Isospin chemical potential in holographic “QCD”

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0709.3948 and in progress

Galileo Galilei Institute, May 6th 2008
AdS/CFT integrability “high precision tests”

non-AdS/non-CFT (direct) applications to realistic gauge theories

zero temperature and chemical potentials ($T = 0, \mu = 0$)
- glueball spectra
- masses of hadrons (mesons)
- hadron form factors

finite-temperature and $\mu = 0$ theories
(viscosity of quark-gluon plasma)

Finite chemical potential THIS TALK!
pure QCD — i.e. no matter do not know geometry

instead, consider 4+1 dim max. susy YM compactify on circle impose anti-periodic bdy. cond. for fermions

in IR, reduces to pure QCD, scalars and fermions decouple

dual to near-horizon geometry of non-extremal D4-brane, doubly Wick rotated
The geometry

\[ ds^2 = \left( \frac{u}{R} \right)^{3/2} \left[ \eta_{\mu\nu} dX^\mu dX^\nu + f(u) d\theta^2 \right] + \left( \frac{R}{u} \right)^{3/2} \left[ \frac{d\theta^2}{f(u)} + u^2 d\Omega_4 \right] \]

- **World-volume**
  - Our 3+1 world

- **Function**
  - \( f(u) = 1 - \left( \frac{u_\Lambda}{u} \right)^3 \)

- **\( \theta \)**
  - A compact Kaluza-Klein circle

- **\( u \)**
  - Radial direction bounded from below, \( u \geq u_\Lambda \)

\[ u: \text{radial direction} \]

\[ S^4 \]

\[ U \ (\text{energy scale}) \]
Several remarks

Solution characterised by two parameters:

\[ \begin{align*}
R_{D4} & : \\
R & : \\
\end{align*} \]

\[ R_{D4}^3 = \pi g_s l_s^3 N_c \]

\[ R = \frac{2\pi}{3} \left( \frac{R_{D4}^3}{u_\Lambda} \right)^{1/2} \rightarrow M_\Lambda = \frac{2\pi}{R} \]

Relation to gauge-theory parameters:

- size of \( S^1 \) on D4 (i.e. \( M_{KK} \)) set by \( R \)
- \( \lambda \equiv g_{YM}^2 N_c = \frac{R_{D4}^3}{\alpha' R} \)

Regime of validity:

- sugra OK if \( R^2 \equiv \frac{R_{D4}^3}{R} \gg \alpha' \rightarrow \lambda \gg 1 \)
  (max curvature at the wall)

- valid as long as \( e^\phi = g_s (u/R_{D4})^{3/4} < 1 \)
  (min coupling at the wall)

Problem: \( M_{KK} \sim M_{\text{glueball}} \sim M_{\text{meson}} \sim M_\Lambda \sim 1/R \)

non-extremality of D-brane: angle \( \theta \) identified with period \( R \) to avoid conical singularity

cannot decouple KK modes!
$u = \text{energy scale}$

\[
\left( \frac{R_{D4}}{u_\Lambda} \right)^{3/2} u_\Lambda^2 = R_{D4}^{3/2} u^{1/2}_\Lambda
\]
Add D8 flavour (probe) branes to D4 stack

strings between flavour & colour branes in fund. rep. of flavour & colour group

Solve for the shape of the D8

\[ D4 : \quad 0 \quad 1 \quad 2 \quad 3 \quad \theta \quad - \quad - \quad - \quad - \quad - \]

\[ D8 : \quad 0 \quad 1 \quad 2 \quad 3 \quad - \quad 5 \quad 6 \quad 7 \quad 8 \quad 9 \]

curve \( u(\theta) \)

coord. sys. adapted to D8 \( \theta, \Omega^4 \)

Solution to the 1st order equation gives embedding \( u(\theta) \)
**Symmetry encoded in geometry**

- Asymptotically exhibits full chiral symmetry $\text{SU}(N_f)_L \times \text{SU}(N_f)_R$
- Bending of the brane encodes spontaneous symmetry breaking in gauge theory in a geometrical way

\[ \text{SU}(N_f)_L \rightarrow \text{SU}(N_f)_{\text{isospin}} \rightarrow \text{SU}(N_f)_R \]

spectrum of fluctuations contains $(\pi^\pm, \pi^0)$ Goldstone bosons

- Brane geometry also reproduces chiral symmetry restoration above $T > T_c$
Low spin mesons

Spectrum is known only in the limits:

**Low-spin mesons:**
fluctuations on and of the flavour brane

Fluctuations governed by Dirac-Born-Infeld action of the flavour brane

\[
S = V_{S^4} \int d^5x \, e^{-\phi} \sqrt{- \det (g_{\mu\nu} + 2\pi \alpha' F_{\mu\nu})} + S_{\text{Wess-Zumino}}
\]

\[
= V_{S^4} \int d^4x \, dz \sqrt{-g} \, F_{\mu\nu} F_{\rho\lambda} g^{\mu\rho} g^{\nu\lambda} + \ldots
\]

Expand world-volume fields in modes \rightarrow \text{meson spectrum & action}
Effective action for light mesons

- Decompose the gauge fields

\[ F_{\mu\nu} = \sum_n G_{\mu\nu}^{(n)}(x) \psi(n)(u), \]
\[ F_{u\mu} = \sum_n B_{\mu}^{(n)}(x) \partial_u \psi(n)(u), \]

- Fourier transform & factor out polarisation vectors,

\[
\int d^4k \tilde{B}_{\mu}^{(m)} \tilde{B}_{\mu}^{(n)} \left[ u^{-1/2} \gamma^{1/2} (\omega^2 - \vec{k}^2) \psi(n) - \partial_u \left( u^{5/2} \gamma^{-1/2} \partial_u \psi(n) \right) \right] = 0.
\]

a Sturm-Liouville problem
mass spectrum of mesons
High spin mesons

- Spectrum is known only in the limit:
- Sigma model \((\text{semiclass})\) → high-spin glueballs \((\text{closed})\) & mesons \((\text{open})\)

\[ q\bar{q} \text{ meson:} \]

\[ u_{f_1} \]

\[ u_{f_2} \]

\[ u_\Lambda \]

\( \text{region I} \)

\( \text{region II} \)

\[ m_q \sim u_{f_1} - u_\Lambda \]

\[ m_q \sim \sqrt{\lambda} M_\Lambda \]

\( \text{vs. low spin mass} \quad M_{\text{low}} \sim M_\Lambda \sim M_{KK} \)

\[ N.B. \quad \text{High spin mass} \quad M_{\text{high}} \sim \sqrt{\lambda} M_\Lambda \]
Part II:

Turning on an *isospin* chemical potential

Chiral Lagrangian
Why isospin chemical potential is easier in holographic models than baryon chemical potential:

- large $N_c$ → baryons much heavier than at finite $N_c$
  mesons closer to the real-world

- baryons complicated solitons, mesons elementary fields

- so far only singular solitons known

- potentially comparable with the lattice (no sign problem)

**Bad feature**: Artificial, no pure isospin systems exist in nature (weak decays)
neutron stars
At small $\mu I$ chiral Lagrangian (with $m_q = 0$) to get a feeling what happens

$$\mathcal{L}_{\text{chiral}} = \frac{f^2}{4} \text{Tr}(D_\nu UD_\nu U^\dagger), \quad U \in U(N_f).$$

$$U \equiv e^{i \frac{\pi}{f} \pi_a(x) T^a} \quad T_a = - U(N_f) \quad \text{generators}$$

Invariant under separate

$$U \to g_L^{-1}U, \quad U \to Ug_R$$

The vacuum $U = I$ preserves the vector-like $U(N_f)$ symmetry,

$$U \to g_L Ug_R^{-1} \quad \Rightarrow \quad g_L = g_R.$$  

In $U = I$ want to turn on a vector chemical potential $\mu_L = \mu_R$.

Other global transformations move us around on the moduli space of vacua,

$$\mathcal{M} = \frac{U(N_f) \times U(N_f)}{U(N_f)}$$
Chiral Lagrangian and $\mu \neq 0$

As usual, chemical potentials via

$D_\nu U = \partial_\nu U - \frac{1}{2}\delta_{\nu,0}(\mu_L U - U \mu_R) = \partial_\nu U - \frac{1}{2}\delta_{\nu,0}([\mu_V, U] - \{\mu_A, U\})$

$(\mu_L = \mu_V - \mu_A, \mu_R = \mu_V + \mu_A)$.

$V_\chi = \frac{f_\pi^2}{4} \text{Tr} \left( ([\mu_V, U] - \{\mu_A, U\})([\mu_V, U^\dagger] + \{\mu_A, U^\dagger\}) \right)$

From $V_\chi$ minima:

1. $\mu_V = 0$, $\mu_A$-any $\rightarrow$ $V_\chi$-const. $\rho_A \sim f_\pi^2 \mu_A$
2. $\mu_A = 0$, $\mu_V = \mu_I \sigma_3/2$ $\rightarrow$
   
   $U_{\text{max}} = e^{i\alpha}(\cos(\beta)I + i\sin(\beta)\sigma_3)$ and $U_{\text{min}} = e^{i\alpha}(\cos(\beta)\sigma_1 + \sin(\beta)\sigma_2)$
   
   in the $U_{\text{min}}$: $\rho_V \sim f_\pi^2 \mu_I$ $\rho_{A,I} = 0$.

3. $\mu_V = \mu_I \sigma_3/2$, $\mu_A = \mu_{A,I} \sigma_3/2$:
   \[
   \begin{cases} 
   \mu_{A,I}^2 < \mu_V^2 \rightarrow U_{\text{min}} \\
   \mu_{A,I}^2 > \mu_V^2 \rightarrow U_{\text{min}} \text{ opposite}
   \end{cases}
   \]
Vectorial isospin potential

$U = I, \langle \pi \rangle = 0$

$\mu_R = \mu_L \propto \sigma_3$ (vector)

$U = I, \langle \pi \rangle = 0$

$\mu_R = -\mu_L \propto \sigma_3$ (axial)

Effects of $\mu_V$ in $U = U_{\text{min}} \Leftrightarrow$ effects of $\mu_A$ in $U = I$ vacuum
Aside: non-zero pion mass

The chiral Lagrangian gives us the behaviour of the pions for small $\mu_I$.

However, Sakai-Sugimoto has $m_\pi = 0$, so we will at small $\mu_I$ see
Chiral Langrangian, valid up to the first massive vector meson,
\[ \mu_I \ll m_\rho \]

Other operators are relevant, e.g. Skyrme term
\[ L_{\text{Skyrme}} = \frac{1}{32e^2} \text{Tr} \left( [U^{-1} \partial_\mu U, U^{-1} \partial_\nu U]^2 \right) . \]

This leads to a dispersion relation for pions
\[ -\omega^2 + k^2 + \mu_I^2 - \frac{k^2 \mu_I^2}{e^2 f_\pi^2} = 0 . \]

This suggests massive pions eventually become unstable.
But, does not explain what the $\rho$ does.

Sakai-Sugimoto has pions and fixed couplings to other mesons.
Study $\pi$'s and $\rho$ in this model as function of $\mu_I$. 
Part III:

Holographic isospin chemical potential
Cigar-shaped subspace with D8’s embedded,

\[ u = (1 + z^2)^{1/3} \]

No chemical potential \( \rightarrow \) no background field, trivial \( A_\mu = 0 \) vacuum.

Meson masses from linearised DBI action around trivial vacuum.

\[
A_\mu(x^\mu, z) = U^{-1}(x) \partial_\mu U(x) \psi_+(z) + \sum_{n \geq 1} B^{(n)}_\mu(x) \psi_n(z),
\]

\( A_z = 0 \)

Can go beyond \( \chi \)-perturbation theory: have \( \chi \)-Langrangian interacting with infinite tower of massive modes.
Beyond Chiral Langrangian $\mu_I = 0$

Effective action we use come from the truncated string effective action

$$S = \tilde{T} \int d^4x \, du \left[ u^{-1/2} \gamma^{1/2} \text{Tr}(F_{\mu\nu} F^{\mu\nu}) + u^{5/2} \gamma^{-1/2} \text{Tr}(F_{\mu\nu} F^{\mu\nu}_u) \right] + \ldots$$

where ignored DBI corrections to the YM, $((l_s F)^n)$ and beyond $O(l_s^3 \partial F)$

For eg., just for pion this gives

$$F_{z\mu} = U^{-1} \partial_\mu U \phi_0(z) + \text{B-stuff}$$

$$F_{\mu\nu} = [U^{-1} \partial_\mu U, U^{-1} \partial_\nu U] \psi_+(z) (\psi_+(z) - 1) + \text{B-stuff}.$$  

which gives chiral Lagrangian plus Skyrme term,

$$S = \int d^4x \, \text{Tr} \left( \frac{f_\pi^2}{4} (U^{-1} \partial_\mu U)^2 + \frac{1}{32e^2} [U^{-1} \partial_\mu U, U^{-1} \partial_\nu U]^2 \right) + \text{"}\pi \leftrightarrow B\text{"}$$

$$f_\pi^2 \sim \lambda N_c M_{KK}^2, \quad e^2 \sim \frac{1}{\lambda N_c}, \ldots$$
Sakai-Sugimoto and chiral symmetry

In Sakai-Sugimoto, global symmetry is realised as large gauge transformation of bulk field,

\[ A_\mu \rightarrow gA_\mu g^{-1} + ig\partial_\mu g^{-1} \]

\[ \lim_{z \to -\infty} g(z, x^\mu) = g_L \in SU(N_f)_L, \]

\[ \lim_{z \to +\infty} g(z, x^\mu) = g_R \in SU(N_f)_R. \]
Sakai-Sugimoto and chiral symmetry

- And changes holonomy

\[ U = P \exp (i \int_{-\infty}^{\infty} dz A_z) \rightarrow g_L g_R^{-1}. \]

changes the pion expectation value, since

\[ U = \exp (i \pi_a(x) \sigma^a / f_\pi). \]

- So if start with trivial vacuum \( A_\mu = A_z = 0 \), the vectorial transformation \( g_L = g_R \) preserves vacuum, does not change \( U \)

- If \( g_L \neq g_R \), does not preserve vacuum

  i.e. changes holonomy \( \chi \)-symmetry breaking
Turning on $\mu_I \neq 0$

For SS model, bulk field $A_\mu(x, u)$

$$A_\nu(x, u) \to B_\nu(x) \left(1 + O\left(\frac{1}{u}\right)\right) + \rho_\nu(x) u^{-3/2} \left(1 + O\left(\frac{1}{u}\right)\right).$$

Here

- $B_\mu(x) \leftrightarrow$ source term for gauge theory current $J^\nu(x)$ ($\int d^4 x B_\mu J^\nu(x)$)
- $\rho_\nu(x) \leftrightarrow$ vev of $J^\mu$

To add vectorial/axial chemical potential, solve for the even/odd bulk field with b.c.:

$$A_\mu(x, z \to -\infty) = \mu_L \delta_{\mu,0}$$
$$A_\mu(x, z \to +\infty) = \mu_R \delta_{\mu,0}$$
First ansatz, assume that condensate is \( x \)-independent \( A_0(z), A_i = 0 \)

Isospin chemical potential background satisfies 5d YM equation (in \( A_u = 0 \) gauge),

\[
\partial_z \left[ (1 + z^2) \partial_z A_0^{(3)} \right] = 0
\]

\[
\begin{align*}
V: & \quad A_0^{(3)} = \mu_V, \\
A: & \quad A_0^{(3)} = \mu_A \arctan z.
\end{align*}
\]

N.B Soln to YM action, neglect DBI corrections, i.e. valid for \( \mu_I \ll \lambda/L \)

Spectrum around vectorial soln (V) tachyonic i.e. free energy is unaffected, but fluctuations are affected!

roll down to \( \langle \pi^{(1)} \rangle \neq 0 \), then rotate back to trivial vacuum

Effectively work with axial solution (A)

Properties of new vacuum:

\( f_\pi \) unmodified, two massive and one massless pion
Instability of isotropic solution

- Soln found is unique **isotropic** soln: pions condensed.

**What about ρ et al?**
Are there any other ground states which dominate for higher μ_I?

**Analyse general stability of soln**

- For μ_I ≪ λ_5/L^2 can still use just nonabelian YM
  expand YM around axial solution \( \bar{A}_0 \) in \( U = I \) vacuum

\[
A_0 = \bar{A}_0(u) + \delta A_0^{(a)}(\omega, \vec{k}, u) \sigma_a e^{i\omega t + i\vec{k} \cdot \vec{x}}, \\
A_i = \delta A_i^{(a)}(\omega, \vec{k}, u) \sigma_a e^{i\omega t + i\vec{k} \cdot \vec{x}}, \\
A_u = 0
\]
Transverse vectors and scalars

- The transverse vectors \( (\delta A_0 = 0, \partial_i \delta A^i = 0) \) develop an instability: at \( \vec{k} = 0 \) the dispersion relation is

- The scalars (fluctuations transverse to the brane) are unstable too, but only for much larger \( \mu \),

- The main question: what about the pions & longitudinal vectors?
Both pions and longitudinal vectors are governed by $A_i \equiv ik_i A_T$ and $A_0$.

Equations diagonal for

$$\delta A_i^{(1)} = \pm i \delta A_i^{(2)} , \quad \delta A_0^{(1)} = \pm i \delta A_0^{(2)} .$$

The difference is the boundary conditions

- The pion is “pure large gauge”, so impose $F_{0i} = 0$,

  $$A_T(z \to +\infty) = \frac{\pi}{2} + \frac{c_3}{z} + \ldots$$

  $$A_0(z \to +\infty) = (\omega + \pi \mu) \frac{\pi}{2} + \left( \frac{c_3 k^2}{\omega + \pi \mu} - \pi \mu \right) \frac{1}{z} + \ldots$$

- Vectors asymptote to zero at $z \to \pm\infty$,

  $$A_T(z \to +\infty) = \frac{1}{z} + \ldots$$

  $$A_0(z \to +\infty) = \frac{k^2}{\omega + \pi \mu} \frac{1}{z} + \ldots$$

N.B $\mu_I = 0$ recover Lorentz inv. rels. ($\delta A_0 = \omega \delta A_T$ pion and $\delta A_0 = k^2/\omega A_T$, long. vec.)
Similarly, imposing appropriate b.c. at $z = -\infty$ fixes $\omega(\mu, k)$. So the spectrum is

- $\pi$'s, for small $\mu_I$ mass up (as from $\chi - L$)
- modes change “nature”
- no-crossing for $k \neq 0$
- $k = 0$ special crossing of $\rho$ and $\pi$ → $\rho$ condenses
The value of $\mu_{\text{crit}}$ the same as for transverse $\rho$

all components of $\rho$ vector for $k = 0$ condense at $\mu_{\text{crit}} \approx 1.7m_\rho$
Finding a new ground state

What is the new ground state?

Ansatz (inspired by linear analysis):

\[ A_3^{(1)}(z) = \pm i A_3^{(2)}(z) , \quad A_i^{(1)}(z) = A_i^{(2)}(z) = 0 \quad (i = 1, 2) , \]
\[ A^{(3)}_\mu = \delta_{\mu,0} A_0^{(3)}(z) \quad A_u = 0 , \]

with b.c.

\[ A_0^{(3)}(z = \pm \infty) = \pm \mu I/2 , \quad A_3^{(1)}(z = \pm \infty) = 0 \]

Solution of the nonlinear equations

\[ \partial_u \left[ u^{5/2} \gamma^{-1/2} \partial_u A_0^{(3)} \right] = 4(A_3^{(1)})^2 A_0^{(3)} u^{-1/2} \gamma^{1/2} , \]
\[ \partial_u \left[ u^{5/2} \gamma^{-1/2} \partial_u A_3^{(1)} \right] = -4(A_0^{(3)})^2 A_3^{(1)} u^{-1/2} \gamma^{1/2} . \]

Have two solutions

\[ \left\{ \begin{array}{l}
\mu < \mu_{\text{crit}} : \quad A_3^{(1)} = 0 \quad A_0^{(3)} = \frac{\mu I}{\pi} \arctan \left( \frac{z}{u}\Lambda \right) \\
\mu > \mu_{\text{crit}} : \quad A_3^{(1)} \neq 0 \quad A_0^{(3)} \neq 0 .
\end{array} \right. \]
The new ground state

A numerical solution yields:

\[ \mu_{\text{crit}} \approx 1.7 m_\rho, \quad \langle \rho \rangle \propto \sqrt{\mu - \mu_{\text{crit}}}. \]

\( \rho \)-meson condensate forms:
- breaking rotational \( \text{SO}(3) \rightarrow \text{SO}(2) \)
- breaking the residual flavour \( \text{U}(1) \)

(in addition, the pion condensate remains present)
Summary and todo

- Can we include the pion mass (using tachyon)?
- How does this depend on $L$ (constituent quark masses)?
- Are there further instabilities at even higher $\mu$?
- Corrections due to DBI and Chern-Simons?
- Behaviour in deconfined phase, as function of temperature?