30c_6 \left( \frac{\mu_q}{T} \right)^4
\]

an estimator for the radius of convergence

\[
\left( \frac{\mu_q}{T} \right)_{crit} = \lim_{n \to \infty} \left| \frac{c_{2n}}{c_{2n+2}} \right|^{1/2}
\]

\( c_n > 0 \) for all \( n \);
singularity for real \( \mu \)
Bulk thermodynamics for small $\mu_q/T$
on $16^3 \times 4$ lattices

- Taylor expansion of pressure up to $O((\mu_q/T)^6)$

$$\frac{p}{T^4} = \frac{1}{V T^3} \ln Z = \sum_{n=0}^{\infty} c_n(T) \left(\frac{\mu_q}{T}\right)^n \simeq c_0 + c_2 \left(\frac{\mu_q}{T}\right)^2 + c_4 \left(\frac{\mu_q}{T}\right)^4 + c_6 \left(\frac{\mu_q}{T}\right)^6$$

---

F. Karsch, apeNEXT, Florence 2007 – p.27/32
**Bulk thermodynamics for small $\mu_q/T$ on $16^3 \times 4$ lattices**

Taylor expansion of **pressure** up to $\mathcal{O}\left((\mu_q/T)^6\right)$

$$\frac{p}{T^4} = \frac{1}{VT^3} \ln \mathcal{Z} = \sum_{n=0}^{\infty} c_n(T) \left(\frac{\mu_q}{T}\right)^n \simeq c_0 + c_2 \left(\frac{\mu_q}{T}\right)^2 + c_4 \left(\frac{\mu_q}{T}\right)^4 + c_6 \left(\frac{\mu_q}{T}\right)^6$$

$c_n > 0$ for all $n$ and $T \lesssim 0.95 \ T_c \iff$ singularity for real $\mu$ (positive $\mu^2$)
Bulk thermodynamics for small $\mu_q/T$
on 16$^3 \times 4$ lattices

Taylor expansion of pressure up to $\mathcal{O}\left((\mu_q/T)^6\right)$

$$
\frac{p}{T^4} = \frac{1}{VT^3} \ln \mathcal{Z} = \sum_{n=0}^{\infty} c_n(T) \left(\frac{\mu_q}{T}\right)^n \simeq c_0 + c_2 \left(\frac{\mu_q}{T}\right)^2 + c_4 \left(\frac{\mu_q}{T}\right)^4 + c_6 \left(\frac{\mu_q}{T}\right)^6
$$

irregular sign of $c_n$ for $T \gtrsim T_c$ $\iff$ singularity in complex plane

F. Karsch, apeNEXT, Florence 2007 – p.27/32
The pressure for $\mu_q/T > 0$

C.R. Allton et al. (Bielefeld-Swansea), PRD68 (2003) 014507

$\mu_q = 0$, $16^3 \times 4$ lattice improved staggered fermions;
$n_f = 2$, $m_\pi \simeq 770$ MeV

corresponding from $\mu_q/T > 0$

Taylor expansion, $O(\mu/T)^4$

High-T, ideal gas limit

$$\left. \frac{p}{T^4} \right|_{\infty} = n_f \left( \frac{7\pi^2}{60} + \frac{1}{2} \left( \frac{\mu_q}{T} \right)^2 + \frac{1}{4\pi^2} \left( \frac{\mu_q}{T} \right)^4 \right)$$

RHIC: $\mu_q/T \lesssim 0.1$
The pressure for $\mu_q/T > 0$

C.R. Allton et al. (Bielefeld-Swansea), PRD68 (2003) 014507

$\mu_q = 0, \ 16^3 \times 4$ lattice
improved staggered fermions;

$n_f = 2, m_\pi \simeq 770$ MeV

NEW: Taylor expansion, $O((\mu/T)^6)$

pattern for $\mu_q = 0$ and $\mu_q > 0$ similar;
quite large contribution in hadronic phase;
$O((\mu/T)^6)$ correction small for $\mu_q/T \lesssim 1$

RHIC: $\mu_q/T \lesssim 0.1$
dense matter created in a HI-collision expands and cools at fixed entropy and baryon number
⇒ lines of constant $S/N_B$ in the QCD phase diagram

for example:
isentropic expansion, "mixed phase model":
V.D. Toneev, J. Cleymans, E.G. Nikonov, K. Redlich, A.A. Shanenko,
EoS on HIC trajectories

dense matter created in a HI-collision expands and cools at fixed entropy and baryon number
⇒ lines of constant $S/N_B$ in the QCD phase diagram

- high T: ideal gas

$$S \frac{N_B}{N_B} = 3 \left( \frac{32\pi^2}{45n_f} + \frac{7\pi^2}{15} + \left( \frac{\mu_q}{T} \right)^2 \right) \frac{\mu_q}{T} + \frac{1}{\pi^2} \left( \frac{\mu_q}{T} \right)^3$$

$S/N_B = \text{constant} \Leftrightarrow \mu_q/T \text{ constant}$

- low T: nucleon + pion gas

$T \rightarrow 0$: $\mu_q/T \sim c/T$
Isentropic Equation of State: $p/\epsilon$


- $p/\epsilon$ vs. $\epsilon$ shows almost no dependence on $S/N_B$
- softest point: $p/\epsilon \simeq 0.075$
- phenomenological EoS for $T_0 \lesssim T \lesssim 2T_0$

$$\frac{p}{\epsilon} = \frac{1}{3} \left( 1 - \frac{1.2}{1 + 0.5 \epsilon \text{ fm}^3/\text{GeV}} \right)$$
Isentropic Equation of State: $p/\epsilon$


- $p/\epsilon$ vs. $\epsilon$ shows almost no dependence on $S/N_B$
- Softest point: $p/\epsilon \simeq 0.075$
- Phenomenological EoS for $T_0 \lesssim T \lesssim 2T_0$
- For $m_\pi \simeq 770$ MeV
- $\mu > 0$:
  - So far analyzed only

\[
\frac{p}{\epsilon} = \frac{1}{3} \left( 1 - \frac{1.2}{1 + 0.5 \epsilon \text{ fm}^3/\text{GeV}} \right)
\]

awaits confirmation in (2+1)-flavor QCD with light quarks

F. Karsch, apeNEXT, Florence 2007 – p.30/32
Conclusions

- non-perturbative QGP
  the QGP is non-perturbative up to high temperatures;
  the running of $\alpha_s$ reflects "remnants of confinement"

- bulk thermodynamics
  the transition between a HG and the QGP is signaled by a rapid change in the energy density;
  calculations with different $O(a^2)$ improved staggered fermions yield a consistent description of the high temperature phase;

- the transition temperature
  at the physical point of (2+1)-flavor QCD our calculation of $T_c$ yields
  
  $$T_c = 192(7)(4)\text{MeV}$$
Taylor expansion \( \Rightarrow \) estimates for radius of convergence

\[
\rho_{2n} = \sqrt{\frac{c_{2n}}{c_{2n+2}}}
\]

\( T < T_0: \rho_n \simeq 1.0 \) for all \( n \) \( \Rightarrow \mu_c^{\text{crit}} \simeq 500 \) MeV
Radius of convergence:
lattice estimates vs. resonance gas

\[ T < T_0: \rho_n \simeq 1.0 \text{ for all } n \Rightarrow \mu_B^{crit} \simeq 500 \text{ MeV} \]

HOWEVER still consistent with resonance gas!!!
HRG analytic, LGT consistent with HRG \( \Rightarrow \) infinite radius of convergence not yet ruled out
Radius of convergence: lattice estimates vs. resonance gas

Taylor expansion $\Rightarrow$ estimates for radius of convergence

$$\rho_{2n} = \sqrt{\frac{c_{2n}}{c_{2n+2}}}$$


$T < T_0$: $\rho_n \simeq 1.0$ for all $n$ $\Rightarrow$ $\mu_{B}^{crit} \simeq 500$ MeV

HOWEVER still consistent with resonance gas!!!

HRG analytic, LGT consistent with HRG $\Rightarrow$ infinite radius of convergence not yet ruled out