Lattice QCD at finite temperature and density

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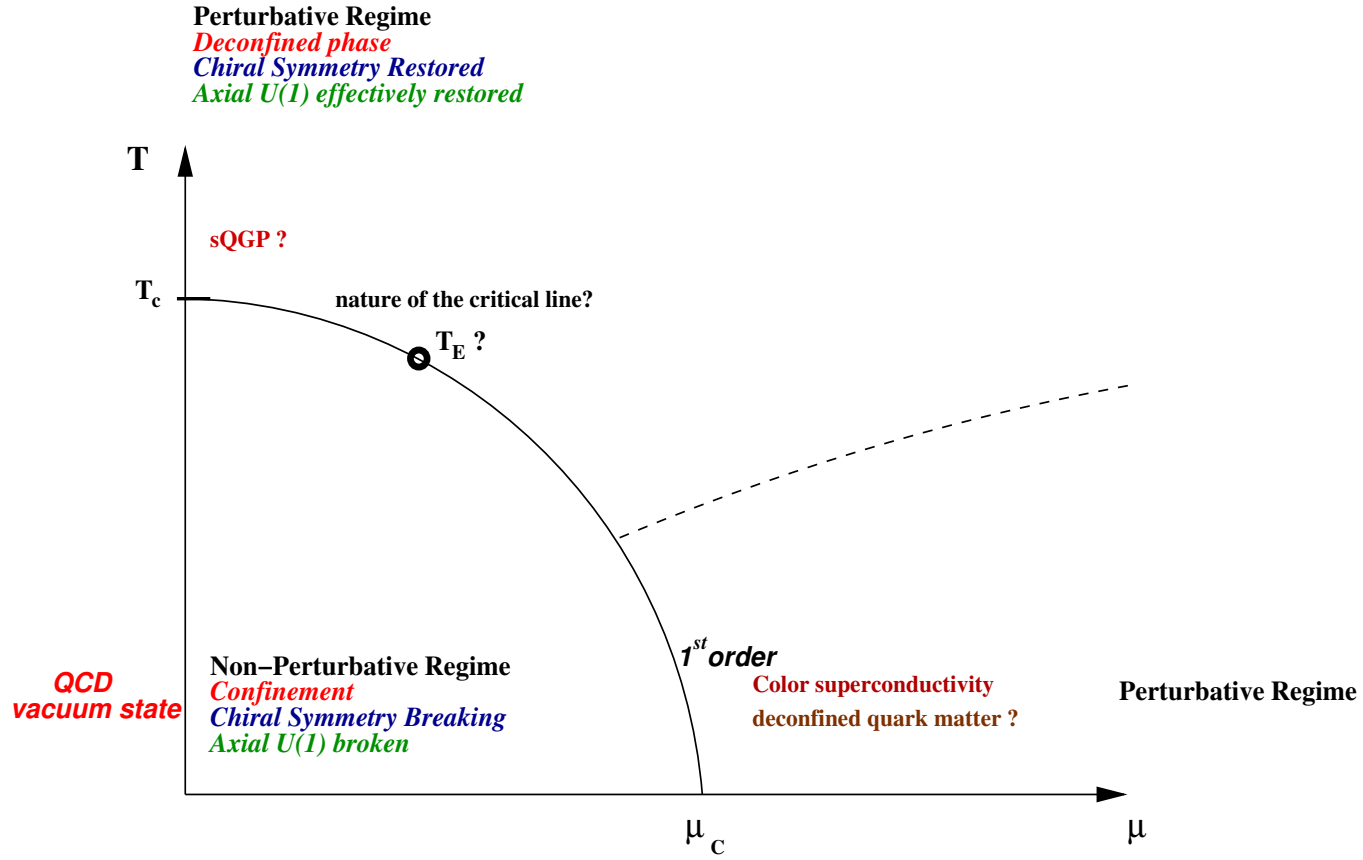
1 – OUTLINE

- QCD phenomenology at finite temperature and baryon density, lattice simulations and the sign problem.

- Existing methods to partially circumvent the sign problem.

- Some lattice results in QCD and QCD-like theories.

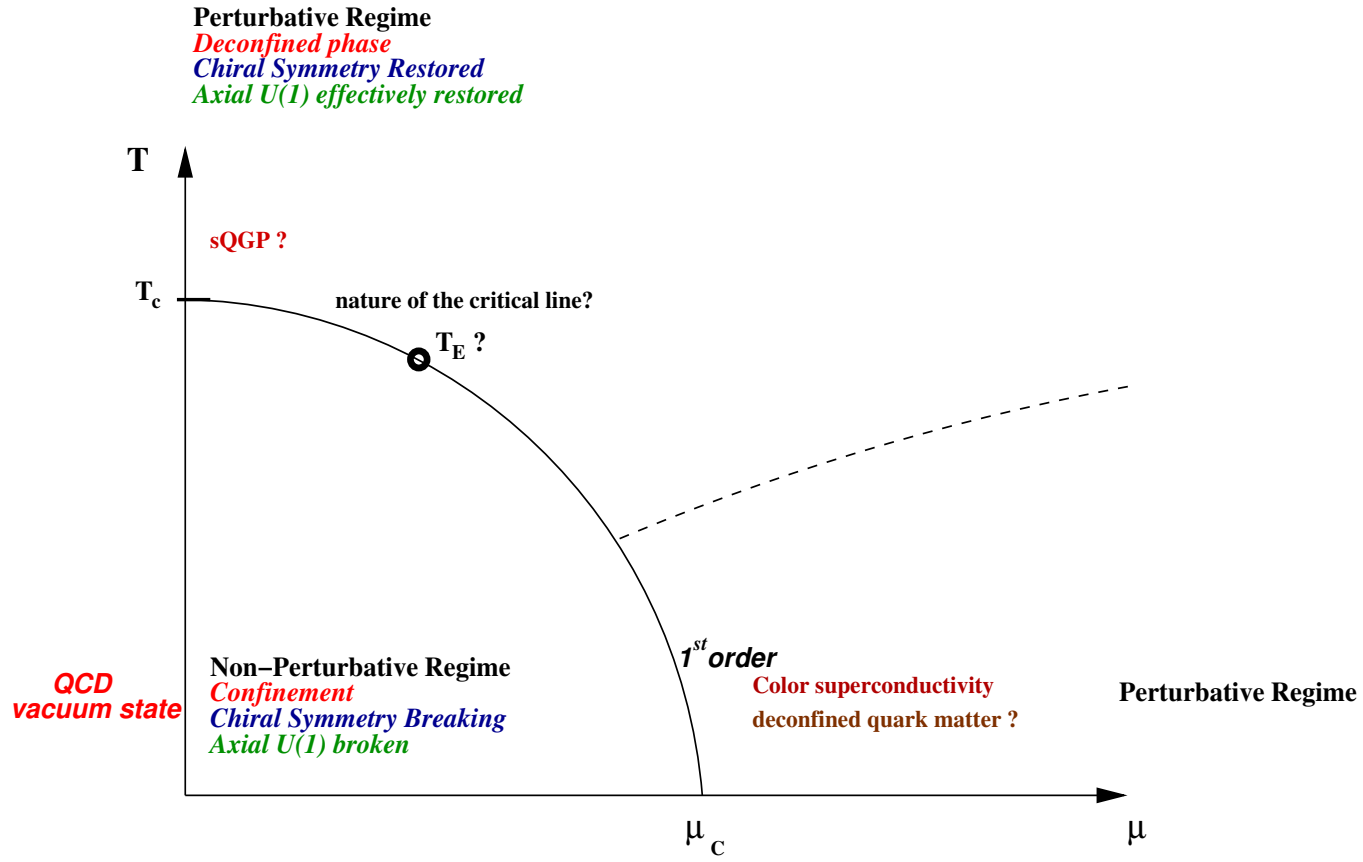
QCD at finite density and the sign problem



The study of QCD at finite temperature and baryon density is relevant to fundamental phenomenological issues: 1) experimental search for the deconfinement transition in heavy ion collisions; 2) properties of compact astrophysical objects.

Several questions need to be clarified: location and nature of the transition lines; properties of strongly interacting matter, specially close to the transition.

QCD at finite density and the sign problem



As a matter of fact, our knowledge of the QCD phase diagram and of the dynamics of strongly interacting matter at finite temperature and baryon density is still partial.

Lattice QCD simulations, which are the best non-perturbative computational tool based only on QCD first principles, are hindered at finite baryonic density by the sign problem.

2 – QCD at finite temperature and density

The finite temperature QCD partition function, $Z(V,T) = \mathrm{Tr}\left(e^{-\frac{H_{\mathrm{QCD}}}{T}}\right)$**, can be written as a functional integral over euclidean space-time with finite temporal extent** $\tau = 1/T$

$$Z = \int \mathcal{D}A\mathcal{D}\psi\mathcal{D}\bar{\psi}e^{-\int_0^{1/T} dt \int d^3x \mathcal{L}_{\mathrm{QCD}}}$$

$$\mathcal{L}_{\mathrm{QCD}} = \frac{1}{4g^2}G^a_{\mu\nu}G^a_{\mu\nu} + \bar{\psi}\left(\gamma^E_\mu(\partial_\mu + iA^a_\mu T^a) + m\right)\psi$$

Periodic boundary condition in time direction are taken for gauge fields, antiperiodic for fermionic fields.

The partition function can be written on a discretized space-time (lattice):

$$Z = \int \mathcal{D}U\mathcal{D}\psi\mathcal{D}\bar{\psi}e^{-(S_G+S_F)} = \int \mathcal{D}U e^{-S_G} \det M[U]$$

$U_\mu(x) = \mathcal{P}\exp\left(i\int_x^{x+\hat{\mu}} dy_\nu A_\nu(y)\right)$ **is the gauge link variable**

$S_G = \beta\sum_{x,\mu<\nu}\left(1 - \frac{1}{N_c}\mathrm{Tr}\Pi_{\mu\nu}(x)\right)$ **is the pure gauge action,** $\beta = \frac{2N_c}{g^2}$

$\Pi_{\mu\nu}(x) = U_\mu(x)U_\nu(x+\hat{\mu})U_\mu^\dagger(x+\hat{\nu})U_\nu^\dagger(x)$ **is the plaquette variable**

$S_F = \bar{\psi}_i M_{ij}\psi_j$ **is the fermionic action with** M **the fermionic matrix, naïvely**

$S_f = \frac{1}{2}\sum_{x,\mu}\bar{\psi}(x)\gamma_\mu^E\left[U_\mu(x)\psi(x+\hat{\mu}) - U_\mu^\dagger(x-\hat{\mu})\psi(x-\hat{\mu})\right] + \sum_x m\bar{\psi}(x)\psi(x)$

Dynamical fermion contributions are encoded in the fermion determinant $\det M[U]$ **which appears after integration of fermion variables.**

$$T = \frac{1}{\tau} = \frac{1}{N_t a(\beta, m)}$$

$a \to 0$ **as** $\beta \to \infty$**, therefore** T **is a monotonic increasing function of** β**.**

The thermal expectation value of a generic operator O is written as

$$\langle O \rangle = \frac{\int DU \, \det M[U] \, e^{-S_g[U]} \, O[U]}{\int DU \, \det M[U] \, e^{-S_g[U]}} \tag{1}$$

if $\det M[U] \, e^{-S_g[U]} > 0$ this has a probabilistic interpretation and Monte Carlo methods can be applied to numerically determine it: only few gauge field configurations give sensible contribution to the functional integral and one looks for a good algorithm to sample them (importance sampling).

The reality of the fermion determinant is guaranteed by γ_5 hermiticity

$$\gamma_5 M^\dagger \gamma_5 = M \implies (\det M)^* = \det M$$

A finite baryonic density can be introduced by adding a finite chemical potential

$$Z(\mu) = \mathrm{Tr}\left(e^{-\frac{H_{\mathrm{QCD}}-\mu N}{T}}\right)$$

where $N = \int d^3x\psi^\dagger\psi = \int d^3x\bar{\psi}\gamma_0\psi$ is the quark (baryonic) number operator. In the euclidean path integral formulation the fermionic part of $\mathcal{L}_{\mathrm{QCD}}$ is modified as follows:

$$\bar{\psi}\left(\gamma_\mu(\partial_\mu + iA_\mu) + m\right)\psi \rightarrow \bar{\psi}\left(\gamma_\mu(\partial_\mu + iA_\mu) + m + \mu\gamma_0\right)\psi$$

The added baryon chemical potential can be viewed as the temporal component of a constant $U(1)$ imaginary background field. This is also the way it is usually implemented on the lattice to avoid perturbative divergences (P. Hasenfratz F. Karsch, Phys. Lett. B125 (1983) 308 J.B. Kogut et al., Nucl. Phys. B225 (1983) 93 R. V. Gavai Phys. Rev. D32 (1985) 519):

$$D_t(n,m) = U_t(n)\delta_{m,n+\hat{t}} - U_t^\dagger(n-\hat{t})\delta_{m,n-\hat{t}} \rightarrow e^{a\mu}U_t(n)\delta_{m,n+\hat{t}} - e^{-a\mu}U_t^\dagger(n-\hat{t})\delta_{m,n-\hat{t}}$$

Where U_t is the temporal link (elementary parallel transport)

In this way the hermiticity properties of the fermion matrix in general are lost and the residual surviving symmetry is

$$(\det M(\mu))^* = \det M(-\mu)$$

$\det M$ **is in general complex** $\Longrightarrow$ **Monte Carlo simulations are unfeasibile.**

This is usually known as the sign problem. It is a problem of technical nature but is strictly related to the fact that we want to create a net unbalance between particles and antiparticles, i.e. suppress antiquark and enhance quark propagation.The problem is known also in other contexts dealing with fermionic systems.

Consider for instance the expectation value of the Polyakov loop and of its hermitian conjugate:

- **It is known that $L = \langle \mathrm{Tr} P \rangle$ and $\bar{L} = \langle \mathrm{Tr} P^\dagger \rangle$ are both real at real chemical potential, but $L \neq \bar{L}$, and in particular $L(\mu) = \bar{L}(-\mu)$ (see e.g. F. Karsch and H. W. Wyld, 1985). This is indeed what we expect since L and $\bar{L}$ describe propagation of a static quark (antiquark) respectively, so their values must be different in presence of a baryon chemical potential.**

- **However, configuration by configuration $\mathrm{Tr} P$ and $\mathrm{Tr} P^\dagger$ are always the complex conjugate of each other, the fact that $\mathrm{Re}\langle \mathrm{Tr} P \rangle_\mu \neq \mathrm{Re}\langle \mathrm{Tr} P^\dagger \rangle_\mu$ is strictly related to the complex nature of the "measure" which is meant $\langle \cdot \rangle_\mu$ A complex measure is necessary in order to have $L \neq \bar{L}$.**

Notice that the partition function is, of course, still real: different complex contributions cancels out. It could be possible in principle to find a smart change of variables to perform the sum avoiding the sign problem. If it exists, it has not yet been found, so we have to deal with partial solutions to the problem.

3 – Trying to evade the sign problem

The importance of studying the QCD phase diagram from first principles lattice gauge theory simulations has been worth a lot of efforts in trying to evade the sign problem.

Most methods try to extract information from simulations where the sign problem is absent (like at $\mu = 0$) and some results have been obtained in the last few years, mostly in the region of small μ and high T, which is the one relevant for heavy ion collisions.
As an alternative one can study sign problem free theories, like two color QCD (real sign) or QCD in presence of a finite isospin density.

We will review these methods, then focussing on results obtained via the imaginary chemical potential approach and on results regarding the theory within two-color QCD.

Theories without the sign problem

QCD at finite isospin density

$(\det M(\mu))^* = \det M(-\mu)$, **that means that a theory with a finite isospin density** μ_{is} **(i.e.** $\mu_u = -\mu_d = \mu_{is}/2$**) has no sign problem (assuming** $m_u = m_d$**)**

$$\det M(\mu) \det M(-\mu) = \det M(\mu)(\det M(\mu))^* = |\det M(\mu)|^2 > 0$$

This is like ignoring the determinant phase, $\det M(\mu) = |\det M(\mu)|e^{i\theta}$ **(phase quenched QCD).**

QCD with two colours

$N_c = 2$ **is a special case of gauge theory: the fundamental representation for quarks is real (**$\sigma_2 U_\mu \sigma_2 = U_\mu^*$**).**

$\Longrightarrow$ **Traces over closed loops are always real**

$\Longrightarrow$ $\det M[U]$**, being expressible in terms of traces over closed loops, is real.**

Such theories are interesting laboratories for exploring some properties or methods, however one should always be aware that they are different physical systems

(mesons $\sim$ **baryons in 2-color QCD, meson matter instead of baryon matter in finite isospin QCD)**

Reweighting

Gauge configurations sampled at $\mu = 0$ can in principle be used to obtain expectation values at $\mu > 0$, using the following identity:

$$\langle O \rangle = \frac{\int DU\, e^{-S_g[U]}\, \det M(\mu)\, O[U]}{\int DU\, e^{-S_g[U]}\, \det M(\mu)} = \frac{\int DU\, e^{-S_g[U]}\, \det M(0) \frac{\det M(\mu)}{\det M(0)}\, O[U]}{\int DU\, e^{-S_g[U]}\, \det M(0)\, \frac{\det M(\mu)}{\det M(0)}} = \frac{\left\langle \frac{\det(M(\mu))}{\det(M(0))} O \right\rangle_{\mu=0}}{\left\langle \frac{\det(M(\mu))}{\det(M(0))} \right\rangle_{\mu=0}}$$

The method, proposed in the 80's by the Glasgow group, fails. Major problems are:

- **Bad sampling: configurations sampled ad $\mu = 0$ give poor sampling of the integral at $\mu \neq 0$. The two statistical ensembles at $\mu \neq 0$ and $\mu = 0$, being related to different physical situations, may have very poor overlap. The problem worsens as volume increases.**
- **Large errors coming from the oscillating factor $\frac{\det M(\mu)}{\det M(0)}$ (which is also not so easily computable): $\langle \det M(\mu) / \det M(0) \rangle \sim 0$ at large μ's, the statistics required for a given accuracy grow exponentially with the volume**

Multiparameter reweighting

(Z. Fodor and S.D. Katz, 2002 . . .)

An improvement consists in reweighting both in β and μ instead that only in μ.

$$Z(\mu,\beta) = \int \mathcal{D}U e^{-S_g(\beta_0)} \det M(0) \left\{ e^{-S_g(\beta)+S_g(\beta_0)} \frac{\det M(\mu)}{\det M(0)} \right\},$$

Configurations sampled at a transition point used to reconstruct the integral at a different transition point: overlap is greatly enhanced.

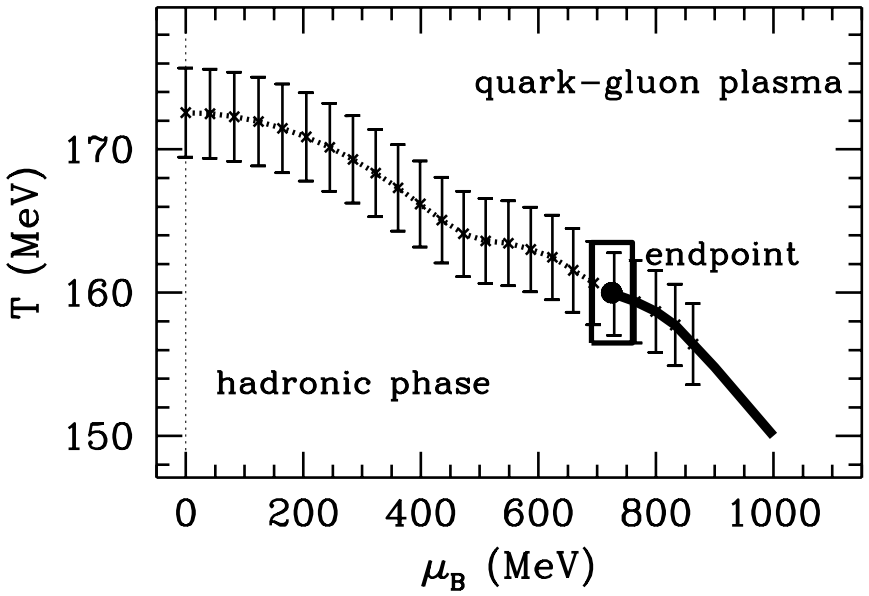
Determination of the critical endpoint based on a Lee-Yang zeroes analysis:

from Z. Fodor, S. D. Katz, JHEP 0203 (2002) 014

Caveat:

Splittorff (2005) about a possible failure of reweighting because of severe sign problem;

Ejiri (2005) about the validity of the Lee-Yang zeroes analysis in presence of a complex factor. Small volumes used and thermodynamical limit hardly feasible: overlap vanishes anyway.



Taylor expansion

(Bielefeld-Swansea Collaboration 2002 . . . ; Gavai and Gupta 2003 . . .)

For small densities, physical quantities can be naturally expanded as a Taylor series in μ around $\mu = 0$. The coefficients of the series can obtained as expectation values of local operators at $\mu = 0$. For instance consider the quark number $n(\mu)$ expanded for small values of μ:

$$n \equiv T \frac{\partial}{\partial \mu} \ln Z(T, \mu) \; ; \quad n(\mu) = n_1 \, \mu + n_3 \, \mu^3 + O(\mu^5)$$

the first coefficient n_1 is the quark number susceptibility computed at $\mu = 0$:

$$n_1 = T \frac{\partial^2}{\partial \mu^2} \ln Z(T, \mu)|_{\mu=0}$$

No particular limitation in approaching the thermodynamical limit, but the method is naturally well suited only for the region of small μ and high T.

Density of states method

Partition function rewritten in terms of the density of states for some quantity (plaquette, quark number, . . .). sign problem might be better under control.

Bhanot, Bitar, Salvador (1987), Karliner, Sharpe, Chang (1988), Azcoiti, Di Carlo, Grillo (1990), Luo (2001), Ambjorn *et al* (2002), Fodor, Katz, Schmidt (2006)

Heavy quark expansions

sign problem still present, but numerical simulations and the computation of the fermion determinant is much easier, allowing high precision reweighting.

T. C. Blum, J. E. Hetrick, D. Toussaint, 1996; J. Engels, O. Kaczmarek, F. Karsch, E. Laermann, 1999; R. De Pietri, A. Feo, E. Seiler, I. O. Stamatescu, 2006.

Imaginary chemical potential

Consider the partition function $Z(i\mu_I)$ defined by an imaginary chemical potential

$$U_t \to e^{ia\mu_I} U_t \qquad\qquad U_{-t} \to e^{-ia\mu_I} U_{-t} = (e^{ia\mu_I} U_t)^\dagger$$

this is like adding a constant and real $U(1)$ background field. $\det M[U] > 0 \implies$ Monte Carlo simulations are feasible. What can we learn from imaginary $\mu's$?

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Reconstruction of the canonical partition function

$Z(i\mu_I)$ can be used to reconstruct the canonical partition function $Z(n)$ at fixed quark number n (Roberge, Weiss, 1986) ($\theta = \mu_I/T$)

$$Z(n) = \mathrm{Tr}\left((e^{-\frac{H_{\mathrm{QCD}}}{T}}\delta(N-n)\right) = \frac{1}{2\pi}\mathrm{Tr}\left(e^{-\frac{H_{\mathrm{QCD}}}{T}}\int_0^{2\pi} \mathrm{d}\theta e^{i\theta(N-n)}\right) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\theta e^{-i\theta n} Z(i\theta T)$$

A. Hasenfratz, D. Toussaint, 1990; Alford *et al.*, 1992 (2-d Hubbard model); de Forcrand, Kratochvila, 2004, 2006; A. Alexandru *et al.*, 2005 Thermodynamical limit not easily reachable.

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Analytic continuation to real μ

Away from critical points $Z(T,\mu)$ is a regular function of μ^2. Results at μ_I ($\mu^2 < 0$) can be used to fit the expected dependence continued from real $\mu' s$ ($\mu^2 > 0$).

M.P. Lombardo, 2000, A. Hart *et al*, 2000, de Forcrand, Philipsen, 2002, D'E., Lombardo, 2003

see also V. Azcoiti *et al.*, 2004 for a more general formulation

The method of analytic continuation from an imaginary chemical potential has some similarities and advantages when compared with other methods:

- **compared with reweighting, it does not suffer much from increasing the spatial volume $\longrightarrow$ easier thermodynamical limit**

- **it is very similar in spirit to Taylor expansion: analyticity around $\mu^2 = 0$ is assumed. But it gathers information from a wider region than just at $\mu = 0$**

It must be used with some care:

- **the phase structure of the theory in the whole $T - \mu^2$ must be known to understand where analytic continuation can be applied**

- **special care must be paid to the choice of the best interpolating functions (physical intuition and models may serve as a guidance) and to the fact that unfortunate cases may happen (e.g. Taylor series in μ^2 which get suppressed at $\mu^2 < 0$ because of cancellations, more later ...)**

Phase structure in the $T - \mu_I$ plane

At small μ_I we expect the continuation of the physical transition at $\mu = 0$, but an additional phase structure appears, related to Polyakov loop dynamics at high T.

The fermionic determinant in the QCD partition function can be interpreted in terms of an effective action

$$Z = \int DU \, \det M[U] \, e^{-S_g[U]} = \int DU \, e^{\mathrm{Tr} \ln M[U]} \, e^{-S_g[U]}$$

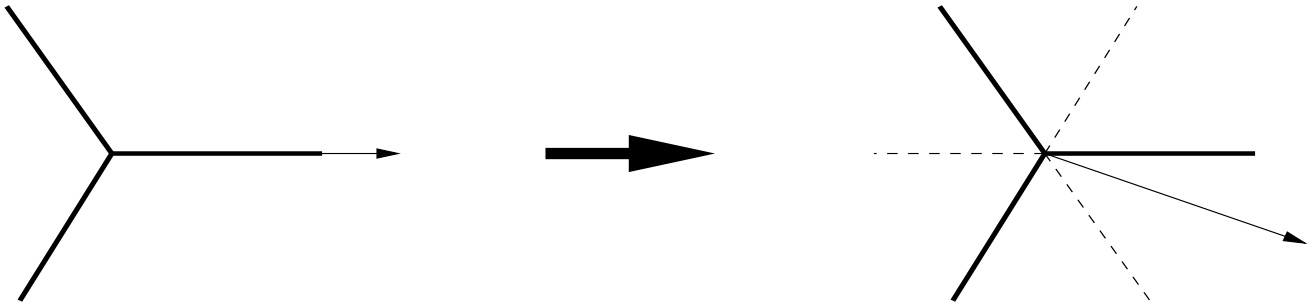
a loop expansion clearly shows that this introduces couplings to the Polyakov loop L, precisely to $(\mathrm{Tr}L + \mathrm{Tr}L^\dagger)$, which explicitly breaks the Z_3 center symmetry (i.e. multiplying all temporal links at a given time slice by a global element of the center of the gauge group), acting like a magnetic field pointing along the real Z_3 root:

- **at low temperatures $\langle L \rangle \neq 0$ and real, even if small**
- **at high temperatures the symmetry breaking term removes the Z_3 degeneracy selecting one particular Z_3 vacuum (the real one)**

The introduction of an imaginary chemical potential ($\det M[U] \rightarrow \det M[U, \mu_I/T]$**) modifies the loop expansion by multiplying all loops winding around the temporal direction by** $\exp(i n_w \mu_I/T)$**, where** n_w **is the winding number. That is equivalent to a rotation of the effective magnetic field by an angle** $\theta = -\mu_I/T$**.**

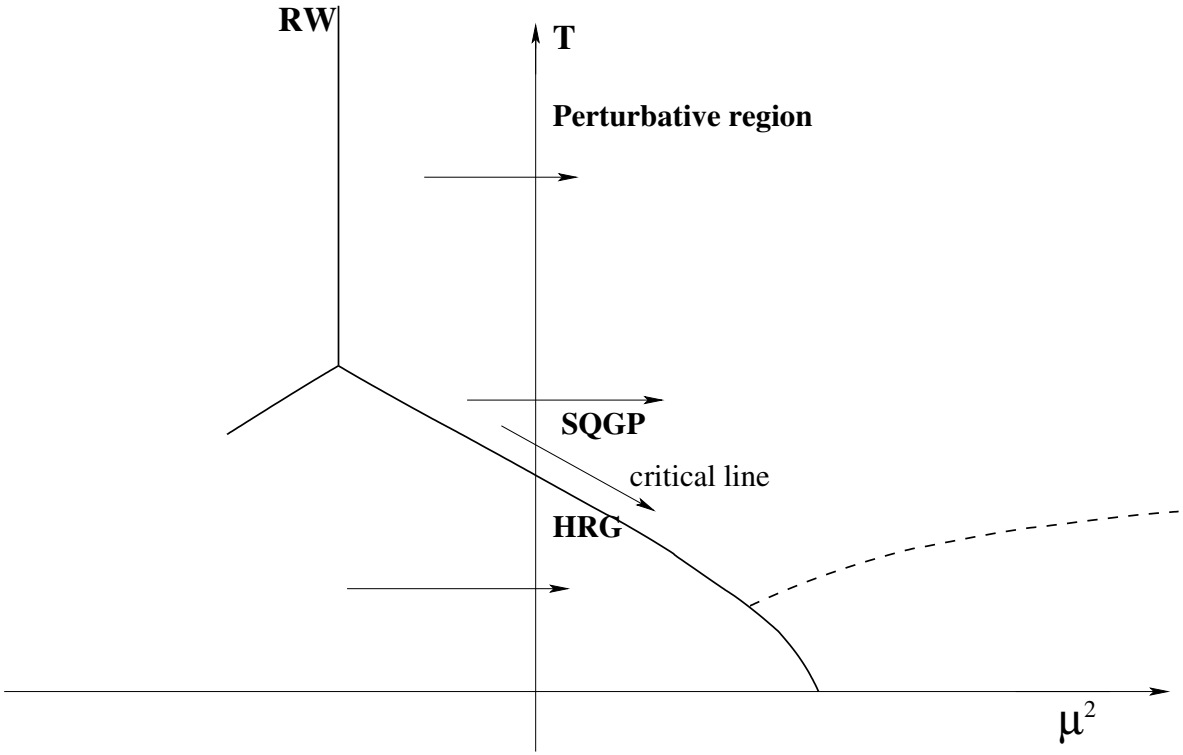
A rotation $\theta \rightarrow \theta + 2\pi/N_c$ **can always be reabsorbed by means of change of variables (center transformation):** $Z(\theta + 2\pi/N_c) = Z(\theta)$ **(Roberge and Weiss, 1980)**

- **At low temperatures there is no phase transition driven by the "external field".** $Z(i\mu_I)$ **is a smooth periodic function of** μ_I/T **with period** $2\pi/N_c$
- **At high temperatures** $\langle L \rangle$ **is constrained to one of the** Z_3 **vacua, but the orientation of the external field selects the true vacuum: first order phase transitions are present for** $\mu_I/T = 2(k + 1/2)\pi/N_c$ **where** k **is integer.** $\Longrightarrow$ **RW transitions**



zero chemical potential **non zero imaginary chemical potential**

The following issues can be therefore analyzed within the analytic continuation approach



- **physical properties of hadronic matter below the transition (HRG model)**
- **physical properties of deconfined matter above the transition (QGP)**
- **continuation of the critical line for small chemical potentials**

I will show in the following some results regarding these issues and compare different methods.

4 – Properties of hadronic matter below T_c: testing the HRG model

In the Hadron Resonance Gas model the free energy density is described as a sum of free particle contributions:

$$\ln Z(T,\mu_B,\mu_{\rm iso}) = \sum_{i\in{\rm mesons}} \ln Z^B_{m_i}(T,\mu_{\rm iso}) + \sum_{j\in{\rm baryons}} \ln Z^F_{m_j}(T,\mu_B,\mu_{\rm iso})$$

where Z^F (Z^B) stands for free particle partition functions for fermions (bosons) and $\mu_{\rm iso}$ is the isospin chemical potential. The model works quite well in describing the statistical distribution of hadronized matter in heavy ion collisions.

Assuming $\mu_{\rm iso} = 0$ and contributions with baryon number ≤ 1, one obtains, in the Boltzmann approximation (valid as $m_i/T \gg 1$), the simple prediction for the pressure (free energy density), i.e. for the equation of state:

$$\frac{\ln Z}{VT^3} = \frac{p(T,\mu_B)}{T^4} = F(T)\cosh(\frac{\mu_B}{T}) = F(T)\cosh(\frac{3\mu}{T})$$

hence the baryon number density is predicted to be

$$n_B(\mu) \equiv \frac{T}{V}\frac{\partial}{\partial\mu}\ln Z \propto \sinh(\frac{3\mu}{T})$$

.

The model can be confronted with QCD predictions using lattice simulations with different approaches.

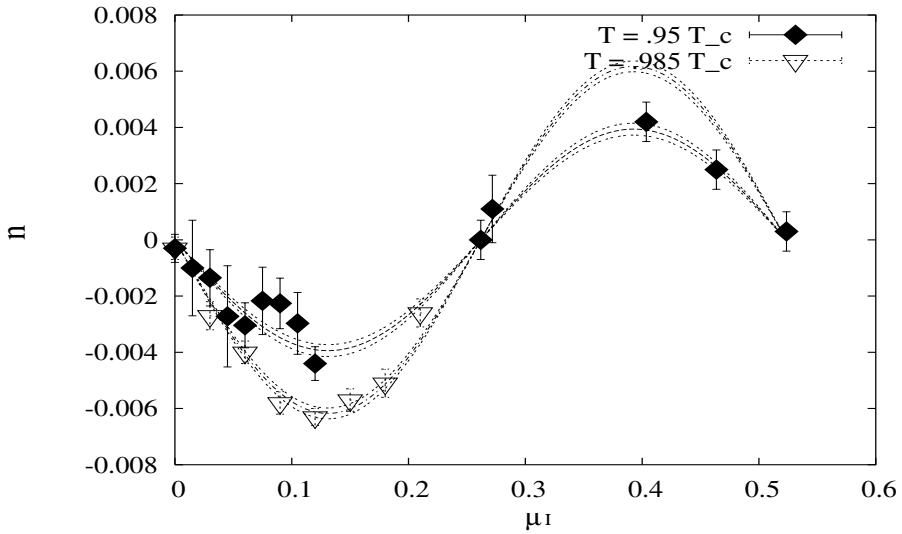
Some results have been obtained within the Taylor expansion approach, where the comparison can be made by measuring various different susceptibilities measured at $\mu = 0$. (see e.g. C.R. Allton et al., Phys. Rev. D 71, 054508 (2005)).

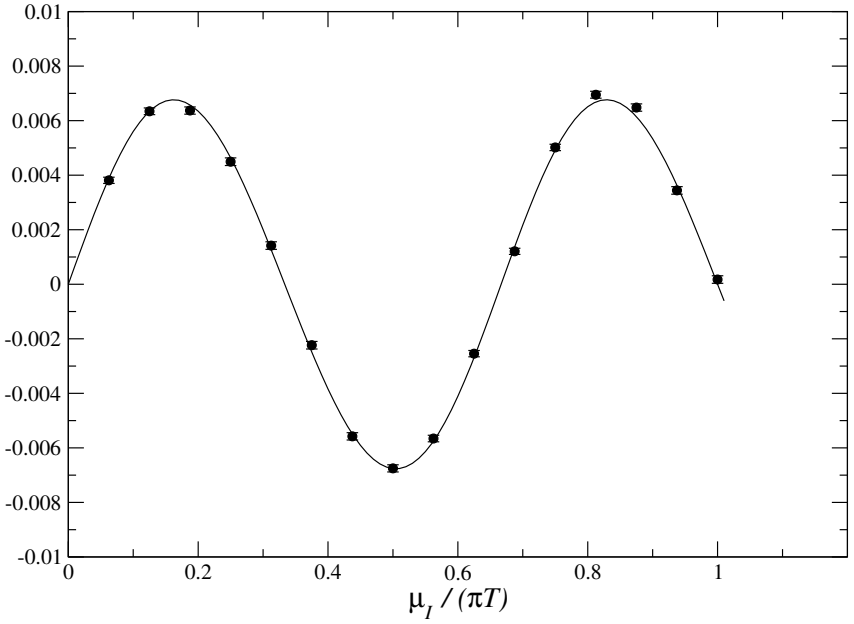
In the analytic continuation approach HRG prediction are simply continued to imaginary μ leading to the following expressions:

$$\frac{p(T, i\mu_I)}{T^4} = F(T)\cos(\frac{3\mu_I}{T}) \; ; \quad \mathrm{Im}(n_B(\mu_I)) \propto \sin(\frac{3\mu_I}{T})$$

which are expected to hold for every value of μ_I.

These predicitions are nicely tested at $\mu = i\mu_I$: single Fourier terms well fit the data





The baryon number density is well described by a single sine term. Results obtained for $N_f = 4$. **M.D'E. and M.P. Lombardo, 2004**

Some preliminary results from a high precision study (M. D'E., F. Sanfilippo).

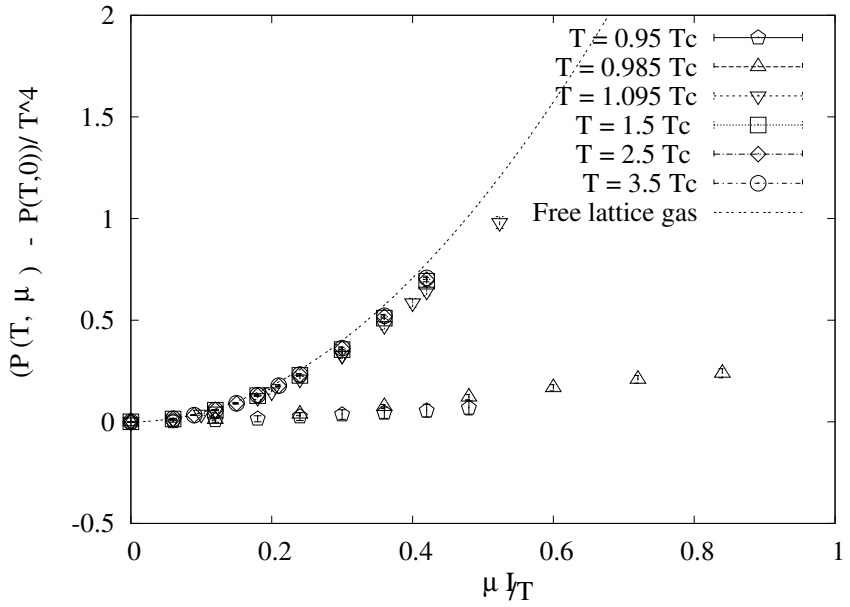
5 – Exploring the Deconfined Phase

A number of quantities can be studied and confronted with model predictions:

- **baryon number density**
- **pressure**

$$\Delta P/T^4 = (P(T,\mu,m_q) - P(T,\mu=0,m_q))/T^4$$

A gas of free particles is expected for very high temperatures, and a strongly interacting plasma getting closer to T_c.



At high temperatures the pressure determined by lattice simulations at imaginary values of μ approaches asymptotically free lattice gas predicition. (M.D'E. and M.P. Lombardo, 2004 for $N_f = 4$). Deviations are larger and larger as T_c is approached.

Properties of sQGP close to T_c

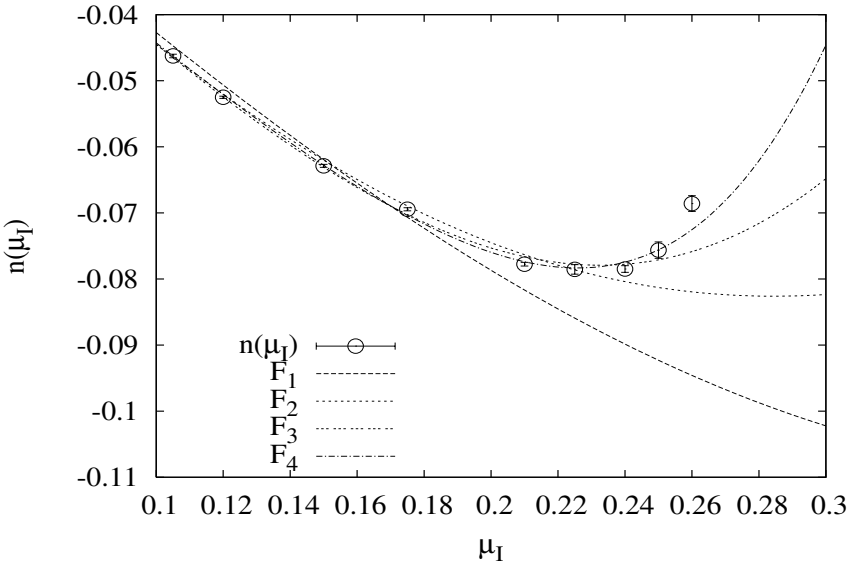
(M. D'E., F. Di Renzo, M.P. Lombardo, Phys. Rev. D76, 114509 (2007))

Data at $T \sim 1.1 T_c$ show sensible deviations from the free particle predicitons. Mimicking the HRG model, we have tried to describe the sQGP as a sum of non-interacting bound states. Setting $\mu_u = \mu_d = \mu$ we have:

$$\frac{\Delta P}{T^4} = F_q(T)(1 - \cosh(\mu/T)) + F_{qq}(T)(1 - \cosh(2\mu/T)) + F_{qqq}(T)(1 - \cosh(3\mu/T)) + dots$$

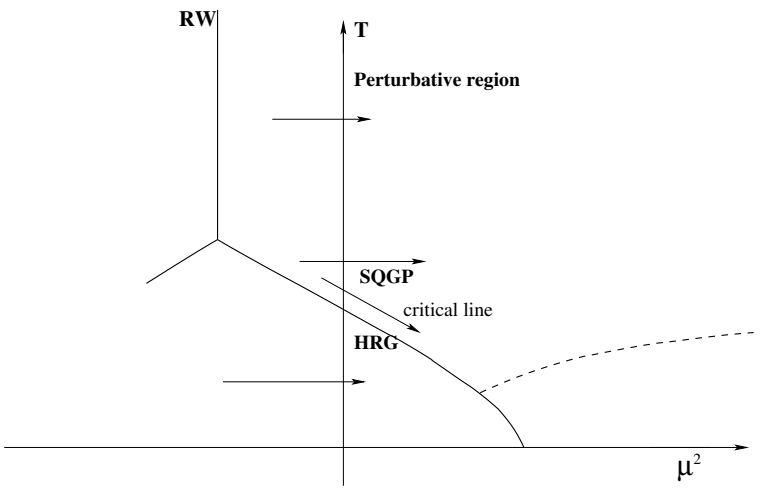
giving in turn

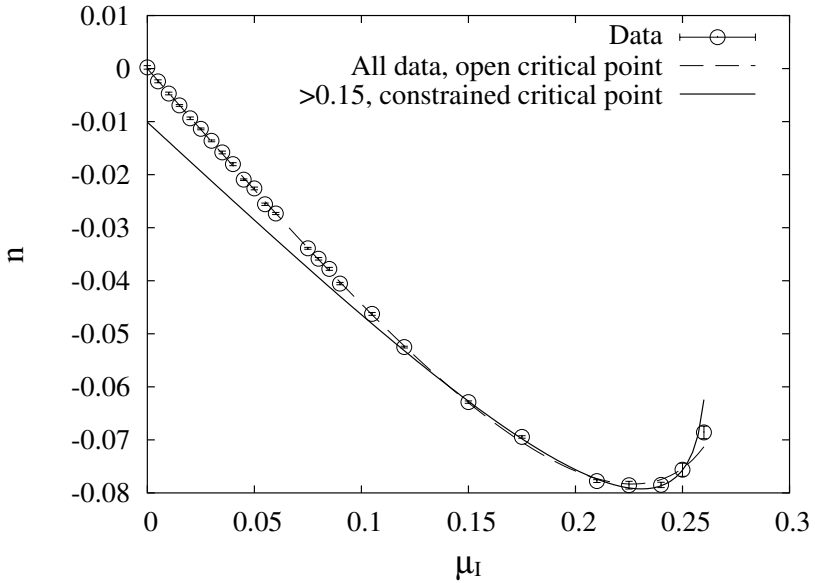
$$n(\mu_I, T) = F_q(T) \sin(\mu_I/T) + 2F_{qq}(T) \sin(2\mu_I/T) + 3F_{qqq}(T) \sin(3\mu_I/T) + \ldots$$



Data are fitted better and better as more terms are introduced, however fit parameters are unstable and one cannot get a $\chi^2/\mathrm{d.o.f.}$ better that ~ 3.

Hypothesis of non-interacting bound states could be too strong





As a different hypothesis, we have tried interpreting results in terms of a critical behaviour induced by the nearby endpoint of the Roberge-Weiss transition line happening at $\mu_I/T = \pi/3$.

A critical behaviour for the fermion number density (imaginary part) like

$$n(\mu_I) = A\mu_I(\mu_I^{c\,2} - \mu_I^2)^\alpha \rightarrow n(\mu) = A\mu(\mu_I^{c\,2} + \mu^2)^\alpha$$

with $\alpha \sim 0.3$, $\mu_I^{c\,2} \sim 0.08$ well reproduces the numerical data for the baryon number density.

That means an increase in density slower than a Stefan-Boltzmann-like law would predict ($\alpha = 1$), as expected for a strongly interacting QGP.

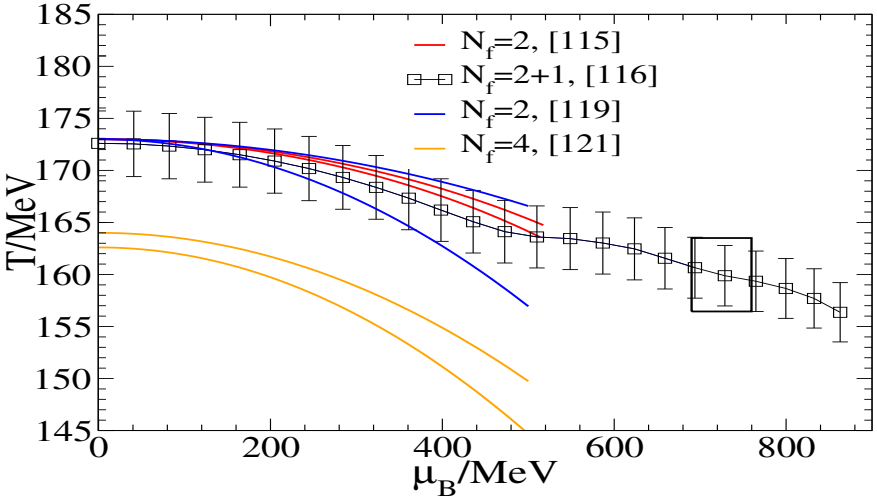
Can the physical properties of the sQGP be described in terms of this critical behaviour at $\mu^2 < 0$? Further studies are needed to test this hypothesis.

6 – Continuation of the critical line

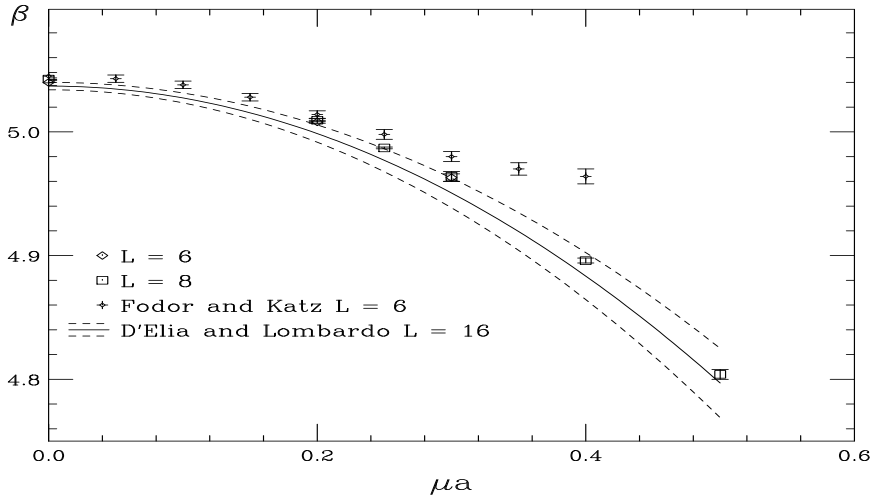
Determine locations of the critical line at $\mu^2 < 0$, to be interpolated by suitable functions (e.g. Taylor expansions) and continued to $\mu^2 > 0$.

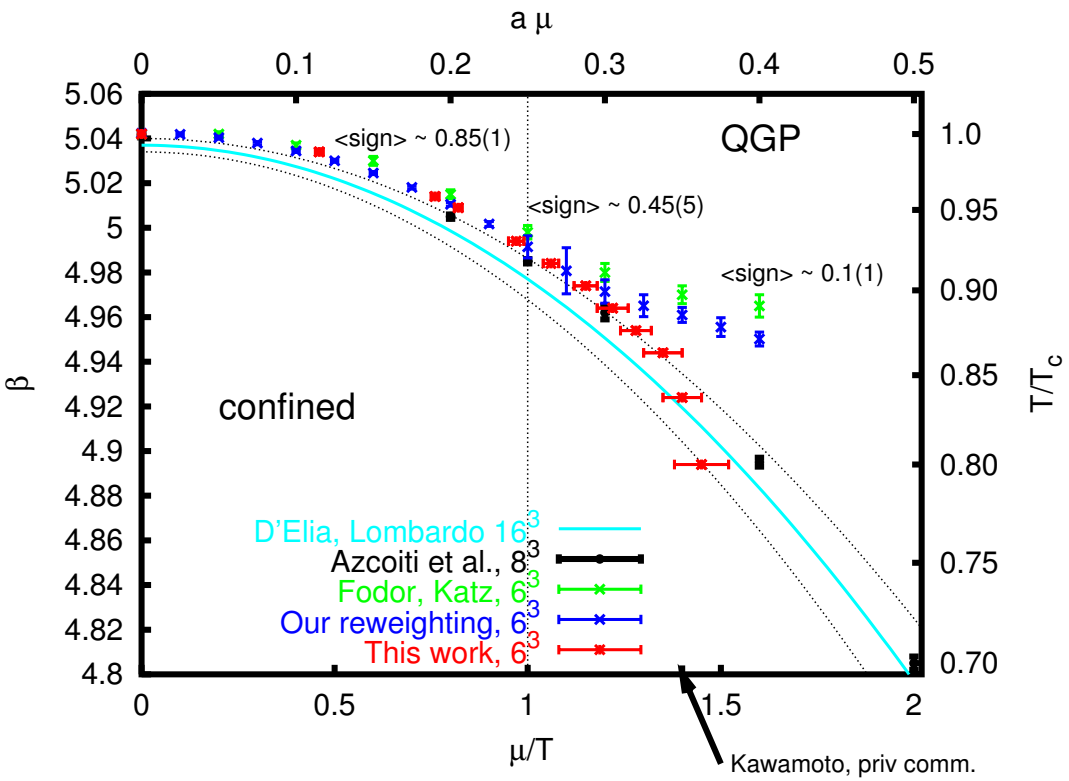
Several results obtained, allowing a detailed comparison with other methods:

The critical line obtained for $N_f = 2$ by de Forcrand and Philipsen by analytic continuation (red) is compared to the determination by Taylor expansion (Bielefeld-Swansea, blue) and reweighting (Fodor and Katz). from E. Laermann and O. Philipsen, hep-ph/0303042.



The critical line obtained for $N_f = 4$ obtained by analytic continuation (M. D'E., M.P. Lombardo) is compared to that obtained by Azcoiti *et al* and by reweighting (Fodor and Katz). from V. Azcoiti *et al*, Nucl. Phys. B 723, 77 (2005).





The critical line for 4 staggered flavours: various methods compared to what obtained by the canonical approach.

from S. Kratochvila and P. de Forcrand, PoS LAT2005, 167 (2006).

7 – Two color QCD as a laboratory for QCD at finite density

QCD with two colors is suitable for numerical simulations at finite baryon chemical potential (the determinant is real): that makes it a very appealing model, however it has significant differences with respect to ordinary QCD, especially regarding hadron matter (baryons are degenerate with mesons).

Differences are more pronounced at low temperatures, where the finite density transition of two color QCD is associated with Bose-Einstein condensation

Nevertheless it may be regarded as an interesting laboratory for studying issues related to gluodynamics in presence of finite baryon density and for testing approximate methods used in real QCD. I will briefly touch two examples:

- fate of dual superconductivity (confinement) at finite density;
- direct test of analytic continuation.

Fate of dual superconductivity at finite density

Dual superconductivity of the QCD vacuum is an appealing mechanism for color confinement, originally proposed by t' Hooft and Mandelstam:
according to it the QCD vacuum behaves as a superconductor expelling chromoelectric fields via dual Meissner effect ($\longrightarrow$ flux tube formation).

A related order parameter is the vacuum expectation value of a magnetically charged operator $\langle \mu \rangle$: contrary to other parameters (like the Polyakov loop) it is a good order parameter for confinement also in presence of dynamical fermions.

L. Del Debbio, A. Di Giacomo and G. Paffuti, Phys. Lett. B 349, 513 (1995)

A. Di Giacomo, B. Lucini, L. Montesi and G. Paffuti, Phys. Rev. D 61, 034503 (2000); Phys. Rev. D 61, 034504 (2000)

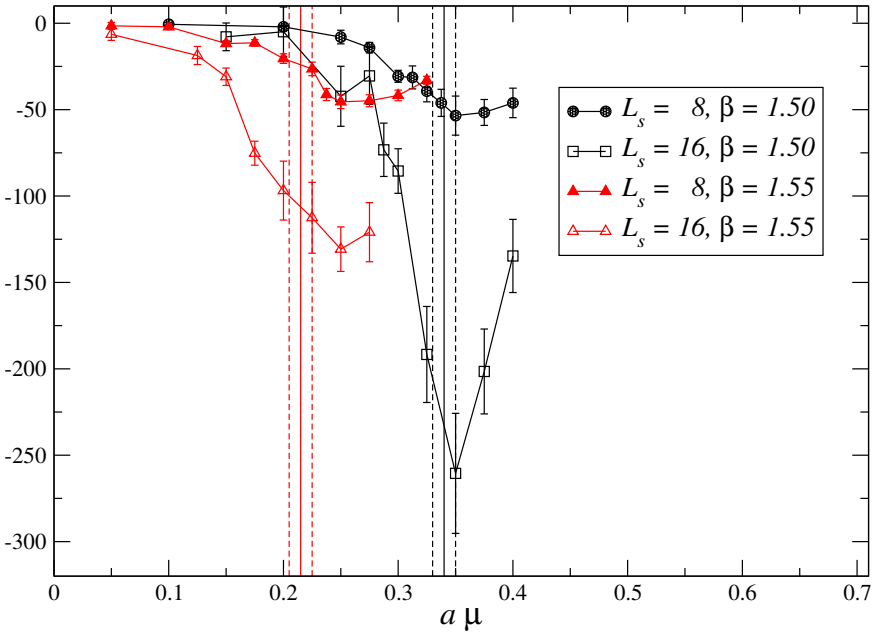
M. D'E., A. Di Giacomo, B. Lucini, C. Pica and G. Paffuti, Phys. Rev. D 71, 114502 (2005)

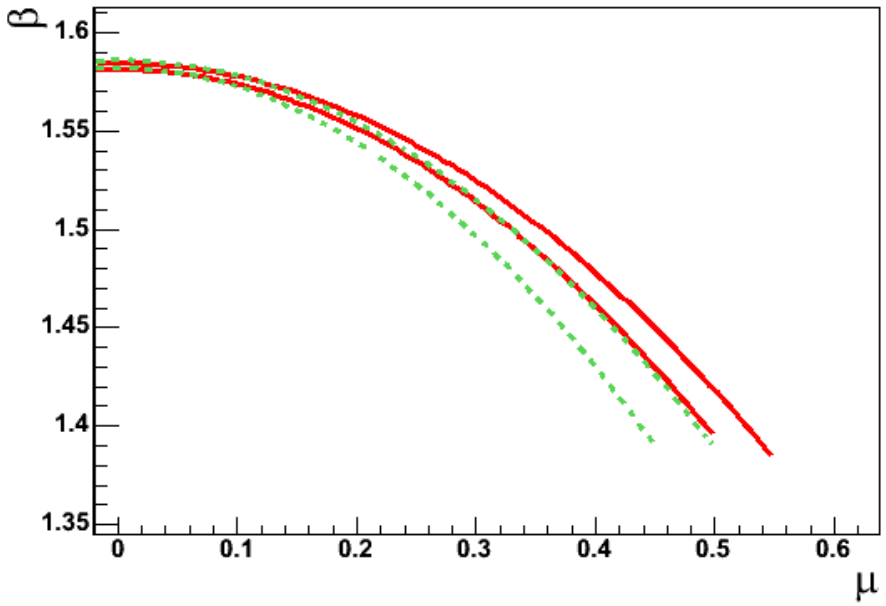
P. Cea and L. Cosmai, JHEP 0111, 064 (2001), P. Cea, L. Cosmai and M. D'Elia, JHEP 0402, 018 (2004).

Can disappearance of dual superconductivity (deconfinement) be induced by a finite baryon density?

The answer is positive. Moreover, in the temperature range explored ($T > 0.4T_c$) the disappearance of dual superconductivity is always associated with chiral symmetry restoration.

S. Conradi, A. D'Alessandro, M. D'E, Phys. Rev. D76, 054504 (2007)





Analytic continuation under scrutiny

In QCD-like theories, where simulations are feasible both at imaginary and real chemical potentials, analytic continuation can be carefully tested. An example is two-color QCD

P. Giudice, A. Papa, 2005; P. Cea, L. Cosmai, M. D'E. and A. Papa, JHEP 0702 (2007) 066, Phys. Rev. D 77, 051501(R) (2008)

A few specific points to address:

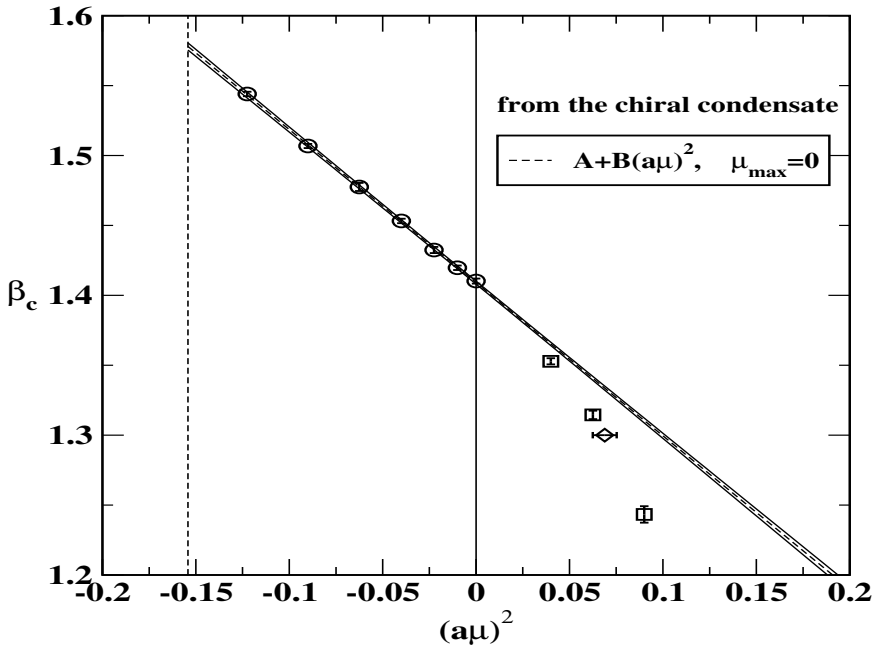
- **To what extent can we trust analytic continuation?**
- **What is the optimal way to extract information?**

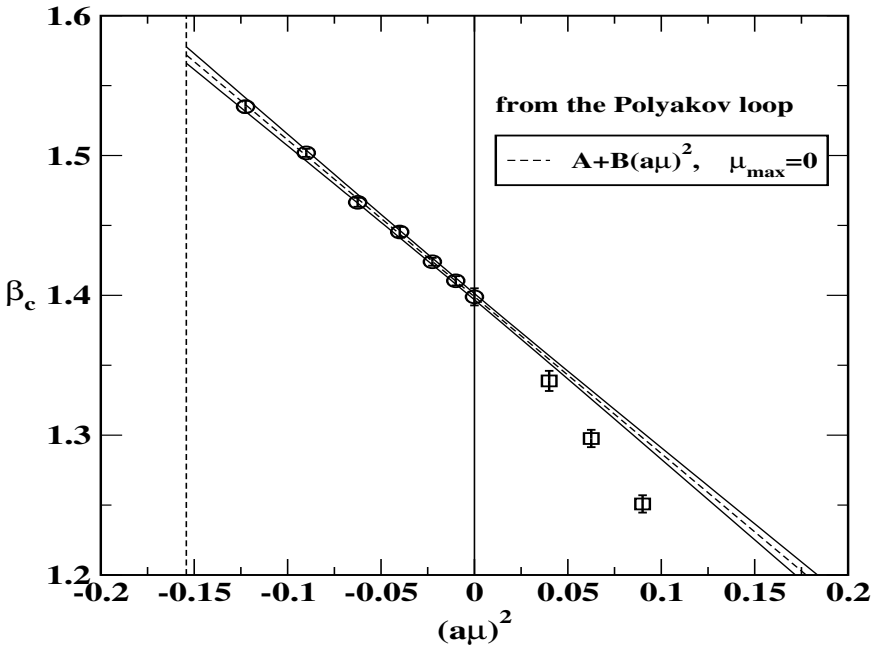
The outcome is that analytic continuation works well, for physical observables, in the expected analyticity ranges.

Special care must be payed to the analytic continuation of the transition line ...

We determine the critical line looking at the peaks of the susceptibilities of various physical quantities: **chiral condensate, Polyakov loop, plaquette.**

The assumption underlying analytic continuation in this case is that the locations of such peaks are an analytic function of μ^2 around $\mu^2 = 0$.



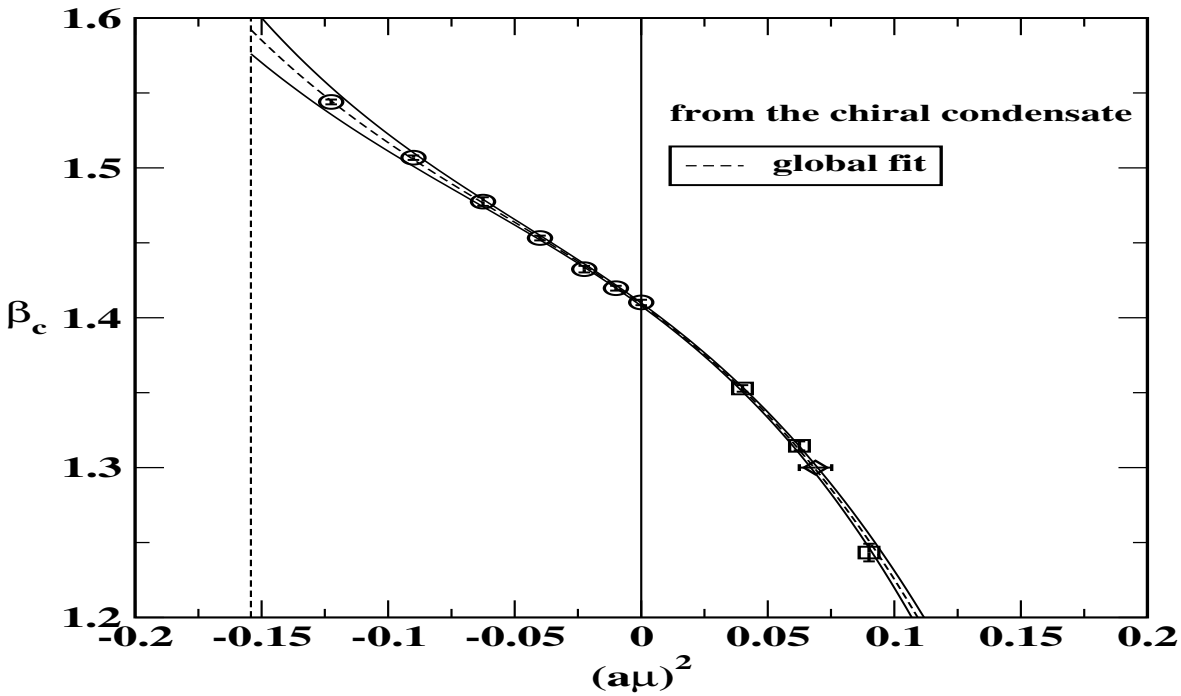


A linear dependence in μ^2 fits well data at $\mu^2 < 0$.

chiral condensate: $\beta_c(\mu^2) = A + B\mu^2 + C\mu^4 \implies B = 1.06(6), C = 0.3(5)$

But: direct determinations at $\mu^2 > 0$ in disagreement with analytic continuation!

Analytic continuation fails? not exactly ...



a higher order polynomial $\beta_c(\mu^2) = A + B\mu^2 + C\mu^4 + D\mu^6$ **well describes all data, with** $B = -1.23(3), C = -3.8(4), D = -23(4)$

An important non-linear contribution at $\mu^2 > 0$ **is largely suppressed (and therefore completely missed) at** $\mu^2 < 0$**.**

May that problem be relevant also for studies regarding real QCD? More extensive studies (for instance in QCD with finite isospin density) of this effect must be done ...

8 – SUMMARY

- **The study of the QCD phase diagram is of fundamental importance in several fields (Early Universe, H.I. Collisions, Neutron Stars, . . .)**

  Lattice QCD simulations are the candidate first principles tool, but the sign problem makes life difficult in presence of a finite baryon density.

- **Recent efforts by several groups using different methods are giving consistent information on the phase diagram, starting from the region high T and low μ, which is the one relevant for the experimental search for deconfined states of matter in Heavy Ion Experiments.**

  Progress is not very fast, but steady and much work can yet be done with existing methods: the critical line can be located, properties of the QGP and of baryon matter below T_c can be studied systematically.

- **Of course a real breakthrough in solving the sign problem would lead to much more progress ...**