Toward understanding superstring theory in $AdS_5 \times S^5$

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- recent progress in perturbative GS superstring:

  first 2-loop computation

  R. Roiban and A. T., arXiv:0709.0681

- Reformulation of $AdS_5 \times S^5$ superstring in terms of currents: "Pohlmeyer reduction"

  M. Grigoriev and A. T., arXiv:0711.0155

AdS/CFT

$\mathcal{N} = 4$ SYM at $N = \infty$

dual to type IIB superstrings in $AdS_5 \times S^5$

$\lambda = g^2_{_{YM}} N$ related to string tension

$2\pi T = \frac{R^2}{\alpha'} = \sqrt{\lambda}$

$g_s = \frac{\lambda}{4\pi N} \to 0$

need to go beyond BPS states and

"supergravity + classical probes" approximation

Problems:

- spectrum of states (exact energies in λ)
- construction of vertex opeartors (closed and open string ones)
- computation of their correlation functions (graviton scattering, application to DIS in QCD ?)
- expectation values of various Wilson loops
- gluon scattering amplitudes
- generalizations to simplest less supersymmetric cases

  – orbifolds, exactly marginal deformations, ...
- strings at finite temperature in $AdS_5 \times S^5$ (without black hole and with it ...)
- solution of type 0 theory in $AdS_5 \times S^5$ ...
- non-critical superstrings: $AdS_5 \times S^1$, ...

$AdS_5 \times S^5$

Recent remarkable progress in quantitative understanding
interpolation from weak to strong ‘t Hooft coupling
based on using perturbative gauge theory (4-loop in λ)
and perturbative string theory (2-loop in $\frac{1}{\sqrt{\lambda}}$) “data”
and assumption of exact integrability
string energies = dimensions of gauge-invariant operators

$$E(\sqrt{\lambda}, J, m, ...) = \Delta(\lambda, J, m, ...)$$

J - charges of $SO(2,4) \times SO(6)$:
spins $S_1, S_2;\ J_1, J_2, J_3$
m - windings, folds, cusps, oscillation numbers, ...
Operators: $\mathrm{Tr}(\Phi_1^{J_1}\Phi_2^{J_2}\Phi_3^{J_3}D_+^{S_1}D_\perp^{S_2}...F_{mn}...\Psi...)$

Solve susy 4-d CFT = string in R-R background:
compute $E = \Delta$ for any λ (and J,m)

Perturbative expansions are opposite:

$\lambda \gg 1$ in perturbative string theory

$\lambda \ll 1$ in perturbative planar gauge theory

use perturbative results on both sides

and other properties (integrability, susy,...)

to come up with an exact answer – Bethe ansatz

Last 5 years: remarkable progress:

"semiclassical" string states with large quantum numbers

dual to "long" gauge operators (BMN, GKP, ...)

$E = \Delta$ – same dependence on $J, m, ...$

coefficients = interpolating functions of λ

SYM: dilatation operator that determines Δ

is same as an integrable spin chain Hamiltonian

integrability at both perturbative gauge ($\lambda \ll 1$)

and string ($\lambda \gg 1$) sides

suggests Bethe ansatz for the spectrum at any λ

Heisenberg-model type BA

(Beisert, Dippel, Staudacher 04; Staudacher 05)

$$e^{ip_k J} = \prod_{j\neq k}^{M} S(p_k, p_j; \lambda), \qquad S = S_1\, e^{i\theta}$$

$$S_1 = \frac{u_k - u_j + i}{u_k - u_j - i}\,, \qquad \theta = \theta(p_k, p_j; \lambda)$$

scattering of elementary excitations (magnons)

with 1-d momenta p_j and rapidities u_j

$u_j(p_j, \lambda) = \frac{1}{2}\cot\frac{p_j}{2}\,\sqrt{1 + \frac{\lambda}{\pi^2}\sin^2\frac{p_j}{2}}$

$E = J + \sum_{j=1}^{M}\left(\sqrt{1 + \frac{\lambda}{\pi^2}\sin^2\frac{p_j}{2}} - 1\right)$

What about phase θ ?

structure fixed by symmetries (Beisert 05)

$$\theta(p, p'; \lambda) = \sum_{r=2}^{\infty}\sum_{s=r+1}^{\infty} c_{rs}(\lambda)\left[q_s(p')q_r(p) - q_s(p)q_r(p')\right]$$

$$q_{r+1}(p) = \frac{2}{r}\sin\frac{rp}{2}\Big(\frac{\sqrt{1+\frac{\lambda}{\pi^2}\sin^2\frac{p}{2}}-1}{\frac{\lambda}{\pi^2}\sin\frac{p}{2}}\Big)^r \ ,$$

$c_{rs}(\lambda) = ?$

crucial input from string theory:

$c_{rs}(\lambda \gg 1) = \lambda^{\frac{r+s-1}{2}}\left[\delta_{r,s-1} + \frac{1}{\sqrt{\lambda}}a_{rs} + \frac{1}{(\sqrt{\lambda})^2}b_{rs} + ...\right]$

String 1-loop corrections to string energies

(Frolov, AT 03; Park, Tirziu, AT 05) $\rightarrow a_{rs} \neq 0$ (Beisert, AT 05)

1-loop string results translate into (Hernandez, Lopez 06)

$a_{rs} = \frac{2}{\pi}[1-(-1)^{r+s}]\frac{(r-1)(s-1)}{(r-1)^2-(s-1)^2}$

Consistent (Arutyunov, Frolov 06; Beisert 06)

with “crossing” (Janik 06)

All-order guess for strong coupling expansion

(Beisert, Hernandez, Lopez 06)

A year ago finally fixed completely (Beisert, Eden, Staudacher 06)

comparing to weak-coupling results (4-loop result of Bern et al)

But first-principles derivation remains to be given

Problem:

solve string theory in $AdS_5 \times S^5$

in particular, on an infinite line $\rightarrow$

determine the magnon (BMN excitation) scattering S-matrix $\rightarrow$

derive BA with the right BHL/BES phase

String Theory in $AdS_5 \times S^5$

bosonic coset $\frac{SO(2,4)}{SO(1,4)} \times \frac{SO(6)}{SO(5)}$

generalized to supercoset $\frac{PSU(2,2|4)}{SO(1,4)\times SO(5)}$ (Metsaev, AT 98)

$$S = T \int d^2\sigma \Big[G_{mn}(x)\partial x^m \partial x^n + \bar\theta(D + F_5)\theta\partial x$$

$$+ \bar\theta\theta\bar\theta\theta\partial x\partial x + ... \Big]$$

tension $T = \frac{R^2}{2\pi\alpha'} = \frac{\sqrt{\lambda}}{2\pi}$

Conformal invariance: $\beta_{mn} = R_{mn} - (F_5)^2_{mn} = 0$

Classical integrability of coset σ-model (Luscher-Pohlmeyer 76)

same for $AdS_5 \times S^5$ superstring (Bena, Polchinski, Roiban 02)

Progress in understanding of implications of (semi)classical integrability (Kazakov, Marshakov, Minahan, Zarembo 04; Beisert et al 05; Dorey, Vicedo 06,...)

Explicit computation of 1-loop quantum superstring corrections to
classical string energies (Frolov, AT 02-4, ...)
results were used as input for 1-loop term
in strong-coupling expansion of the phase θ in BA

Tree-level S-matrix of BMN states from $AdS_5 \times S^5$ GS string
agrees with limit of elementary magnon S-matrix
(Klose, McLoughlin, Roiban, Zarembo 06)

Semiclassical S-matrix in different limits:
string solitons on an infinite line – Giant magnons
(Hofman, Maldacena 06; Dorey 06, ...)
"Near-flat" limit (Hofman, Maldacena 07)
studied at 1-loop level with consistent results...

Last year:

2-loop string corrections (Roiban, Tirziu, AT; Roiban, AT 07)

2-loop check of finiteness of the GS superstring;

agreement with BA

– implicit check of integrability of quantum string theory

– non-trivial confirmation of BES exact phase in BA

– comparison to strong-coupling expansion

of BES equation (Basso, Korchemsky, Kotansky 07)

should extend to higher loop level

Universal scaling function = Cusp anomalous dimension

gauge theory: $\mathrm{Tr}(\Phi D_+^S \Phi)$

$$\Delta = S + 2 + f(\lambda) \ln S + ..., \qquad S \gg 1$$

$$f(\lambda \ll 1) = c_1 \lambda + c_2 \lambda^2 + c_3 \lambda^3 + c_4 \lambda^4 + ...$$

c_n are given by Feynmann graphs of 4d CFT – N=4 SYM

string theory: GKP folded string with spin S in AdS_5

$$f(\lambda \gg 1) = \frac{\sqrt{\lambda}}{\pi}[a_0 + \frac{a_1}{\sqrt{\lambda}} + \frac{a_2}{(\sqrt{\lambda})^2} + ...]$$

a_n are given by Feynmann graphs of 2d CFT – $AdS_5 \times S^5$ string

Explicitly:

$$f_{\lambda \ll 1} = \frac{1}{2\pi^2}\Big[\lambda - \frac{\lambda^2}{48} + \frac{11\lambda^3}{2^8 \times 45} - (\frac{73}{630} + \frac{4(\zeta(3))^2}{\pi^6})\frac{\lambda^4}{2^7} + ...\Big]$$

c_3: Kotikov, Lipatov, et al 03; c_4: Bern, Dixon, et al 06

$$f_{\lambda \gg 1} = \frac{\sqrt{\lambda}}{\pi}[1 - \frac{3\log 2}{\sqrt{\lambda}} - \frac{K}{(\sqrt{\lambda})^2} + ...]$$

a_0: Gubser, Klebanov, Polyakov 02;

a_1: Frolov, AT 02

a_2: Roiban, AT 07

$K = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} = 0.915...$ – Catalan's constant

appears from 2-loop sigma model integrals

Smooth interpolation from weak to strong coupling

Remarkably, both expansions are reproduced from single Beisert-Eden-Staudacher integral equation for $f(\lambda)$ obtained using the exact BES phase in the BA

Beyond 2-loop order in string theory ?

Deeper understanding of quantum string theory from integrability point of view?

Exact string S-matrix?

Proof of the BES Bethe ansatz ?

Green-Schwarz superstring in $AdS_5 \times S^5$

Superstring in curved type II supergravity background

$\int d^2\sigma\, G_{MN}(Z)\partial Z^M \partial Z^N + ... \,, \qquad Z^M = (x^m, \theta^I_\alpha)$

$m = 0, 1, ...9, \quad \alpha = 1, 2..., 16, \quad I = 1, 2$

Explicit form of action is generally hard to find

$AdS_5 \times S^5$: coset space symmetry facilitates explicit construction

Algebraic construction of unique κ-invariant action as in flat space

GS superstring in flat space:

$R^{1,9} = \frac{G}{H} = \frac{\text{Poincare}}{\text{Lorentz}}$

Flat superspace = $\frac{\widehat{G}}{H}$= $\frac{\text{SuperPoincare}}{\text{Lorentz}}$

structure of action is fixed by superPoincare algebra (P, M, Q)

$[P, M] \sim P, \;\; [M, M] \sim M, \;\; [M, Q] \sim Q, \;\; \{Q, Q\} \sim P$

$g^{-1}dg = J^m P_m + J^I_\alpha Q^\alpha_I + J^{mn} M_{mn}$

Supercoset action= $\int \mathrm{Tr}(g^{-1}dg)^2_{G/H}$ + fermionic WZ-term

$I = \int d^2\sigma (J^m J^m + a\bar{J}^I J^I) + b \int \; J^m \wedge \bar{J}^I \Gamma_m J^J s_{IJ}$

$s_{IJ} = (1, -1)$

$J^m = dx^m - i\bar{\theta}^I \Gamma^m \theta^I, \qquad J^I_\alpha = d\theta^I_\alpha$

Manifest superPoincare symmetry, but

unitarity and right fermionic spectrum iff $a = 0,\ b = \pm 1$:

κ-invariance $\rightarrow$ Green-Schwarz action:

$$L = -\tfrac{1}{2}(\partial_a x^m - i\bar{\theta}^I \Gamma^m \partial_a \theta^I)^2$$

$$+ i\epsilon^{ab} s_{IJ} \bar{\theta}^I \Gamma_m \partial_a \theta^J (\partial_b x^m - \tfrac{i}{2}\bar{\theta}^K \Gamma^m \partial_b \theta^K)$$

peculiar "degenerate" Lagrangian: no $\partial\bar{\theta}\partial\theta$ term

$L \sim \partial x \partial x + \partial x \bar{\theta}\partial\theta + (\bar{\theta}\partial\theta)^2$

perturbative expansion is well-defined

near $\bar{x}$ background, e.g., $x^m = N^m_a \sigma^a$

$x = \bar{x} + \xi,\ \ \theta' = \sqrt{\partial\bar{x}}\,\theta$

$L \sim \partial\xi\partial\xi + \bar{\theta}'\partial\theta' + \frac{1}{\sqrt{\partial\bar{x}}}\partial\xi\bar{\theta}'\partial\theta' + \ldots$

non-renormalizable by power counting

but κ-symmetry (uniqueness of action) implies finiteness

direct check of cancellation of 2-loop logarithmic UV divergences
and trivial partition function (Roiban, Tirziu, AT 07)
preservation of κ-symmetry implies that semiclassical loop (α')
expansion must be finite also in curved space
but regularization issues are non-trivial starting with 2 loops

$AdS_5 \times S^5 = \frac{SO(2,4)}{SO(1,4)} \times \frac{SO(6)}{SO(5)}$

Killing vectors and Killing spinors of $AdS_5 \times S^5$:

$PSU(2,2|4)$ symmetry

replace G/H=SuperPoincare/Lorentz in flat GS case by

$$\frac{PSU(2,2|4)}{SO(1,4) \times SO(5)}$$

generators: $(P_q, M_{pq}); (P'_r, M'_{rs}); Q^I_\alpha, \quad m = (q,r)$

$$[P,P] \sim M, \;\; [P,M] \sim P, \;\; [M,M] \sim M,$$

$$[Q,P_q] \sim \gamma_q Q, \;\; [Q,M_{pq}] \sim \gamma_{pq} Q$$

$$\{Q^I, Q^J\} \sim \delta^{IJ}(\gamma \cdot P + \gamma' \cdot P') + \epsilon^{IJ}(\gamma \cdot M + \gamma' \cdot M')$$

$PSU(2,2|4)$ invariant action:

$\int \mathrm{Tr}(g^{-1}dg)^2_{G/H}$ + WZ-term

$J = g^{-1}dg = J^m P_m + J^I_\alpha Q^\alpha_I + J^{mn} M_{mn}$

$$I = \frac{\sqrt{\lambda}}{2\pi}\Big[\int d^2\sigma (J^m J^m + a\bar{J}^I J^I) + b\int \ J^m \wedge \bar{J}^I \Gamma_m J^J s_{IJ}\Big]$$

as in flat space $a = 0, \ \ b = \pm 1$ required by κ-symmetry

unique action with right symmetry and right flat-space limit

Formal argument for UV finiteness (2d conformal invariance):

1. global symmatry – only overall coefficient
   of J^2 term (radius) can run
2. non-renormalization of WZ term (homogeneous 3-form)
3. preservation of κ-symmetry at the quantum level
   – relating coefficients of J^2 and WZ terms

Component form:

coset representative $g(x,\theta) = f(x)e^{\theta Q}$

$J^m = e^m(x) - i\bar\theta^I\Gamma^m D\theta^I + O(\theta^4), \quad J^I = D\theta^I + O(\theta^3)$

solving Maurer-Cartan eqs:

$$J^A_a \;\; = \;\; \partial_a x^m e^A_m - 4i\bar\theta^I\Gamma^A\Big[\frac{\sinh^2(\frac{s}{2}\mathcal{M})}{\mathcal{M}^2}\Big]_{IJ} D_a\theta^J \;, \qquad J^I_a = \Big[\frac{\sinh(s\mathcal{M})}{\mathcal{M}} D_a\theta\Big]^I,$$

$$D\theta^I = \mathcal{D}\theta^I - \frac{i}{2}\epsilon^{IJ}e^A(x)\Gamma_*\Gamma_A\theta^J \;\;, \qquad \mathcal{D}\theta^I = d\theta^I + \frac{1}{4}\omega^{AB}(x)\Gamma_{AB}\theta^I \;,$$

$$(\mathcal{M}^2)^{IL} = -\epsilon^{IJ}\Gamma_*\Gamma^A\theta^J\bar\theta^L\Gamma_A + \frac{1}{2}\epsilon^{LK}(\Gamma^{pq}\theta^I\bar\theta^K\Gamma_{pq}\Gamma_* - \Gamma^{rs}\theta^I\bar\theta^K\Gamma_{rs}\Gamma'_*)$$

$e^A(x) = dx^m\, e^A_m(x), \quad A = (p,r)$

$\Gamma_* = i\Gamma_0\Gamma_1\Gamma_2\Gamma_3\Gamma_4, \quad \Gamma'_* = i\Gamma_5\Gamma_6\Gamma_7\Gamma_8\Gamma_9$

RR coupling: “mass term” in D

D in IIB Killing spinor eq. $D^{IJ}\epsilon^J = 0,\; [D_M, D_N] = 0$

Expansion near string soliton solution $x = \bar{x}$:

conformal gauge and κ-symmetry gauge $\theta^1 = \theta^2$

$I = \frac{\sqrt{\lambda}}{2\pi} \int d^2\sigma \, (L_{\rm kin} + L_{\rm WZ})$

$$
\begin{aligned}
L_{\rm kin} &= -\frac{1}{2}\partial_a x^\mu \partial^a x^\nu G_{\mu\nu}(x) + 2ie_a^A \bar{\theta}\Gamma_A \mathcal{D}^a\theta + 2\bar{\theta}\Gamma^A \mathcal{D}_a\theta\bar{\theta}\Gamma_A\mathcal{D}^a\theta \\
&+ \frac{1}{12}e_a^A e^{aB}\; \bar{\theta}\Gamma_A(\Gamma^{pq}\theta\bar{\theta}\Gamma_{pq} - \Gamma^{rs}\theta\bar{\theta}\Gamma_{rs})\Gamma_B\theta + O(\theta^6) \\
L_{\rm WZ} &= \epsilon^{ab}\Big[- e_a^A e_b^B \bar{\theta}\Gamma_A\Gamma_*\Gamma_B\theta + \frac{4i}{3}e_a^A\bar{\theta}\Gamma_A\Gamma_*\Gamma_B\theta\bar{\theta}\Gamma^B\mathcal{D}_b\theta\Big] + O(\theta^6)
\end{aligned}
$$

Expansion: $x \to x+\xi, \quad L = \xi D^2\xi + \bar{\theta}D\theta + \xi^3 + \xi^4 + \xi\theta^2 + \theta^4 + ...$

1-loop results:

• check of finiteness of GS action for generic $\bar{x}$ solution

• computation of 1-loop quantum string corrections to energies of rigid rotating string solutions (Frolov, AT 02,03; Park, AT 05)

– data for reconstructing 1-loop term in strong-coupling expansion of phase in BA (Beisert, AT 05; Hernandez, Lopez 06)

Simple form of the $AdS_5 \times S^5$ action

special choice of coordinates (Poincare)

and special κ-symmetry gauge: $\theta^1 = \Gamma_{0123}\theta^2$

plus "Killing spinor" redefn of fermions (Kallosh, Rajaraman 98)

$$I = \frac{\sqrt{\lambda}}{4\pi} \int d^2\sigma \big[z^2 (\partial_a x^m - i\bar\theta \Gamma^m \partial_a \theta)^2 + \frac{1}{z^2} \partial^a z^s \partial_a z^s + 4\epsilon^{ab} \bar\theta \partial_a z^s \Gamma_s \partial_b \theta \big]$$

$m = 0, 1, 2, 3;\ s = 4, ..., 9,\ \ z^2 = z^s z^s,\ a, b = 0, 1$

after formal T-duality: $x^m \to \widetilde{x}^m$

action becomes exactly quadratic in θ (Kallosh, AT 98)

$$I = \frac{\sqrt{\lambda}}{4\pi} \int d^2\sigma \big[\frac{1}{z^2} (\partial^a x^m \partial_a x_m + \partial^a z^s \partial_a z^s) + 4\epsilon^{ab} \bar\theta (\partial_a x^m \Gamma_m + \partial_a z^s \Gamma_s) \partial_b \theta \big]$$

starting point of computation of 2-loop string correction

to cusp anomalous dimension (Roiban, AT 07)

check of 2-loop finiteness of $AdS_5 \times S^5$ GS string

check of BES phase proposal against 2-loop string theory

How to solve quantum string theory in $AdS_5 \times S^5$?

GS string on supercoset $\frac{PSU(2,2|4)}{SO(1,4)\times SO(5)}$
not of known solvable type (cf. free oscillators; WZW)
analogy with exact solution of $O(n)$ model (Zamolodchikovs) or
principal chiral model (Polyakov-Wiegmann; KWZ; ...) ?
– 2d CFT – no mass generation

Try as in flat space –
light-cone gauge: analog of $x^+ = p^+\tau,\ p^+ = \text{const},\ \Gamma^+\theta = 0$
Two natural options:
(i) null geodesic parallel to the boundary in Poincare patch –
action/Hamiltonian quartic in fermions (Metsaev, Thorn, AT, 01)
(ii) null geodesic wrapping S^5:
hidden $su(2|2) \times su(2|2)$ symmetry
but complicated action (Callan et al, 03;
Arutyunov, Frolov, Plefka, Zamaklar, 05-06)

Common problem:

lack of manifest 2d Lorentz symmetry

hard to apply known 2d integrable field theory methods –
S-matrix depends on two rapidities, not on their difference only
constraints on it are unclear, etc.

An alternative approach: "Pohlmeyer reduction"
use conf. gauge, solve Virasoro conditions in terms of currents,
find "reduced" action for physical number of d.o.f.,
use it as a starting point for quantization

compare to two related models:

I. "non-abelian dual" for PCM
(Zakharov, Mikhailov 78; Nappi 80)
– solve EOM's in terms of currents,
consider flatness condition (MC) as dynamical

$$L = \mathrm{Tr}(J_a J^a) \ , \qquad J_a = g^{-1}\partial_a g$$

$$\partial_a J^a = 0 \,, \qquad \partial_a J_b - \partial_b J_a + [J_a, J_b] = 0$$

Solve EOM by $J_a = \epsilon_{ab}\partial^b \chi, \;\; \chi \in \mathfrak{g}$
then from flatness (MC)

$$\partial^a \partial_a \chi - \epsilon^{ab} \partial_a \chi \partial_b \chi = 0$$

following from

$$L = \mathrm{Tr}(\partial^a \chi \partial_a \chi + \frac{2}{3} \epsilon^{ab} \chi [\partial_a \chi, \partial_b \chi])$$

corresponds to a gauge-equivalent choice of classical
Lax pair (Mikhailov-Zakharov 78)
But: does not solve Virasoro conditions;
does not define equivalent quantum theory
(Nappi; Fridling, Jevicki 84; Fradkin, AT 85)

Another attempt:

II. FR model (Faddeev, Reshetikhin 86)

express PCM + Virasoro in terms of two

constrained currents as basic variables

fix conf. symm. or add Virasoro for $R_t \times G$ ($X^0 = \mu\tau$)

$$\mathrm{Tr}(J_+ J_+) = \mu^2 \ , \qquad \mathrm{Tr}(J_- J_-) = \mu^2$$

in addition to EOM combined with MC into

$$D_- J_+ = 0 \ , \qquad D_+ J_- = 0 \ , \qquad D_a = \partial_a + [J_a, \ ]$$

e.g. $G = S^3 = SU(2)$: take $n^i_\pm = \mu^{-1} J^i_\pm$

as two unit vectors to solve Virasoro; action:

$$S = \int d^2\sigma [C_+(n_-) + C_-(n_+) + \mu^2 n^i_+ n^i_-] \ ,$$

where $C_a(J^i) \equiv -\frac{1}{2}\int_0^1 dy\ \epsilon_{ijk} n^i \partial_a n^j \partial_y n^k$

get first-order action for 2+2=4 independent d.o.f.

But: 2d Lorentz invariance is missing – broken by constraints

Remarkably, there is an alternative system with standard
2d Lorentz invariant second-order action
for 2 dynamical d.o.f. (1+3-2=2)
describing the same $R_t \times S^3$ string equations of motion
Complex sine-Gordon model found by Pohlmeyer reduction (PR)

$$\widetilde{S} = \int d^2\sigma [\partial_+\varphi\partial_-\varphi + \cot^2\varphi\, \partial_+\theta\partial_-\theta + \frac{\mu^2}{2}\cos 2\varphi]$$

CSG: an example of non-abelian Toda model (Leznov, Saveliev):
related to massive integrable perturbation of a coset WZW model
–here $SO(3)/SO(2)$ (Hollowood, Miramontes, Park 94)
quantum-integrable: S-matrix is known (Dorey, Hollowood 95)

Aim: construct PR version for $AdS_5 \times S^5$ superstring

(i) introduce new fields locally related to supercoset currents

(ii) solve conformal gauge (Virasoro) condition explicitly

(iii) find local 2d Lorentz-invariant

action for independent (8B+8F) d.o.f

– fermionic generalization of non-abelian Toda theory

PR: a nonlocal map that preserves integrable structure

1. gauge-equivalent Lax pairs; map between soliton solutions

gives integrable massive local field theory

2. quantum equivalence to original GS model ?

may expect for full $AdS_5 \times S^5$ string model = CFT

3. integrable theory: semiclassical solitonic spectrum

may essentially determine quantum spectrum

the two solitonic S-matrices should be closely related:

Lorentz-invariant S-matrix of PR-model should effectively

give the complicated magnon S-matrix

Pohlmeyer reduction: bosonic coset models

Prototypical example: S^2-sigma model $\to$ Sine-Gordon theory

$$L = \partial_+ X^m \partial_- X^m - \Lambda(X^m X^m - 1)\,, \qquad m = 1,2,3$$

Equations of motion:

$$\partial_+\partial_- X^m + \Lambda X^m = 0\,, \quad \Lambda = \partial_+ X^m \partial_- X^m\,, \quad X^m X^m = 1$$

Stress tensor: $\mathrm{T}_{\pm\pm} = \partial_\pm X^m \partial_\pm X^m$

$$\mathrm{T}_{+-} = 0\,, \quad \partial_+ \mathrm{T}_{--} = 0\,, \qquad \partial_- \mathrm{T}_{++} = 0$$

implies $\mathrm{T}_{++} = f(\sigma_+)$, $\mathrm{T}_{--} = h(\sigma_-)$
using the conformal transformations $\sigma_\pm \to F_\pm(\sigma_\pm)$ can set

$$\partial_+ X^m \partial_+ X^m = \mu^2\,, \qquad \partial_- X^m \partial_- X^m = \mu^2\,, \qquad \mu = \text{const}\ .$$

3 unit vectors in 3-dimensional Euclidean space:

$$X^m\,, \qquad X^m_+ = \mu^{-1}\partial_+ X^m\,, \qquad X^m_- = \mu^{-1}\partial_- X^m\,,$$

X^m is orthogonal ($X^m\partial_\pm X^m = 0$) to both X^m_+ and X^m_-
remaining $SO(3)$ invariant quantity is scalar product

$$\partial_+ X^m \partial_- X^m = \mu^2 \cos 2\varphi$$

then $\partial_+\partial_-\varphi + \frac{\mu^2}{2}\sin 2\varphi = 0$
following from sine-Gordon action (Pohlmeyer, 1976)

$$\widetilde{L} = \partial_+\varphi\partial_-\varphi + \frac{\mu^2}{2}\cos 2\varphi$$

2d Lorentz invariant despite explicit constraints
Classical solutions and integrable structures
(Lax pair, Backlund transformations, etc) are directly related
e.g., SG soliton mapped into rotating folded string on S^2
"giant magnon" in the $J = \infty$ limit (Hofman, Maldacena 06)
other examples for CSG (Chen, Dorey, Okamura 06;
Okamura, Suzuki, Hayashi, Vicedo 07;
Jevicki, Spradlin, Volovich, et al 07)

Analogous construction for S^3 model gives

Complex sine-Gordon model (Pohlmeyer; Lund, Regge 76)

$$\widetilde{L} = \partial_+\varphi\partial_-\varphi + \cot^2\varphi\,\partial_+\theta\partial_-\theta + \frac{\mu^2}{2}\cos 2\varphi$$

φ, θ are $SO(4)$-invariants:

$$\mu^2 \cos 2\varphi = \partial_+ X^m \partial_- X^m$$

$$\mu^3 \sin^2\varphi\,\partial_\pm\theta = \mp\frac{1}{2}\epsilon_{mnkl} X^m \partial_+ X^n \partial_- X^k \partial_\pm^2 X^l$$

"String on $R_t \times S^n$" interpretation

conformal gauge plus $t = \mu\tau$ to fix conformal diffeomorphisms:

$\partial_\pm X^m \partial_\pm X^m = \mu^2$ are Virasoro constraints

Similar construction for AdS_n case,

i.e. string on $AdS_n \times S^1_\psi$ with $\psi = \mu\tau$

e.g. reduced theory for $AdS_3 \times S^1$

$$\widetilde{L} = \partial_+\phi\partial_-\phi + \coth^2\varphi\,\partial_+\chi\partial_-\chi - \frac{\mu^2}{2}\cosh 2\phi$$

Comments:

- Virasoro constraints are solved by a special choice of variables related nonlocally to the original coordinates
- Although the reduction is not explicitly Lorentz invariant the resulting Lagrangian turns out to be 2d Lorentz invariant
- The reduced theory is formulated in terms of manifestly $SO(n)$ invariant variables: "blind" to original global symmetry
- reduced theory is equivalent to the original theory as integrable system: the respective Lax pairs are gauge-equivalent
- PR may be thought of as a formulation in terms of physical d.o.f. – coset space analog of flat-space l.c. gauge (where 2d Lorentz is unbroken)
- In general reduced theory can not be quantum-equivalent to the original one (e.g., conformal symmetry was assumed in the reduction procedure)

PR for bosonic F/G-coset model

To find reduced theory for $AdS_5 \times S^5$ GS model need to understand PR of F/G coset sigma models as G/H gauged WZW models modified by relevant integrable potential and then generalize to GS supercoset

F/G-coset sigma model:

symmetric space condition ($\mathfrak{f}$, $\mathfrak{g}$ are Lie algebras of F and G)

$$\mathfrak{f} = \mathfrak{p} \oplus \mathfrak{g} \,, \qquad [\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{g} \,, \qquad [\mathfrak{g}, \mathfrak{p}] \subset \mathfrak{p} \,, \qquad [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{g}$$

with $\langle \mathfrak{g}, \mathfrak{p} \rangle = 0$ (choose $\langle a, b \rangle = \mathrm{Tr}(ab)$)

Lagrangian:

$$L = -\mathrm{Tr}(P_+ P_-) \,, \qquad P_\pm = (f^{-1}\partial_\pm f)_\mathfrak{p} \,,$$

$$J = f^{-1} df = \mathcal{A} + P \,, \qquad \mathcal{A} = J_\mathfrak{g} \in \mathfrak{g} \,, \qquad P = J_\mathfrak{p} \in \mathfrak{p} \,.$$

Symmetries: G gauge transformations $f \to fg$;

global F-symmetry: $f \to f_0 f$, $f_0 = \mathrm{const} \in F$

classical conformal invariance

Equations of motion in terms of currents

let $J = \mathcal{A} + P$ be fundamental variables, not f

$D_+ P_- = 0\,, \quad D_- P_+ = 0\,, \qquad D = d + [\mathcal{A},\ ]$ – EOM

$D_- P_+ - D_+ P_- + [P_+, P_-] + \mathcal{F}_{+-} = 0$ – Maurer-Cartan

$\mathrm{Tr}(P_+ P_+) = -\mu^2\,, \quad \mathrm{Tr}(P_- P_-) = -\mu^2$ – Virasoro

Main idea: – first solve EOM and Virasoro and then MC
using special choice of G gauge condition and conformal diffs
then find reduced action giving eqs. resulting from MC
gauge fixing that solves the first Virasoro constraint

$$P_+ = \mu\, T = \text{const}\,, \qquad T \in \mathfrak{p} = \mathfrak{f} \ominus \mathfrak{g}, \qquad \mathrm{Tr}(TT) = -1$$

choice of special element $T \to$ decomposition of the algebra of F

$$\mathfrak{f} = \mathfrak{p} \oplus \mathfrak{g}\,, \qquad \mathfrak{p} = T \oplus \mathfrak{n}\,, \qquad \mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}\,, \qquad [T, \mathfrak{h}] = 0\,,$$

$$[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}\,, \qquad [\mathfrak{m}, \mathfrak{h}] \subset \mathfrak{m}\,, \qquad [T, \mathfrak{m}] \subset \mathfrak{n}\,, \qquad [T, \mathfrak{n}] \subset \mathfrak{m}\,.$$

$\mathfrak{h}$ is a centraliser of T in $\mathfrak{g}$

EOM $D_-P_+ = 0$ is solved by

$$(\mathcal{A}_-)_\mathfrak{m} = 0\ , \qquad \mathcal{A}_- = (\mathcal{A}_-)_\mathfrak{h} \equiv A_-$$

second Virasoro constraint is solved by

$$P_- = \mu\, g^{-1} T g\ , \qquad\qquad g \in G$$

EOM $D_+P_- = 0$ is solved by

$$\mathcal{A}_+ = g^{-1}\partial_+ g + g^{-1} A_+ g$$

To summarise:
solved EOM's and Virasoro constraints introducing
new dynamical field variables

G-valued field g , $\qquad$ $\mathfrak{h}$-valued fields $\quad A_+,\ A_-, \quad [T, A_\pm] = 0$

what remains is the Maurer-Cartan equation on $g, A_\pm$

Relation to G/H gauged WZW model

Maurer-Cartan equation in terms of new parametrization:

$$\partial_-(g^{-1}\partial_+ g + g^{-1}A_+ g) - \partial_+ A_- \\ + [A_-, g^{-1}\partial_+ g + g^{-1}A_+ g] + \mu^2[g^{-1}Tg, T] = 0$$

Recall: $J = f^{-1}df = \mathcal{A} + P,\ P_+ = \mu T,\ P_- = \mu g^{-1}Tg$

$\mathcal{A}_+ = g^{-1}\partial_+ g + g^{-1}A_+ g, \quad \mathcal{A}_- = A_-$

MC eq. has "on-shell" $H \times H$ gauge symmetry:

$$g \to h^{-1} g \bar{h}\,,$$

$$A_+ \to h^{-1}A_+ h + h^{-1}\partial_+ h\,, \quad A_- \to \bar{h}^{-1}A_-\bar{h} + \bar{h}^{-1}\partial_-\bar{h}\,,$$

can choose a gauge: $A_+ = (g^{-1}\partial_+ g + g^{-1}A_+ g)_{\mathfrak{h}}$,

$A_- = (-\partial_- g g^{-1} + gA_- g^{-1})_{\mathfrak{h}}$

remains left-right H gauge symmetry: $h = \bar{h}$

"off-shell" symmetry of corresponding gWZW action

G/H gWZW action with potential:

$$
\begin{aligned}
L = \quad & - \quad \frac{1}{2}\mathrm{Tr}(g^{-1}\partial_+ g g^{-1}\partial_- g) + \text{WZ term} \\
& - \quad \mathrm{Tr}\Big(A_+ \, \partial_- g g^{-1} - A_- \, g^{-1}\partial_+ g - g^{-1}A_+ g A_- + A_+ A_-\Big) \\
& - \quad \mu^2 \mathrm{Tr}(T g^{-1} T g)
\end{aligned}
$$

Pohlmeyer-reduced theory for F/G coset sigma model

(as first proposed by Bakas, Park, Shin 95)

and thus also for strings on $R_t \times F/G$ or $F/G \times S^1_\psi$

integrable potential: relation at the level of Lax pairs

special case of non-abelian Toda theory:

"symmetric space Sine-Gordon model"

(Hollowood, Miramontes et al 96)

Similar reduction for G PCM or $\frac{G\times G}{G}$ coset leads to G/H theory

with $H = [U(1)]^r$= Cartan of G,

"homogeneous Sine-Gordon model", known to be quantum-integrable

generalizes CSG model ($G = S^3 = SO(3)$)

What to do with A_+, A_-: integrate out or gauge-fix

Reduced equation of motion in the "on-shell" gauge $A_\pm = 0$:

On-shell $\partial_- A_+ - \partial_+ A_- + [A_-, A_+] = 0$ so can set $A_\pm = 0$

$$\partial_-(g^{-1}\partial_+ g) - \mu^2 [T, g^{-1} T g] = 0 \,,$$
$$(g^{-1}\partial_+ g)_{\mathfrak{h}} = 0 \,, \qquad\qquad (\partial_- g g^{-1})_{\mathfrak{h}} = 0 \,.$$

$F/G = SO(n+1)/SO(n) = S^n$: $G/H = SO(n)/SO(n-1)$

$$g = \begin{pmatrix} k_1 & k_2 & \dots & k_n \\ \dots & \dots & \dots & \dots \end{pmatrix} \,, \qquad \sum_{1=1}^{n} k_l k_l = 1$$

get (in general non-Lagrangian) EOM for k_m

$$\partial_-\Big(\frac{\partial_+ k_\ell}{\sqrt{1 - \sum_{m=2}^{n} k_m k_m}}\Big) = -\mu^2 k_\ell \,, \qquad \ell = 2, \dots, n \,.$$

Linearising around the vacuum $g = 1$ (i.e. $k_1 = 1,\ k_\ell = 0$)

$$\partial_+\partial_- k_\ell + \mu^2 k_\ell + O(k_\ell^2) = 0$$

massive spectrum: non-trivial S-matrix with H global symmetry

$F/G = SO(n+1)/SO(n) = S^n$:

parametrization of g in Euler angles

$g = e^{T_{n-2}\theta_{n-2}} ... e^{T_1\theta_1} e^{2T\varphi} e^{T_1\theta_1} ... e^{T_{n-2}\theta_{n-2}}$

and integrating out $H = SO(n-1)$ gauge field $A_\pm$

leads to reduced theory that generalizes SG and CSG

$$\widetilde{L} = \partial_+\varphi\partial_-\varphi + G_{pq}(\varphi,\theta)\partial_+\theta^p\partial_-\theta^q + \frac{\mu^2}{2}\cos 2\varphi$$

no B_{mn} coupling

gWZW for $G/H = SO(n)/SO(n-1)$

$$ds^2_{n=2} = d\varphi^2 \,, \qquad\qquad ds^2_{n=3} = d\varphi^2 + \cot^2\varphi\, d\theta^2$$

ironically, return of old metrics of "de Sitter" or "S^n"

gWZW models (Bars, Nemeschansky,...)

$G/H = SO(4)/SO(3)$ (Fradkin, Linetsky 91)

$$ds^2_{n=4} = d\varphi^2 + \cot^2\varphi\,(d\theta_1 + \cot\theta_1\tan\theta_2\, d\theta_2)^2 + \tan^2\varphi\,\frac{d\theta_2^2}{\sin^2\theta_1}$$

change of variables $x = \cos\theta_1 \; \cos\theta_2, \;\; y = \sin\theta_2$

$$ds^2_{n=4} = d\varphi^2 + \frac{\cot^2\varphi \, dx^2 + \tan^2\varphi \, dy^2}{1 - x^2 - y^2}$$

$G/H = SO(5)/SO(4)$ (Bars, Sfetsos 92)

$$\begin{aligned} ds^2_{n=5} &= d\varphi^2 + \cot^2\varphi \, (d\theta_1 + U d\theta_2 + V d\theta_3)^2 \\ &+ \tan^2\varphi [\frac{d\theta_2^2}{\cos^2\theta_1} + \frac{d\theta_3^2}{\sin^2\theta_1}] \\ U &= \frac{\tan\theta_1 \sin 2\theta_2}{\cos 2\theta_2 + \cos 2\theta_3}, \quad V = \frac{\cot\theta_1 \sin 2\theta_3}{\cos 2\theta_2 + \cos 2\theta_3} \end{aligned}$$

no isometries, singularities

similar for $F/G = SO(2, n-1)/SO(1, n-1) = AdS_n$ case:

$G/H = SO(1, n-1)/SO(n-1)$

Bosonic strings on $AdS_n \times S^n$

straightforward generalization:

Lagrangian and the Virasoro constraints

$L = \mathrm{Tr}(P_+^A P_-^A) - \mathrm{Tr}(P_+^S P_-^S)\,,$

$\mathrm{Tr}(P_\pm^S P_\pm^S) - \mathrm{Tr}(P_\pm^A P_\pm^A) = 0$

fix conformal symmetry by

$\mathrm{Tr}(P_\pm^S P_\pm^S) = \mathrm{Tr}(P_\pm^A P_\pm^A) = -\mu^2$

then PR applies independently in each sector:

get direct sum of reduced systems for S^n and AdS_n

linked by Virasoro, i.e. common μ

e.g. for $F/G = AdS_2 \times S^2$:

$$\widetilde{L} = \partial_+\varphi\partial_-\varphi + \partial_+\phi\partial_-\phi + \frac{\mu^2}{2}(\cos 2\varphi - \cosh 2\phi)$$

$AdS_5 \times S^5$ superstring sigma-model

$AdS_5 \times S^5 = \frac{SU(2,2)}{Sp(2,2)} \times \frac{SU(4)}{Sp(4)}$

supercoset GS sigma model (Metsaev, AT 98)

$\frac{\widehat{F}}{G} = \frac{PSU(2,2|4)}{Sp(2,2)\times Sp(4)}$

basic superalgebra $\widehat{\mathfrak{f}} = psu(2,2|4)$

bosonic part $\mathfrak{f} = su(2,2) \oplus su(4) \cong so(2,4) \oplus so(6)$

admits Z_4-grading: (Berkovits, Bershadsky, et al 89)

$$\widehat{\mathfrak{f}} = \mathfrak{f}_0 \oplus \mathfrak{f}_1 \oplus \mathfrak{f}_2 \oplus \mathfrak{f}_3\,, \qquad\qquad [\mathfrak{f}_i, \mathfrak{f}_j] \subset \mathfrak{f}_{i+j \bmod 4}$$

$\mathfrak{f}_0 = \mathfrak{g} = sp(2,2) \oplus sp(4)$

current ($J = f^{-1}\partial_a f,\ f \in \widehat{F}$) decomposes as

$$J_a = f^{-1}\partial_a f = \mathcal{A}_a + Q_{1a} + P_a + Q_{2a}$$

$$\mathcal{A} \in \mathfrak{f}_0, \quad Q_1 \in \mathfrak{f}_1, \quad P \in \mathfrak{f}_2, \quad Q_2 \in \mathfrak{f}_3\,.$$

GS Lagrangian:

$$L_{\rm GS} = \frac{1}{2}{\rm STr}(\sqrt{-g}g^{ab}P_aP_b + \varepsilon^{ab}Q_{1a}Q_{2b})\,,$$

very simple structure – but not standard coset model:

fermionic currents in WZ term only

this leads to local fermionic κ-symmetry:

$$\delta_\kappa J_a = \partial_a\epsilon + [J_a,\epsilon]$$

$$(\delta_\kappa\sqrt{-g}g^{ab})^{ab} = {\rm STr}\left(W([ik^a_{1(-)},Q^b_{1(-)}] + [ik^a_{2(+)},Q^b_{2(+)}])\right)$$

$$\epsilon = \epsilon_1 + \epsilon_2 = \{P_{(+)a},ik^a_{1(-)}\} + \{P_{(-)a},ik^a_{2(+)}\}$$

self-dual 2-vector parameters $k_{1(-)}$ and $k_{2(+)}$

take values in the degree 1 and degree 3 subspaces of $u(2,2|4)$

$W = {\rm diag}(1,\ldots,1,-1,\ldots,-1)$

$V^a_{(\pm)} \equiv \frac{1}{2}(\gamma^{ab}\mp\varepsilon^{ab})V_b$

conformal gauge: $\sqrt{-g}g^{ab} = \eta^{ab}$

$$L_{\rm GS} = {\rm STr}[P_+P_- + \frac{1}{2}(Q_{1+}Q_{2-} - Q_{1-}Q_{2+})]$$

$${\rm STr}(P_+P_+) = 0\,, \qquad\qquad {\rm STr}(P_-P_-) = 0$$

In terms of current $J = \mathcal{A} + P + Q_1 + Q_2$

$$\text{EOM}: \quad \begin{aligned} \partial_+P_- + [\mathcal{A}_+, P_-] + [Q_{2+}, Q_{2-}] = 0\,, \\ \partial_-P_+ + [\mathcal{A}_-, P_+] + [Q_{1-}, Q_{1+}] = 0\,, \\ [P_+, Q_{1-}] = 0\,, \qquad [P_-, Q_{2+}] = 0\,. \end{aligned}$$

$$\text{Virasoro}: \quad {\rm STr}(P_+P_+) = 0\,, \qquad\qquad {\rm STr}(P_-P_-) = 0$$

$$\text{MC}: \quad \partial_-J_+ - \partial_+J_- + [J_-, J_+] = 0\,.$$

PR procedure: solve first EOM and Virasoro

κ-gauge condition: $Q_{1-} = 0\,, \qquad Q_{2+} = 0$

solves the last (fermionic) pair of EOM

remaining EOM:

$$\partial_+ P_- + [\mathcal{A}_+, P_-] = 0\,, \qquad \partial_- P_+ + [\mathcal{A}_-, P_+] = 0$$

Maurer-Cartan:

$$\begin{aligned}
\partial_+\mathcal{A}_- - \partial_-\mathcal{A}_+ + [\mathcal{A}_+, \mathcal{A}_-] + [P_+, P_-] + [Q_{1+}, Q_{2-}] &= 0\,,\\
\partial_- Q_{1+} + [\mathcal{A}_-, Q_{1+}] - [P_+, Q_{2-}] &= 0\,,\\
\partial_+ Q_{2-} + [\mathcal{A}_+, Q_{2-}] - [P_-, Q_{1+}] &= 0\,.
\end{aligned}$$

as in the bosonic F/G case can fix the "reduction gauge"

$$P_+ = \mu\, T\,, \qquad T = \frac{i}{2}\mathrm{diag}(1,1,-1,-1|1,1,-1,-1)$$

$$P_- = \mu\, g^{-1} T g\,, \quad \mathcal{A}_+ = g^{-1}\partial_+ g + g^{-1} A_+ g\,, \quad \mathcal{A}_- = A_-$$

T defines $\mathfrak{h}$ by $[\mathfrak{h}, T] = 0$:

$\mathfrak{h} = su(2) \oplus su(2) \oplus su(2) \oplus su(2)$

new parametrisation: $G = Sp(2,2) \times Sp(4)$-valued field g

and $\mathfrak{h}$-valued field $A_\pm$

MC eqs. become:

$$\begin{aligned}\partial_-(g^{-1}\partial_+ g + g^{-1}A_+ g) - \partial_+ A_- + [A_-, g^{-1}\partial_+ g + g^{-1}A_+ g] \\ = -\mu^2[g^{-1}Tg, T] + [Q_{1+}, Q_{2-}]\,, \\ \partial_- Q_{1+} + [A_-, Q_{1+}] = \mu[T, Q_{2-}]\,, \\ \partial_+ Q_{2-} + [g^{-1}\partial_+ g + g^{-1}A_+ g, Q_{2-}] = \mu[g^{-1}Tg, Q_{1+}]\end{aligned}$$

AdS_5 and S^5 sectors now coupled by fermions

remains residual κ-symmetry to be fixed

use T to generalise decomposition of bosonic part

$\mathfrak{f} = T \oplus \mathfrak{n} \oplus \mathfrak{h} \oplus \mathfrak{m}$ to superalgebra $psu(2,2|4)$

$$\widehat{\mathfrak{f}} = \widehat{\mathfrak{f}}^{\|} \oplus \widehat{\mathfrak{f}}^{\perp}\,, \qquad [T,[T,\widehat{\mathfrak{f}}^{\perp}]] = 0$$

define

$$\Psi_1 = Q_{1+}\,, \qquad \Psi_2 = gQ_{2-}g^{-1}$$

$\Psi_1^{\perp}, \Psi_2^{\perp}$ can be set =0 by residual κ-symmetry remaining fermionic components

$$\Psi_{_R} = \frac{1}{\sqrt{\mu}}\Psi_1^{\parallel}\,, \qquad\qquad \Psi_{_L} = \frac{1}{\sqrt{\mu}}\Psi_2^{\parallel}\,,$$

transform under $H \times H$ as $\Psi_{_R} \to \bar{h}^{-1}\Psi_{_R}\bar{h}\,,\ \Psi_{_L} \to h^{-1}\Psi_{_L}h\,.$
equations of motion of reduced theory are thus:

$$\begin{aligned}\partial_-(g^{-1}\partial_+ g + g^{-1}A_+ g) - \partial_+ A_- + [A_-, g^{-1}\partial_+ g + g^{-1}A_+ g]& \\ = \ -\mu^2[g^{-1}Tg, T] - \mu[g^{-1}\Psi_{_L} g, \Psi_{_R}]\,,&\end{aligned}$$

$$[T, D_-\Psi_{_R}] = -\mu(g^{-1}\Psi_{_L} g)^{\parallel}\,, \qquad [T, D_+\Psi_{_L}] = -\mu(g\Psi_{_R} g^{-1})^{\parallel}\,.$$

Lagrangian of PR theory for $AdS_5 \times S^5$ superstring

(Grigoriev, AT 07; related work: Mikhailov, Schafer-Nameki 07)

fermionic generalization of "gWZW+ potential" theory for

$$\frac{G}{H} = \frac{Sp(2,2)}{SU(2)\times SU(2)} \times \frac{Sp(4)}{SU(2)\times SU(2)}$$

$$\begin{aligned} L &= L_{\rm gWZW}(g, A_+, A_-) + \mu^2 \,{\rm STr}(g^{-1}TgT) \\ &+ {\rm STr}\left(\Psi_{_L}[T, D_+\Psi_{_L}] + \Psi_{_R}[T, D_-\Psi_{_R}]\right) \\ &+ \mu\,{\rm STr}\left(g^{-1}\Psi_{_L} g \Psi_{_R}\right) \end{aligned}$$

direct sum of PR theories for AdS_5 and S^5

"glued together" by components of fermions

$$\begin{aligned} L &= \widetilde{L}_{S^5}(g, A_+, A_-) + \widetilde{L}_{AdS_5}(g, A_+, A_-) \\ &+ \psi_{_L} D_+ \psi_{_L} + \psi_{_R} D_+ \psi_{_R} + \mu\ (\text{interaction terms}) \end{aligned}$$

all gauge symmetries fixed; standard kin. terms (cf. GS action)

The corresponding Lax pair encoding the equations of motion

$$\mathcal{L}_- = \partial_- + A_- + \ell^{-1}\sqrt{\mu} g^{-1}\Psi_{_L} g + \ell^{-2}\mu g^{-1} T g\,,$$
$$\mathcal{L}_+ = \partial_+ + g^{-1}\partial_+ g + g^{-1}A_+ g + \ell\sqrt{\mu}\Psi_{_R} + \ell^2 \mu T\,.$$

use that $[T,[T,\Psi_{_{L,R}}]] = -\Psi_{_{L,R}}$

Comments:

- gWZW model coupled to the fermions interacting minimally and through the "Yukawa term"
- 8 real bosonic and 16 real fermionic independent variables
- 2d Lorentz invariant with $\Psi_{_R}, \Psi_{_L}$ as 2d Majorana spinors
- 2d supersymmetry? yes, at the linearised level, and yes in $AdS_2 \times S^2$ case: $n = 2$ super sine-Gordon
- μ-dependent interaction terms are equal to original GS Lagrangian; gWZW produces MC eq.: path integral derivation via change from fields to currents?
- quadratic in fermions (like susy version of gWZW); integrating out $A_\pm$ gives quartic fermionic terms (reflecting curvature)
- linearisation of EOM in the gauge $A_\pm = 0$ around $g = \mathbf{1}$ describes 8+8 massive bosonic and fermionic d.o.f. with mass μ: same as in BMN limit
- symmetry of resulting relativistic S-matrix: $H = [SU(2)]^4$ – same as bosonic part of magnon S-matrix symmetry $[PSU(2|2)]^2$

Example: superstring on $AdS_2 \times S^2$

Explicit parametrisation:

$$T = \frac{1}{2}\begin{pmatrix} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}.$$

$$g = \begin{pmatrix} \cosh\phi & \sinh\phi & 0 & 0 \\ \sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & \cos\varphi & i\sin\varphi \\ 0 & 0 & i\sin\varphi & \cos\varphi \end{pmatrix}$$

$$\Psi_{_R} = \begin{pmatrix} 0 & 0 & 0 & i\gamma \\ 0 & 0 & -\beta & 0 \\ 0 & i\beta & 0 & 0 \\ \gamma & 0 & 0 & 0 \end{pmatrix}, \quad \Psi_{_L} = \begin{pmatrix} 0 & 0 & 0 & \rho \\ 0 & 0 & -i\nu & 0 \\ 0 & \nu & 0 & 0 \\ i\rho & 0 & 0 & 0 \end{pmatrix}$$

PR Lagrangian: same as $n=2$ supersymmetric sine-Gordon!

$$\widetilde{L} = \partial_+\varphi\partial_-\varphi + \partial_+\phi\partial_-\phi + \frac{\mu^2}{2}(\cos 2\varphi - \cosh 2\phi)$$
$$+ \beta\partial_-\beta + \gamma\partial_-\gamma + \nu\partial_+\nu + \rho\partial_+\rho$$
$$- 2\mu\left[\cosh\phi\ \cos\varphi\,(\beta\nu + \gamma\rho) + \sinh\phi\ \sin\varphi\,(\beta\rho - \gamma\nu)\right] .$$

indeed, equivalent to

$$\widetilde{L} = \partial_+\Phi\partial_-\Phi^* - |W'(\Phi)|^2$$
$$+\psi^*_{L}\partial_+\psi_{L} + \psi^*_{R}\partial_-\psi_{R} + \left[W''(\Phi)\psi_{L}\psi_{R} + W^{*\prime\prime}(\Phi^*)\psi^*_{L}\psi^*_{R}\right] .$$

bosonic part is of $AdS_2 \times S^2$ bosonic reduced model if

$$W(\Phi) = \mu\cos\Phi\ , \qquad |W'(\Phi)|^2 = \frac{\mu^2}{2}(\cosh 2\phi - \cos 2\varphi)\ .$$

$$\psi_{L} = \nu + i\rho\,, \qquad \psi_{R} = -\beta + i\gamma\ ,$$

Example: superstring on $AdS_3 \times S^3$

Green-Schwarz superstring on $AdS_3 \times S^3$
supported by RR 3-form flux: coset model

$$\frac{PSU(1,1|2) \times PSU(1,1|2)}{SU(2) \times SU(1,1)}$$

superalgebra $psu(1,1|2)$ admits a Z_4-grading
complexified algebra $\widehat{\mathfrak{f}}^{\mathbb{C}} = psl(2|2) \oplus psl(2|2)$

$$\begin{pmatrix} a & \alpha & 0 & 0 \\ \beta & b & 0 & 0 \\ 0 & 0 & c & \gamma \\ 0 & 0 & \delta & d \end{pmatrix}$$

a, c, b, d are 2×2 bosonic matrices from $sl(2)$; $\alpha, \beta, \gamma, \delta$ are complex fermionic matrices.

The antiautomorphism determining the Z_4 structure

$$M^{\Omega} = -\mathbf{K}^{-1} M^T \mathbf{K}\,, \qquad \mathbf{K} = \begin{pmatrix} 0 & K \\ K & 0 \end{pmatrix}, \qquad K = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & \mathbf{1} \end{pmatrix},$$

$$\begin{pmatrix} a & \alpha & 0 & 0 \\ \beta & b & 0 & 0 \\ 0 & 0 & c & \gamma \\ 0 & 0 & \delta & d \end{pmatrix}^{\Omega} = - \begin{pmatrix} c^t & -\delta^t & 0 & 0 \\ \gamma^t & d^t & 0 & 0 \\ 0 & 0 & a^t & -\beta^t \\ 0 & 0 & \alpha^t & b^t \end{pmatrix}$$

Z_4 components $\widehat{\mathfrak{f}}_l^{\mathbb{C}}$ are eigenspaces of Ω:

$$M^{\Omega} = i^k M \; , \quad M \in \widehat{\mathfrak{f}}_k^{\mathbb{C}} \; , \quad \widehat{\mathfrak{f}}^{\mathbb{C}} = \widehat{\mathfrak{f}}_0^{\mathbb{C}} \oplus \widehat{\mathfrak{f}}_1^{\mathbb{C}} \oplus \widehat{\mathfrak{f}}_2^{\mathbb{C}} \oplus \widehat{\mathfrak{f}}_3^{\mathbb{C}}$$

Ω induces the Z_4 decomposition of $\widehat{\mathfrak{f}} = psu(1,1|2) \oplus psu(1,1|2)$

$$\widehat{\mathfrak{f}} = \widehat{\mathfrak{f}}_0 \oplus \widehat{\mathfrak{f}}_1 \oplus \widehat{\mathfrak{f}}_2 \oplus \widehat{\mathfrak{f}}_3 \; , \qquad [\widehat{\mathfrak{f}}_i, \widehat{\mathfrak{f}}_j] \subset \widehat{\mathfrak{f}}_{i+j \bmod 4} \; .$$

GS Lagrangian: in terms of Z_4-components of $J_\pm = \widehat{f}^{-1} \partial_\pm \widehat{f}$

$$J_\pm = \mathcal{A}_\pm + P_\pm + Q_{1\pm} + Q_{2\pm} \; ,$$

$$\mathcal{A} \in \widehat{\mathfrak{f}}_0, \quad Q_1 \in \widehat{\mathfrak{f}}_1, \quad P \in \widehat{\mathfrak{f}}_2, \quad Q_2 \in \widehat{\mathfrak{f}}_3$$

$$L_{\mathrm{GS}} = \mathrm{STr}\big[P_+ P_- + \frac{1}{2}(Q_{1+} Q_{2-} - Q_{1-} Q_{2+})\big] \; ,$$

conformal gauge constraints: $\mathrm{STr}(P_+P_+)=0$ and $\mathrm{STr}(P_-P_-)=0$

κ-symmetry partially fixed by the gauge condition

$$Q_{1-}=0\,, \qquad\qquad Q_{2+}=0$$

explicit choice of $T\in\widehat{\mathfrak{f}}_0$

$$T=diag(t,t^T)\,, \qquad t=\frac{i}{2}diag(1,-1,1,-1)\,.$$

choice of T induces decomposition

$\widehat{\mathfrak{f}}=\widehat{\mathfrak{f}}^{\perp}\oplus\widehat{\mathfrak{f}}^{\parallel}$ in each $psu(1,1|2)$: $\widehat{\mathfrak{f}}=\widehat{\mathfrak{f}}^{\parallel}\oplus\widehat{\mathfrak{f}}^{\perp}$, $P^{\parallel}=-[T,[T,\cdot\,]]$

Reduced theory is a fermionic generalization of 2 copies of

$G/H=[SU(1,1)\times SU(2)]/[U(1)\times U(1)]$

Reduced theory Lagrangian

$$L_{tot}=L_{\mathrm{gWZW}}(g,A)+\mu^2\,\mathrm{STr}(g^{-1}TgT)$$
$$+\tfrac{1}{2}\mathrm{STr}\left(\Psi_1[T,D_+\Psi_1]+\Psi_2[T,D_-\Psi_2]\right)+\mu\,\mathrm{STr}\left(g^{-1}\Psi_1 g\Psi_2\right)\,,$$

$g \in SU(1,1) \times SU(2)$, $A_\pm \in u(1) \oplus u(1)$, Ψ_1, Ψ_2 related to $(\ )^\parallel$ parts of fermionic currents Q_{1+}, Q_{2-}

$$g = \begin{pmatrix} g_A & 0 \\ 0 & g_S \end{pmatrix}, \qquad \Psi_{1,2} = \begin{pmatrix} 0 & \psi_{1,2} \\ i\psi^\dagger_{1,2}\sigma_3 & 0 \end{pmatrix}.$$

$$g_A = \begin{pmatrix} e^{i\chi}\cosh\phi & \sinh\phi \\ \sinh\phi & e^{-i\chi}\cosh\phi \end{pmatrix}, \qquad g_S = \begin{pmatrix} e^{i\theta}\cos\varphi & \sin\varphi \\ -\sin\varphi & e^{-i\theta}\cos\varphi \end{pmatrix}$$

$$\psi_1 = \begin{pmatrix} 0 & \lambda + i\nu \\ \rho + i\sigma & 0 \end{pmatrix}, \qquad \psi_2 = \begin{pmatrix} 0 & \alpha + i\beta \\ \gamma + i\delta & 0 \end{pmatrix},$$

Solving for gauge fields $A_\pm$

$$L_{tot} = L_1 + L_2 + L_3 = L_B + \text{fermionic terms}$$

bosonic terms: direct sum of the CSG action and its "hyperbolic" counterpart – reduced bosonic string in $AdS_3 \times S^3$:

$$L_B = \partial_+\varphi\partial_-\varphi + \cot^2\varphi\, \partial_+\theta\partial_-\theta$$
$$+ \partial_+\phi\partial_-\phi + \coth^2\phi\, \partial_+\chi\partial_-\chi + \frac{\mu^2}{2}(\cos 2\varphi - \cosh 2\phi)$$

$$
\begin{aligned}
L_1 &= \partial_+\varphi\partial_-\varphi + \frac{1}{2}(1+\cos 2\varphi)\,\partial_+\theta\partial_-\theta \\
&+\partial_+\phi\partial_-\phi - \frac{1}{2}(1+\cosh 2\phi)\,\partial_+\chi\partial_-\chi + \frac{\mu^2}{2}(\cos 2\varphi - \cosh 2\phi)\ .
\end{aligned}
$$

$$
\begin{aligned}
L_2 &= \alpha\partial_-\alpha + \beta\partial_-\beta + \gamma\partial_-\gamma + \delta\partial_-\delta + \lambda\partial_+\lambda + \nu\partial_+\nu + \rho\partial_+\rho + \sigma\partial_+\sigma \\
&-2\mu\Big(\sinh\phi\sin\varphi(\lambda\beta - \nu\alpha + \rho\delta - \sigma\gamma) + \cosh\phi\cos\varphi\Big[\cos(\chi+\theta)(\sigma\alpha - \rho\beta \\
&+\lambda\delta - \nu\gamma) + \sin(\chi+\theta)(\rho\alpha + \sigma\beta - \lambda\gamma - \nu\delta)\Big]\Big)
\end{aligned}
$$

$$
\begin{aligned}
L_3 &= \frac{[\partial_+\chi\,(1+\cosh 2\phi)\ - 2(\alpha\beta - \gamma\delta)][\partial_-\chi\,(1+\cosh 2\phi)\ + 2(\lambda\nu - \rho\sigma)]}{2(\cosh 2\phi - 1)} \\
&+ \frac{[\partial_+\theta\,(1+\cos 2\varphi)\ + 2(\alpha\beta - \gamma\delta)][\partial_-\theta\,(1+\cos 2\varphi)\ - 2(\lambda\nu - \rho\sigma)]}{2(1-\cos 2\varphi)}\ .
\end{aligned}
$$

identify the fermions $\alpha, \beta, \gamma, \delta$ and $\lambda, \nu, \rho, \sigma$ with 2d MW spinors –
2d supersymmetry ?

Yes for consistent truncation $\chi = \theta = 0,\ \beta = \delta = \lambda = \rho = 0 \rightarrow$
reduced Lagrangian for the $AdS_2 \times S^2$ superstring

$$L = \partial_+\varphi\partial_-\varphi + \partial_+\phi\partial_-\phi + \frac{\mu^2}{2}(\cos 2\varphi - \cosh 2\phi) + \alpha\partial_-\alpha + \gamma\partial_-\gamma$$
$$+ \nu\partial_+\nu + \sigma\partial_+\sigma - 2\mu\big[\cosh\phi\cos\varphi(\gamma\nu - \alpha\sigma) + \sinh\phi\sin\varphi(\gamma\sigma + \alpha\nu)\big]$$

equivalent to $N = 2$ super SG:

$$L = \partial_+\Phi\partial_-\Phi^* - |W'(\Phi)|^2 + \psi^*_{_L}\partial_+\psi_{_L} + \psi^*_{_R}\partial_-\psi_{_R}$$
$$+ \left[W''(\Phi)\psi_{_L}\psi_{_R} + W^{*\prime\prime}(\Phi^*)\psi^*_{_L}\psi^*_{_R}\right],$$

$\Phi = \varphi + i\phi, \psi_{_L} = \nu + i\sigma,\ \psi_{_R} = -\gamma - i\alpha, W = \mu\cos\Phi.$

CSG model and its “hyperbolic” analog each
admit $N = 2$ supersymmetric extensions:
interpret $\xi \equiv \ln\cos\varphi + i\theta$ and $\eta \equiv \ln\cosh\phi + i\chi$ as
complex scalar components of chiral superfields
using that $d\varphi^2 + \cot^2\varphi\, d\theta^2 = \frac{\partial^2 K}{\partial\xi\partial\bar\xi} d\xi d\bar\xi$,
$d\phi^2 + \coth^2\phi\, d\chi^2 = \frac{\partial^2 K'}{\partial\eta\partial\bar\eta} d\eta d\bar\eta$:

K and K' are then Kahler potentials and
μe^{ξ} and μe^{η} as superpotentials
But resulting $N = 2$ supersymmetric Lagrangian
is direct sum of two decoupled $N = 2$ theories –
not equivalent to L_{tot}
(e.g., does not admit the $N = 2$ SG truncation)
2d susy of L_{tot} remains open question...

Open questions

- Quantum equivalence of reduced theory and GS theory?

  Check of UV finiteness? Yes in $AdS_2 \times S^2$. In $AdS_3 \times S^3$?

- Path integral argument of equivalence?

  Potential term is original action

  $\mathrm{Tr}(P_+ P_-) = \mu^2 \mathrm{Tr}(T g^{-1} T g)$

  while gWZW should come from change of variables.

  Rough idea: string in $R_t \times F/G$ coset

  $L = -(\partial t)^2 + \mathrm{Tr}(f^{-1} df + B)^2 \ , \quad f \in F, \ \ B \in \mathfrak{g}$

  string path integral in conformal+ $t = \mu\tau$ gauge:

  $$\int Df DB \, \delta(T_{++} - \mu^2) \, \delta(T_{--} - \mu^2) \ \ e^{iI(f,B)}$$

  then replace $f^{-1} df$ by C

  $$\int DC DB Dv \, \delta(T_{++} - \mu^2) \delta(T_{--} - \mu^2) \exp[i \int (C+B)^2 + v(dC + C \wedge C)]$$

set $(C+B)_+ = \mu T$, $(C+B)_- = \mu g^{-1} T g$; change from C, B, v to $g \in G, A \in \mathfrak{h}$: $[\mathfrak{h}, T] = 0$

Transformation may work only in genuine quantum-conformal ($AdS_n \times S^n$) case.

- Indication of equivalence: semiclassical expansion

  near analog of (S, J) rigid string in $AdS_5 \times S^5$ leads to the same characteristic frequencies – same 1-loop partition function (Roiban, AT 08, to appear)

- Tree-level S-matrix for elementary excitations?

  Manifest $SU(2) \times SU(2) \times SU(2) \times SU(2)$ symmetry?

  Hidden bigger symmetry? Relation to magnon S-matrix in BA?

- better understanding the relationship between the original and the reduced system: symmetries, vacua, values of conserved charges, etc.; which observables can be related?

Conclusion

Pohlmeyer reduction seems most promising approach
towards solution of $AdS_5 \times S^5$ GS superstring
Uncovers remarkable connection to a fermionic
(2d supersymmetric? UV finite?)
integrable deformation of a gWZW model
solvable by Bethe Ansatz?
same BA as on gauge theory side ?
appears to be very likely...

Some additional remarks

Lax pair for a coset model:

found from the zero curvature condition $d\omega + \omega \wedge \omega = 0$ for Lax connection

$$\omega = d\sigma^+(\mathcal{A}_+ + \ell P_+) + d\sigma^-(\mathcal{A}_- + \ell^{-1} P_-)\,,$$

$$[\partial_+ + \mathcal{A}_+ + \ell P_+, \partial_- + \mathcal{A}_- + \ell^{-1} P_-] = 0\,,$$

ℓ is a spectral parameter. The equations of motion follow as the coefficients of order ℓ^{-1} and ℓ terms. The coefficient of the order ℓ^0 term is the $\mathfrak{g}$-component of the zero curvature condition for the connection $J = \mathcal{A} + P$.

$$M(\ell) = P\exp\int_{(-\infty,t)}^{(\infty,t)} \omega(\ell)$$

conserved charges – coefficients of expansion in ℓ

Matrix superalgebras

The algebra $Mat(n|l;\Lambda)$ is that of $(n+l)\times(n+l)$ matrices over Λ whose diagonal block entries are even elements of Grassmann algebra Λ while off-diagonal block entries are odd. The super-transposition st is defined as follows:

$$\begin{pmatrix} A & X \\ Y & B \end{pmatrix}^{st} = \begin{pmatrix} A^t & -Y^t \\ X^t & B^t \end{pmatrix}, \qquad (MN)^{st} = N^{st}M^{st}\,.$$

$$(M^{st})^{st} = WMW\,, \qquad W = \mathrm{diag}(1,\ldots,1,-1,\ldots,-1)$$

A real form of a complex matrix Lie (super)algebra: antilinear anti-automorphism $*$

$$(MN)^* = M^*N^*\,, \qquad (M^*)^* = M\,, \qquad (aM)^* = \bar{a}M^*\,,\;\; a\in\mathbb{C}$$

The real subspace of elements satisfying $M^* = -M$ is then a real Lie superalgebra.

The case of $n=l$, i.e. $Mat(n|n,\Lambda)$. define $*$ on arbitrary super-matrices according to $M^* = \mathbf{\Sigma}^{-1} M^\dagger \mathbf{\Sigma}$

$$\mathbf{\Sigma} = \begin{pmatrix} \Sigma & 0 \\ 0 & \mathbf{1} \end{pmatrix}, \qquad \begin{pmatrix} A & X \\ Y & B \end{pmatrix}^\dagger = \begin{pmatrix} A^\dagger & -iY^\dagger \\ -iX^\dagger & B^\dagger \end{pmatrix}$$

$\Sigma^2 = \mathbf{1}$ and $\Sigma^\dagger = \Sigma$. Note that $(MN)^\dagger = N^\dagger M^\dagger$ and $(M^\dagger)^\dagger = M$.
$(M^\dagger)^{st} = W(M^{st})^\dagger W$

To define Z_4 anti-automorphism consider

$$\begin{pmatrix} A & X \\ Y & B \end{pmatrix}^\Omega = -\begin{pmatrix} K^{-1}A^tK & -K^{-1}Y^tK \\ K^{-1}X^tK & K^{-1}B^tK \end{pmatrix}$$

where $K^2 = \pm\mathbf{1}$ and $K^t = \pm K^{-1}$.

$$M^\Omega = -\mathbf{K}^{-1} M^{st} \mathbf{K}, \qquad \mathbf{K} = \begin{pmatrix} K & 0 \\ 0 & K \end{pmatrix}, \quad (MN)^\Omega = -N^\Omega M^\Omega$$

explicit form in the case of $psu(2,2|4)$

$$\Sigma = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad K = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

$\mathfrak{f}^{\mathbb{C}}$ admits Z_4 grading if can be decomposed

$$\mathfrak{f}^{\mathbb{C}} = \mathfrak{f}_0^{\mathbb{C}} \oplus \mathfrak{f}_1^{\mathbb{C}} \oplus \mathfrak{f}_2^{\mathbb{C}} \oplus \mathfrak{f}_3^{\mathbb{C}}$$

where $\mathfrak{f}_m^{\mathbb{C}}$ denotes the eigenspace with eigenvalue i^m

$$M^\Omega = i^m M\,, \quad ([M,N])^\Omega = i^{m+n}[M,N]\,, \qquad M \in \mathfrak{f}_m^{\mathbb{C}}, \quad N \in \mathfrak{f}_n^{\mathbb{C}}$$

Ω is compatible with the reality condition $(M^*)^\Omega = i^m M^*$

Superalgebra $\mathrm{psu}(2,2|4)$

$\mathrm{su}(2,2|4)$ is spanned by 8×8 matrices M: in terms of 4×4 blocks

$$M = \begin{pmatrix} A & X \\ Y & D \end{pmatrix}$$

required to have vanishing supertrace $\mathrm{str}M = \mathrm{tr}A - \mathrm{tr}D = 0$ and to satisfy the following reality condition

$$HM + M^\dagger H = 0\,, \quad H = \begin{pmatrix} \Sigma & 0 \\ 0 & -\mathbb{I} \end{pmatrix}\,, \quad \Sigma = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

A and D span the subalgebras $\mathrm{u}(2,2)$ and $\mathrm{u}(4)$, while $Y = X^\dagger\Sigma$

$\mathrm{su}(2,2|4)$ also contains the $\mathrm{u}(1)$ generator $i\mathbb{I}$

the bosonic subalgebra of $\mathrm{su}(2,2|4)$ is

$\mathrm{su}(2,2) \oplus \mathrm{su}(4) \oplus \mathrm{u}(1)$

$\mathrm{psu}(2,2|4)$ is defined as the quotient algebra of $\mathrm{su}(2,2|4)$

over this $\mathrm{u}(1)$ factor

su(2, 2|4) has $\mathbb{Z}_4$ grading

$$M = M^{(0)} \oplus M^{(1)} \oplus M^{(2)} \oplus M^{(3)}$$

defined by the automorphism $M \to \Omega(M)$

$$\Omega(M) = \begin{pmatrix} KA^tK & -KY^tK \\ KX^tK & KD^tK \end{pmatrix},$$

$$K = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

$M^{(0)}$ is the $\mathrm{so}(4,1) \times \mathrm{so}(5)$ subalgebra

$M^{(2)}$ is the $AdS_5 \times S^5$ coset

$M^{(1)}, M^{(3)}$ contain odd fermionic variables

Dirac matrices for SO(4,1) and SO(5) γ_a and Γ_a, $a = 1, \ldots, 5$

$$K\gamma_a^t K = -\gamma_a\,, \qquad K\Gamma_a^t K = -\Gamma_a$$

span the orthogonal complements to so(4,1) and so(5)

Coset Representative (Arutyunov et al 05)

$$g = g(\theta,\eta)g(x,y)$$

$g(x,y)$ describes an embedding of AdS into SU(2,2) $\times$ SU(4)

$$g(x,y) = \underbrace{\exp\frac{1}{2}(x_a\gamma_a)}_{g(x)}\underbrace{\exp\frac{i}{2}(y_a\Gamma_a)}_{g(y)}$$

x_a parametrize the AdS_5 space while y_a – S^5

$g(x,y)$ is 8 by 8 block-diagonal matrix:

upper 4 by 4 block $g(x)$, and lower block $g(y)$

$g(\theta,\eta)$ incorporates the original 32 fermionic degrees of freedom

$$g(\theta,\eta) = \exp\begin{pmatrix} 0 & 0 & 0 & 0 & \eta^5 & \eta^6 & \eta^7 & \eta^8 \\ 0 & 0 & 0 & 0 & \eta^1 & \eta^2 & \eta^3 & \eta^4 \\ 0 & 0 & 0 & 0 & \theta^1 & \theta^2 & \theta^3 & \theta^4 \\ 0 & 0 & 0 & 0 & \theta^5 & \theta^6 & \theta^7 & \theta^8 \\ \eta_5 & \eta_1 & -\theta_1 & -\theta_5 & 0 & 0 & 0 & 0 \\ \eta_6 & \eta_2 & -\theta_2 & -\theta_6 & 0 & 0 & 0 & 0 \\ \eta_7 & \eta_3 & -\theta_3 & -\theta_7 & 0 & 0 & 0 & 0 \\ \eta_8 & \eta_4 & -\theta_4 & -\theta_8 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Here θ^i and η^i are $8+8$ complex fermions: $\theta^{i\,*} = \theta_i$, $\eta^{i\,*} = \eta_i$

alternative: $g = \mathrm{diag}(g_a, g_s)$

in terms of 6+6 embedding coordinates of AdS_5 and S^5 in $\mathbb{R}^{4,2}$ and $\mathbb{R}^6$

$$
\begin{aligned}
g_a(v) &= \begin{pmatrix} 0 & -iv_5 - v_6 & v_1 - iv_4 & -iv_2 - v_3 \\ iv_5 + v_6 & 0 & -iv_2 + v_3 & v_1 + iv_4 \\ -v_1 + iv_4 & iv_2 - v_3 & 0 & iv_5 - v_6 \\ iv_2 + v_3 & -v_1 - iv_4 & -iv_5 + v_6 & 0 \end{pmatrix} \\
g_s(u) &= \begin{pmatrix} 0 & -iu_5 - u_6 & -iu_1 - u_4 & -u_2 + iu_3 \\ iu_5 + u_6 & 0 & -u_2 - iu_3 & -iu_1 + u_4 \\ iu_1 + u_4 & u_2 + iu_3 & 0 & iu_5 - u_6 \\ u_2 - iu_3 & iu_1 - u_4 & -iu_5 + u_6 & 0 \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
v_1^2 + v_2^2 + v_3^2 + v_4^2 - v_5^2 - v_6^2 = -1 \\
u_1^2 + u_2^2 + u_3^2 + u_2^2 + u_5^2 + u_6^2 = 1
\end{aligned}
$$

so $g_a(v)$ and $g_s(u)$ belong to SU(2,2) and SU(4) respectively:

on (u, v) conformal and $SO(6)$ transformations act linearly