Fundamental Parameters of QCD from the Lattice

Hubert Simma

Milano Bicocca, DESY Zeuthen

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QCD Lagrangian and Parameters

$$\mathcal{L}_{\mathrm{QCD}}(g_0, m_0) = \frac{1}{4} F_{\mu\nu} F_{\mu\nu} + \sum_{f=1}^{N_f} \overline{\psi}(i \not{D} - m_0^{(f)})\psi$$

$$\overbrace{\begin{bmatrix} F_\pi \\ m_\pi \\ m_{\mathrm{K}} \\ m_{\mathrm{D}} \\ m_{\mathrm{B}} \end{bmatrix}}^{\text{Experiment}} \quad \overset{\mathcal{L}_{\mathrm{QCD}}(g_0, m_0)}{\Longrightarrow} \quad \overbrace{\begin{bmatrix} \Lambda_{\mathrm{QCD}} \\ \hat{M} = (M_{\mathrm{u}} + M_{\mathrm{d}})/2 \\ M_{\mathrm{s}} \\ M_{\mathrm{c}} \\ M_{\mathrm{b}} \end{bmatrix}}^{\textit{parameters (RGI)}} \quad + \quad \overbrace{\begin{bmatrix} \xi \\ F_B \\ B_B \\ \vdots \end{bmatrix}}^{\textbf{Predictions}}$$

Renormalisation

At high energies: PT and $\overline{MS}$

$$\Phi(q,r) = C_0(q,r) + C_1(q,r,\mu)\cdot\alpha_{\overline{MS}}(\mu) + C_2(q,r,\mu)\cdot\alpha^2_{\overline{MS}}(\mu) + \cdots$$

$\Rightarrow \alpha_{\overline{MS}}(\mu) \equiv \frac{g^2_{\overline{MS}}}{4\pi}$ (depends on Φ, choice of $\mu \approx q$, and order of PT)

$\Rightarrow \overline{m}_{\overline{MS}}(\mu)$ (may require additional assumptions, e.g. QCD sum rules)

Renormalisation

At low energies: Simulation at finite lattice spacing a

$$\mathcal{S}_{\mathrm{W}} = \frac{1}{g_0^2} \sum_p tr(1 - U_p) + \sum_f \sum_x \overline{\psi}_x (D_W + m_0^{(f)}) \psi_x$$

Hadronic scheme

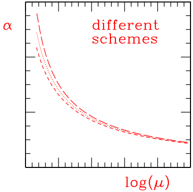
$$m_H^{exp} = \lim_{\mathrm{a} \to 0} \frac{(am_H)}{a(g_0)}$$

depending on choice of m_H,
and on N_f ratios $m_{H'}/m_H$ (to be kept at physical values)

Renormalization Group and Λ-Parameter

RGE for mass-independent scheme: $\overline{g} \equiv \overline{g}(\mu)$

$$\mu \frac{\partial \overline{g}}{\partial \mu} = \beta(\overline{g}) \overset{\bar{g}\to 0}{\sim} -\bar{g}^3 \left\{ b_0 + b_1 \bar{g}^2 + b_2 \bar{g}^4 + \ldots \right\}$$



- exact equation for "integration constant" Λ

$$\Lambda = \mu \left(b_0 \overline{g}^2 \right)^{-b_1/2b_0^2} \mathrm{e}^{-1/2b_0\overline{g}^2} \exp \left\{ - \int_0^{\overline{g}} \mathrm{d}g \left[\frac{1}{\beta(g)} + \frac{1}{b_0 g^3} - \frac{b_1}{b_0^2 g} \right] \right\}$$

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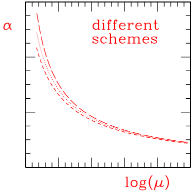

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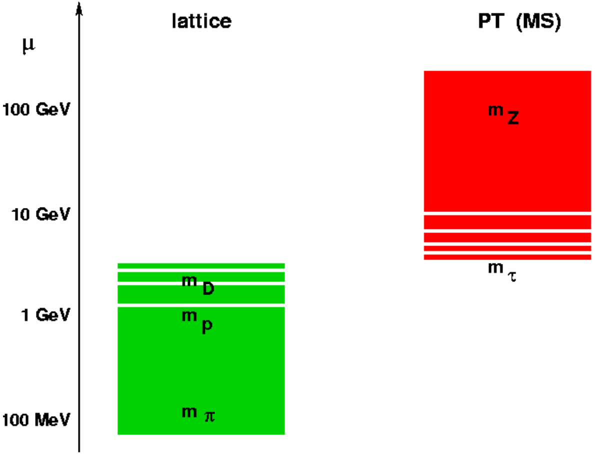
- trivial scheme dependence

$$\alpha_a = \alpha_b + c_{ab}\, \alpha_b^2 + O(\alpha_b^3) \;\Rightarrow\; \Lambda_a/\Lambda_b = \mathrm{e}^{c_{ab}/(4\pi b_0)}$$

- use a suitable physical coupling (scheme) and non-perturbative $\beta(\overline{g})$

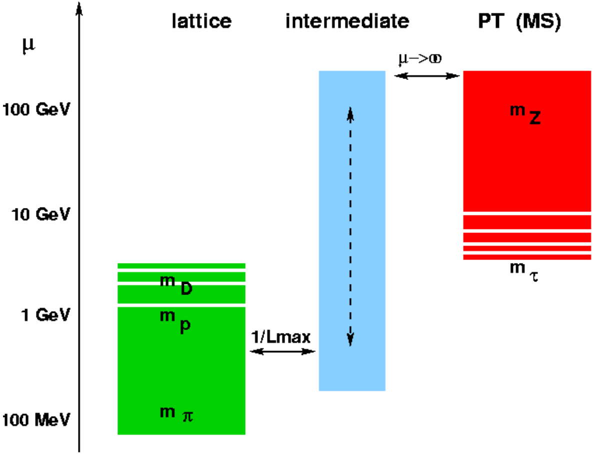
Connecting Hadronic and High-Energy Physics

Problem: Large scale differences $a^{-1} \gg \mu_{PT} \gg \mu_H \gg L^{-1}$



Connecting Hadronic and High-Energy Physics

Solution: Intermediate Renormalisation Scheme



ALPHA Collaboration Project

Use Schrödinger Functional (SF) as intermediate scheme

Calculate relation between low- and high-energy quantities in QCD with $N_f = 0, 2, \ldots$ flavors:

- define and compute NP renormalisation and running
- implementation and test of Symanzik improvement
- perform reliable continuum limit
- verify that systematic errors are under control

Not only applicable to fundamental parameters, but also to effective operators (B_K, ...)

ALPHA Collaboration

...initiated through key work and ideas of M. Lüscher et. al

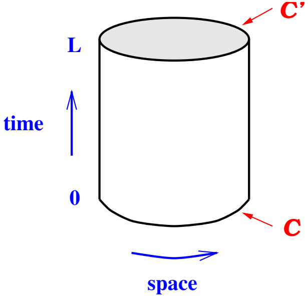
Univ. Bern	S. Dürr
CERN	M. Della Morte, C. Pena
Univ. Colorado	R. Hoffmann
DESY, Zeuthen	D. Guazzini, B. Leder, H.S., R. Sommer
Univ. Dublin	S. Sint
Univ. Edinburgh	J. Wennekes
Humboldt Univ. Berlin	J. Rolf, O. Witzel, U. Wolff
NIC, Zeuthen	K. Jansen, I. Wetzorke, A. Shindler
Univ. Mainz	F. Palombi, H. Wittig
MIT	H. Meyer
Univ. Münster	P. Fritzsch, J. Heitger
MPI München	P. Weisz
Univ. Roma II	P. Dimopoulos, R. Frezzotti, M. Guagnelli, A. Vladikas
Univ. Southampton	A. Jüttner
Univ. Wuppertal	F. Knechtli

`http://www-zeuthen.desy.de/alpha/`

Definition of Schrödinger Functional

- finite physical volume L^4, $T = L$
- Dirichlet b.c. $C(\eta)$, $C'(\eta)$ at $x_0 = 0, T$
- periodic b.c. in space (up to phase θ)

$$Z_{SF}(C, C') = e^{-\Gamma(\eta)} = \int_{\text{fields}} e^{-S(\eta)}$$



(LxLxL box with periodic b.c.)

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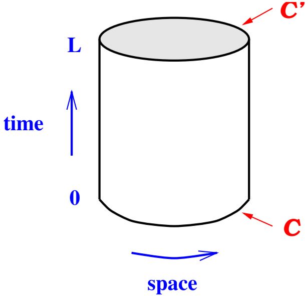
- renormalised coupling

$$\left.\frac{\partial\Gamma(\eta)}{\partial\eta}\right|_{\eta=0} \equiv \frac{k}{\bar{g}^2_{SF}(L)}$$



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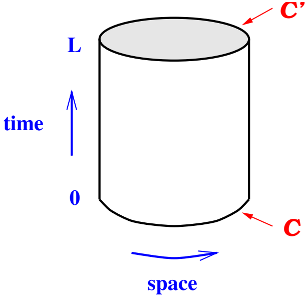
- mass-independent scheme

$$m_{PCAC} = 0$$

- renormalisation scale

$$\mu = 1/L$$



(LxLxL box with periodic b.c.)

Properties of Schrödinger Functional

- NP definition in continuum
- $\overline{g}_{SF}$ is local (plaquette-like) observable on the lattice
- spectral gap $\sim 1/L$ allows simulation with massless quarks
- known perturbative expansion

  (can use PT for running at very large μ
  after checking that it coincides with NP running)

Step Scaling Function (SSF)

- "discrete" β-function

$$\sigma(\overline{g}^2(L)) \equiv \overline{g}^2(2L)$$

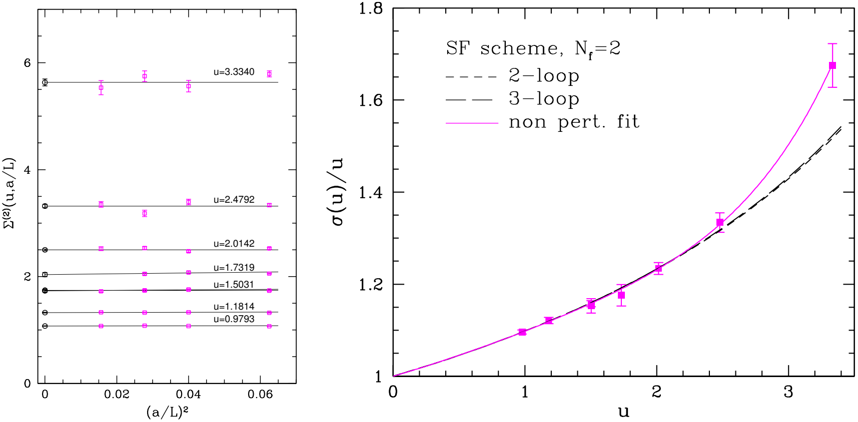
- determines NP running

$$u_k = \overline{g}^2\left(L_{max}/2^k\right)$$
$$\updownarrow$$
$$u_0 = \overline{g}^2\left(L_{max}\right)$$

- computation on the lattice

$$\Sigma(u, a/L) = \sigma(u) + O(a/L)$$

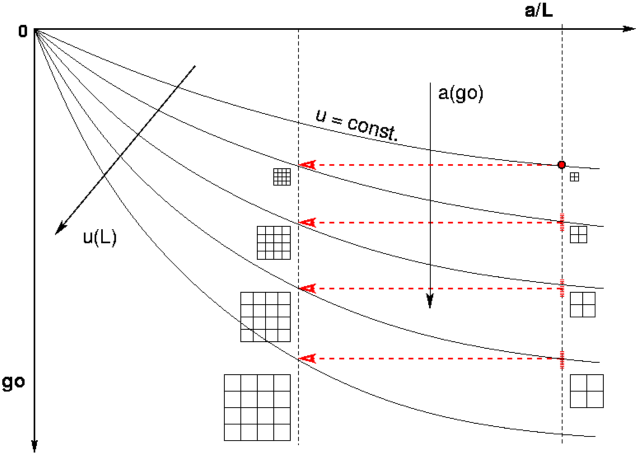
SSF for $N_f = 2$



u=3.3340
u=2.4792
u=2.0142
u=1.7319
u=1.5031
u=1.1814
u=0.9793
Σ⁽²⁾(u,a/L)
(a/L)²
0
0.02
0.04
0.06
0
2
4
6
SF scheme, N_f=2
2−loop
3−loop
non pert. fit
σ(u)/u
u
1
1.2
1.4
1.6
1.8
0
1
2
3

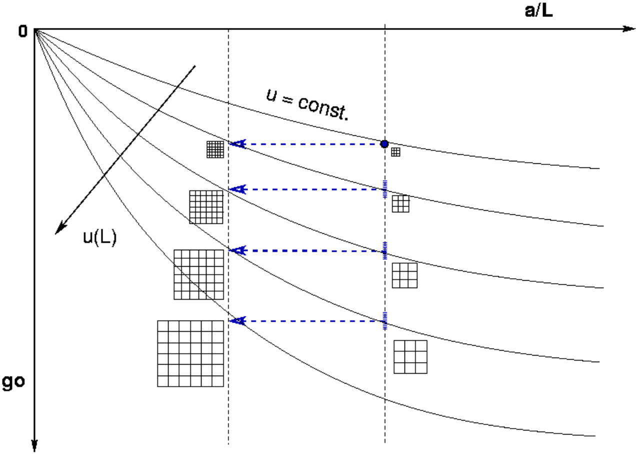
Simulation Parameters of SSF

$$(g_0, a/L) \quad \longrightarrow \quad u \equiv \bar{g}^2(L)$$

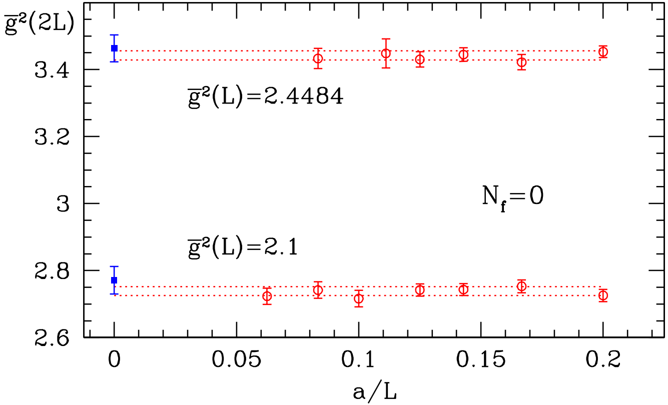


Simulation Parameters of SSF

Repeat for decreasing $a/L = 1/6, 1/8, \ldots \rightarrow$ continuum limit

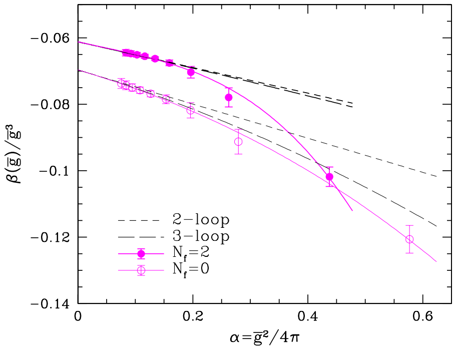


Precision test of the continuum extrapolation



$\Rightarrow$ procedure of continuum limit (with NP improved SF) is safe

Conversion of SSF to Beta Function



by solving

$$-2ln2 = \int_u^{\sigma(u)} \frac{\mathrm{d}x}{\sqrt{x}\beta(\sqrt{x})}$$

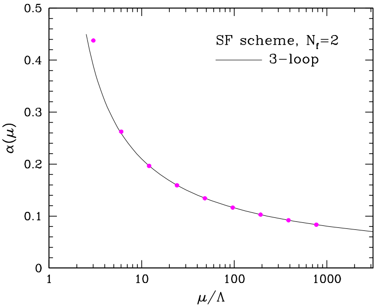
with parametrised SSF

- clear effect of N_f
- strong deviation from 3-loop PT for $\alpha_{SF} \geq 0.25$
- without indication from within PT

Running of α

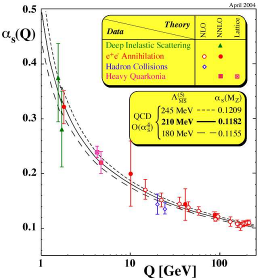
$N_f = 2$, NP + PT, SF scheme

error bars smaller than symbol size



Experiment + PT, $\overline{MS}$ scheme

[Bethke 2000]



Matching to Hadronic Scheme

- SSF yields precise ΛL_{max} (e.g. 7 % on Λ)

$$N_f = 0, \quad u_{max} = 3.48 : \quad ln(\Lambda_{\overline{MS}} L_{max}) = -0.84(8)$$
$$N_f = 2, \quad u_{max} = 4.61 : \quad ln(\Lambda_{\overline{MS}} L_{max}) = -0.40(7)$$

- For Λ in MeV need scale from aF_K (or aF_π)

$$\Lambda = (\Lambda L_{max}) \lim_{g_0 \to 0} \underbrace{\left(\frac{a}{L_{max}}\right)}_{\mathrm{SF}} \cdot \frac{F_K^{exp}}{\underbrace{(aF_K)}_{\mathrm{large}\,V}}$$

keeping N_f suitable flavoured mass ratios m_H/F_K fixed.

N.B.: "standard" values of $\beta = 6/g_0^2$ may need non-integer a/L_{max} from interpolation of $u(g_0, a/L) = u_{max}$

... $N_f = 2$ simulations with large volumes running on apeNEXT

Setting the Scale by r_0

- Currently need to use $r_0 \approx 0.5$ fm

$$\Lambda = (\Lambda L_{max}) \left(\frac{a}{L_{max}} \right) \left(\frac{r_0}{a} \right) \frac{1}{0.5 fm}$$

e.g. with QCDSF data for r_0/a (extrapolated to chiral limit)

- Summary of $\Lambda_{\overline{MS}}\, r_0$ for different N_f

	$N_f = 0$	$N_f = 2$	$N_f = 4$	$N_f = 5$
SF (ALPHA)	0.60(5)	0.62(6)	—	—
DIS (NLO)	—	—	0.57(8)	—
world av.	—	—	0.74(10)	0.54(8)

RGI Mass Parameter

RGE in mass-independent scheme

$$\mu \frac{\partial \overline{m}}{\partial \mu} = \tau(\overline{g}) \cdot \overline{m}$$

$$\overset{\bar{g}\to 0}{\sim} -\bar{g}^2 \left\{ d_0 + d_1 \bar{g}^2 + d_2 \bar{g}^4 + \ldots \right\}$$

RGI mass (integration constant of RGE)

$$M^{(f)} = \lim_{\mu\to\infty} (2b_0\overline{g})^{-d_0/2b_0} \overline{m}^{(f)}(\mu)$$

$$= \overline{m}^{(f)}(\mu) \cdot (2b_0\overline{g}^2)^{-d_0/2b_0} \times exp\left\{ -\int_0^{\overline{g}} \mathrm{d}g \left[\frac{\tau(g)}{\beta(g)} - \frac{d_0}{b_0 g} \right] \right.$$

RGI Mass Parameter

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- scale and scheme independent parameter
- use non-perturbative $\beta(\overline{g})$ and $\tau(\overline{g})$

Renormalised Quark Mass

- In a mass-independent scheme

$$\overline{m}^{(f)}(\mu) = \underbrace{Z_m(\mu a, g_0)}_{\text{flavour independent}} \cdot m^{(f)}_{\text{bare}}(g_0)$$

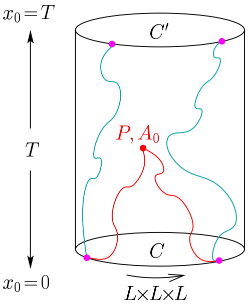
- can solve running once and for all

$$\frac{M^{(f)}}{M^{(j)}} = \frac{\overline{m}^{(f)}(\mu)}{\overline{m}^{(j)}(\mu)} = \frac{m^{(f)}_{\text{bare}}(g_0)}{m^{(j)}_{\text{bare}}(g_0)}$$

- defining $m^{(f)}_{\text{bare}}$ e.g. by PCAC relation $\partial_\mu A^{(f)}_\mu = 2\, m^{(f)}_{PCAC} P^{(f)}$

$$Z_m(\mu a, g_0) = \frac{Z_A(g_0)}{Z_P(g_0, L/a)}$$

Definition of Z_P in the SF



$$T = L, \; C = C' = 0$$

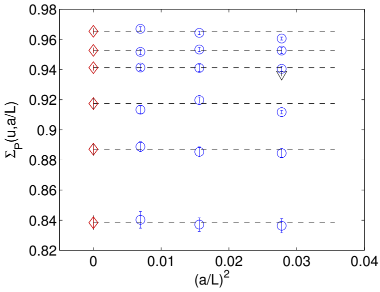
$$\theta = 1/2, \; m = 0$$

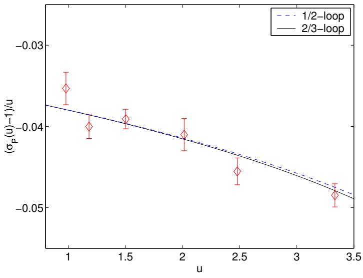
$$Z_P(L) \equiv c \frac{\sqrt{f_1}}{f_P(L/2)}$$

$$= 1 + O(g^2)$$

SSF for Quark Mass

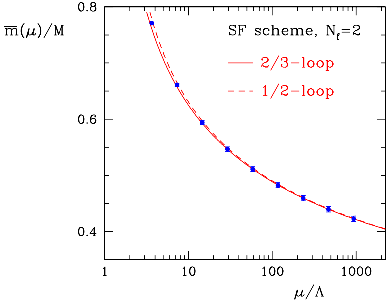
$$\Sigma_P(u, a/L) \equiv \left. \frac{Z_P(2L)}{Z_P(L)} \right|_{\bar{g}^2(L)=u} \qquad \sigma_P(u) \equiv \lim_{a\to 0} \Sigma_P(u, a/L)$$





NP Running of the Quark Mass

solve combined recursion for $\sigma_P(u)$ and $\sigma(u)$
(and PT from $L_{max}/2^k$ to "∞" for $k = 6$)



$$N_f = 2, \quad u_{max} = 4.61 : \quad \frac{M_{\rm RGI}}{\overline{m}(\mu)} = 1.297(16)$$

Determination of the Quark Masses

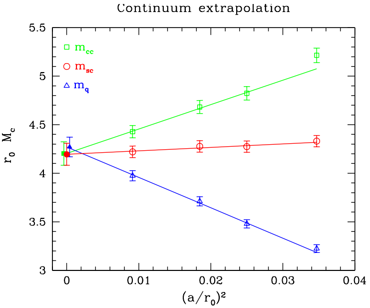
$$M_{\mathrm{RGI}}^{(f)} = \frac{M_{\mathrm{RGI}}}{\overline{m}(\mu)} \cdot \lim_{g_0 \to 0} Z_m(g_0, a\mu)\, m_0^{(f)}(\kappa^f, g_0, a/L)$$

Only $m_0^{(f)}$ is flavour-dependent, i.e. must be determined by matching (a ratio of) flavoured hadron masses

- M_s: m_K
- M_c: m_D (so far only $N_f = 0$)
- M_b: m_B (after matching to NP renormalised HQET)

Charm Quark Mass ($N_f = 0$)

- large mass renders $O(a)$ improvement essential
- different definitions of $m_0^{(c)}$ differ by $O(a^2 m_c^2)$ errors
- difficult continuum extrapolation



$$M_c = 1.65(5)\ \text{GeV}, \quad \text{or} \quad \overline{m}_c^{\overline{MS}}(\overline{m}_c) = 1.30(3)\ \text{GeV} \qquad (N_f = 0)$$

Strange Quark Mass ($N_f = 2$)

- determine reference quark mass m_{ref}, s.t.

$$m_{\mathrm{PS}}(m_{ref}, m_{ref}) = m_K$$

- from QCDSF data for r_0/a and am_{PS}
  determine $\kappa_{\mathrm{ref}}(\beta)$ at $\beta = 5.2, 5.29, 5.4$
- computing $\overline{m}(L_{max})$ at $\kappa_{\mathrm{ref}}(\beta)$ yields

$$M_{\mathrm{ref}} = 72(3)(13)\ \mathrm{MeV} \quad (\beta = 5.4)$$

Strange Quark Mass (cont.)

- lowest order 3-flavour χPT

$$m_K^2 = \frac{1}{2}\left(m_{K^+}^2 + m_{K^0}^2\right) = (\hat{M} + M_s)B_{\mathrm{RGI}}$$

- yields for 2 degenerate quarks

$$2M_{\mathrm{ref}} = (\hat{M} + M_s)$$

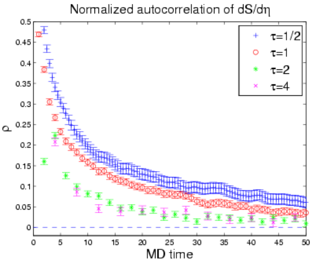
- use $M_s/\hat{M} = 24.4(1.5)$ [Leutwyler 1996] , i.e. $M_s = 48/25 M_{\mathrm{ref}}$

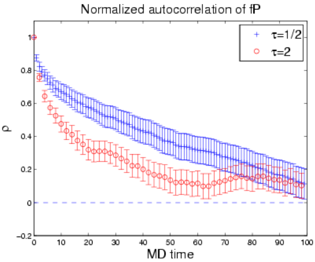
$M_s = 138(5)(26)$ MeV, or $\overline{m}_s^{\overline{MS}}(2\mathrm{GeV}) = 97(23)$ MeV

Simulation Algoritms

Various algorithmic improvements investigated on APE, e.g.

- Polynomial HMC [Frezzotti, Jansen]
- Hasenbusch trick
- Multiple time scale integration [Lüscher, Urbach et al.]
- Trajectory lengths





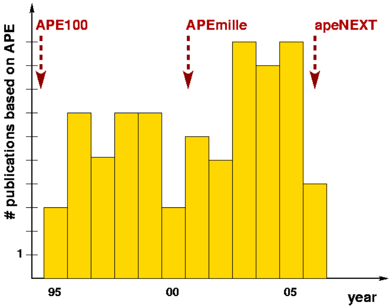
Main Steps

year	$N_f = 0$	hep-lat
1993...	SSF running coupling	9309005, 0110201
1996	NP improvement	9609035, ...
1996	Z_A, Z_V	9611015
1997	SSF running mass	9709125, 9810063
1998	L_{ref}/r_0	9806005
	$N_f = 2$	
1997	NP improvement	9709022, ...
2001...	SSF running coupling	0105003, 0411025
2005	Z_A, Z_V	0505026
2005	SSF running mass	0507035
?	$L_{ref} \cdot F_\pi$	

Computing Resources

ALPHA Collaboration is running (almost) exclusively on APE since 1994!

- significant fraction of APE installation at DESY
- contribution to APE development
  (O(25) man years out of ALPHA, QCDSF, NIC)
- early physics codes for qualification of APEmille/apeNEXT
- O(80) publications based on numerical results from APE



Outlook

Challenges:

- matching to hadronic scheme (Λ in MeV) for $N_f = 2$
- heavy quark physics
  (HQET, M_c for $N_f = 2$, f_B, M_b, B decays, ...)
- $N_f > 2$

...unlikely to be completed on apeNEXT