# Non-Abelian strings and monopoles

## in supersymmetric gauge theories

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# 1 Introduction

*Seiberg and Witten 1994 :* Abelian confinement in $\mathcal{N} = 2$ QCD

Cascade gauge symmetry breaking:

- SU(N)$\to$ U(1)$^{N-1}$ VEV's of adjoint scalars
- U(1)$^{N-1}$ $\to$ 0 (or discrete subgroup) VEV's of quarks/monopoles

At the last stage Abelian Abrikosov-Nielsen-Olesen flux tubes are formed.

$$\pi_1(U(1)^{N-1}) = \mathcal{Z}^{N-1}$$

$(N-1)$ infinite towers of strings. In particular $(N-1)$ elementary strings

$\longrightarrow$ Too many degenerative hadron states

In search for non-Abelian confinement non-Abelian strings were suggested in $\mathcal{N} = 2$ U(N) QCD

*Hanany, Tong 2003*

*Auzzi, Bolognesi, Evslin, Konishi, Yung 2003*

*Shifman Yung 2004*

*Hanany Tong 2004*

$Z_N$ Abelian string: Flux directed in the Cartan subalgebra, say for $SO(3) = SU(2)/Z_2$

$$flux \sim \tau_3$$

Non-Abelian string : **Orientational zero modes**

Rotation of color flux inside SU(N).

## 2 Bulk theory

$\mathcal{N}=2$ QCD with gauge group $U(N)=SU(N)\times U(1)$ and

$N_f=N$ flavors of fundamental matter – quarks

$$+$$

Fayet-Iliopoulos term of U(1) factor

The bosonic part of the action

$$S = \int d^4x \left[\frac{1}{4g_2^2}\left(F^a_{\mu\nu}\right)^2 + \frac{1}{4g_1^2}\left(F_{\mu\nu}\right)^2 + \frac{1}{g_2^2}\left|D_\mu a^a\right|^2 + \frac{1}{g_1^2}\left|\partial_\mu a\right|^2 \right.$$
$$\left. + \left|\nabla_\mu q^A\right|^2 + \left|\nabla_\mu \bar{\tilde{q}}^A\right|^2 + V(q^A, \tilde{q}_A, a^a, a)\right].$$

Here

$$\nabla_\mu = \partial_\mu - \frac{i}{2}\, A_\mu - i A^a_\mu\, T^a\,.$$

The potential is

$$
\begin{aligned}
V(q^A, \tilde{q}_A, a^a, a) &= \frac{g_2^2}{2}\left(\frac{i}{g_2^2}\, f^{abc}\bar{a}^b a^c + \bar{q}_A\, T^a q^A - \tilde{q}_A T^a\, \bar{\tilde{q}}^A\right)^2 \\
&+ \frac{g_1^2}{8}\left(\bar{q}_A q^A - \tilde{q}_A \bar{\tilde{q}}^A - N\xi\right)^2 \\
&+ 2g_2^2 \left|\tilde{q}_A T^a q^A\right|^2 + \frac{g_1^2}{2}\left|\tilde{q}_A q^A\right|^2 \\
&+ \frac{1}{2}\sum_{A=1}^{N}\left\{\left|(a + \sqrt{2}m_A + 2T^a a^a)q^A\right|^2\right. \\
&+ \left.\left|(a + \sqrt{2}m_A + 2T^a a^a)\bar{\tilde{q}}^A\right|^2\right\} .
\end{aligned}
$$

## Vacuum

$$\langle\Phi\rangle = \langle\frac{1}{2}\,a + T^a\,a^a\rangle = -\frac{1}{\sqrt{2}}\begin{pmatrix} m_1 & \ldots & 0 \\ \ldots & \ldots & \ldots \\ 0 & \ldots & m_N \end{pmatrix},$$

For special choice

$$m_1 = m_2 = ... = m_N$$

U(N) gauge group is classically unbroken.

$$\langle q^{kA}\rangle \;=\; \sqrt{\xi}\begin{pmatrix} 1 & \ldots & 0 \\ \ldots & \ldots & \ldots \\ 0 & \ldots & 1 \end{pmatrix}, \qquad \langle\bar{\tilde{q}}^{kA}\rangle = 0,$$

$$k \;=\; 1, ..., N \qquad A = 1, ..., N\,,$$

# Note

- Color-flavor locking

  Both gauge $U(N)$ and flavor $SU(N)$ are broken, however diagonal $SU(N)_{C+F}$ is unbroken

$$\langle q \rangle \to U \langle q \rangle U^{-1}$$

$$\langle a \rangle \to U \langle a \rangle U^{-1}$$

- Two ways to make it valid in quantum regime:
  - $N_f > 2N$ The theory is not assymptotically free and stays at weak coupling (*Argyres, Plesser and Seiberg,1996*).
  - Another way to stay at weak coupling:

$$\sqrt{\xi} \gg \Lambda$$

$$\frac{8\pi^2}{g_2^2(\xi)} = N \log \frac{\sqrt{\xi}}{\Lambda} \gg 1$$

# 3 Non-Abelian strings

$Z_N$ string solution

$$q = \begin{pmatrix} \phi_2(r) & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & \phi_2(r) & 0 \\ 0 & 0 & \dots & e^{i\alpha}\phi_1(r) \end{pmatrix},$$

$$A_i^{\mathrm{SU}(N)} = \frac{1}{N} \begin{pmatrix} 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & -(N-1) \end{pmatrix} (\partial_i \alpha) \left[-1 + f_{NA}(r)\right],$$

$$A_i^{\mathrm{U}(1)} \equiv \frac{1}{2}\, A_i = \frac{1}{N}\, \left(\partial_i \alpha\right) \left[1 - f(r)\right]$$

Magnetic U(1) flux of this $Z_N$ string is

$$\int d^2 x F_{12} = \frac{4\pi}{N}$$

BPS string.

First order equations

$$r\frac{d}{dr}\,\phi_1(r) - \frac{1}{N}\,\left(f(r) + (N-1)f_{NA}(r)\right)\phi_1(r) = 0\,,$$

$$r\frac{d}{dr}\,\phi_2(r) - \frac{1}{N}\,\left(f(r) - f_{NA}(r)\right)\phi_2(r) = 0\,,$$

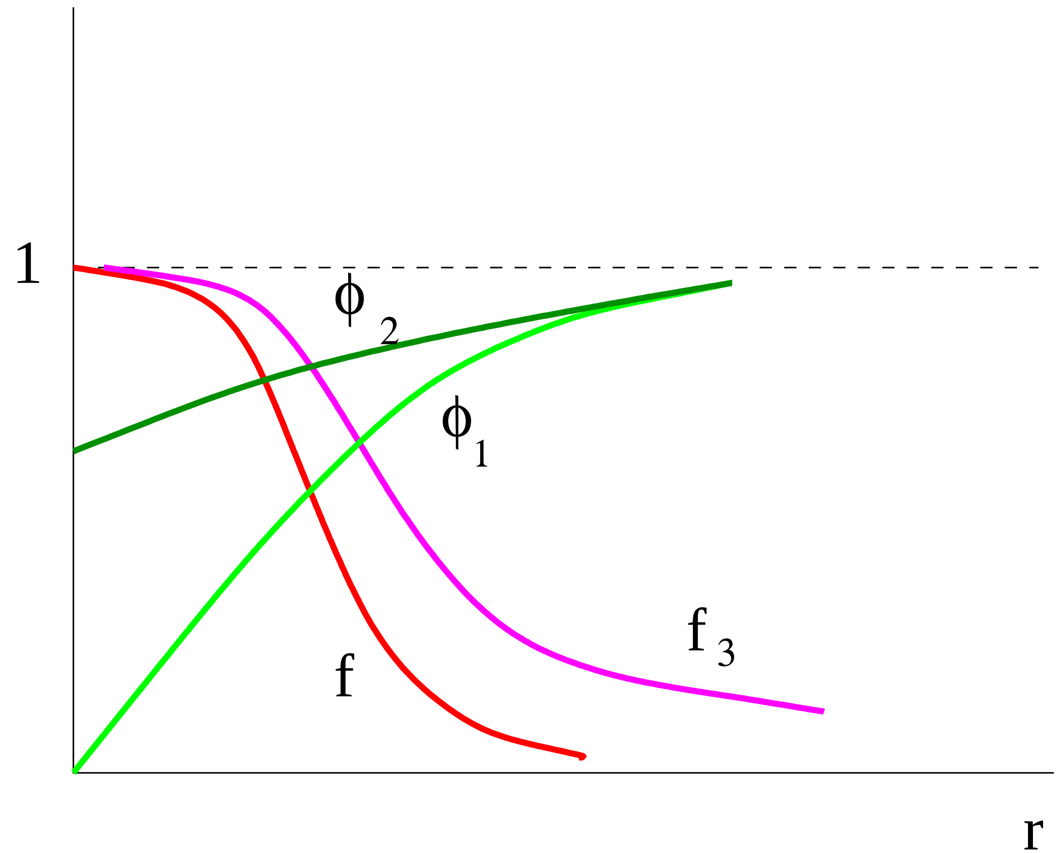
$$-\frac{1}{r}\,\frac{d}{dr}f(r) + \frac{g_1^2 N}{4}\,\left[(N-1)\phi_2(r)^2 + \phi_1(r)^2 - N\xi\right] = 0\,,$$

$$-\frac{1}{r}\,\frac{d}{dr}f_{NA}(r) + \frac{g_2^2}{2}\,\left[\phi_1(r)^2 - \phi_2(r)^2\right] = 0\,.$$

Tension of the elementary $Z_N$ string

$$T = 2\pi\,\xi$$

Profile functions of the string (for $N = 2$)



## Non-Abelian string

$$\frac{1}{N}\left\{U\begin{pmatrix} 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & -(N-1) \end{pmatrix}U^{-1}\right\}_p^l = -n^l n_p^* + \frac{1}{N}\delta_p^l \ ,$$

with

$$n_l^* n^l = 1$$

Then

$$\begin{aligned} q &= \frac{1}{N}[(N-1)\phi_2 + \phi_1] + (\phi_1 - \phi_2)\left(n \cdot n^* - \frac{1}{N}\right), \\ A_i^{\mathrm{SU}(N)} &= \left(n \cdot n^* - \frac{1}{N}\right)\varepsilon_{ij}\,\frac{x_j}{r^2}\,f_{NA}(r)\,, \\ A_i^{\mathrm{U}(1)} &= \frac{1}{N}\varepsilon_{ij}\,\frac{x_j}{r^2}\,f(r)\,, \end{aligned}$$

## 4 $CP(N)$ model on the string

String moduli: $x_{0i}$, $i = 1, 2$ and $n^l$, $l = 1, ..., N$

Make them $t, z$-dependent

$Z_N$ solution breaks $SU(N)_{C+F}$ down to $SU(N-1) \times U(1)$ Thus the orientational moduli space is

$$\frac{\mathrm{SU}(N)}{\mathrm{SU}(N-1) \times \mathrm{U}(1)} \sim \mathrm{CP}(N-1)$$

$$S^{(1+1)} = 2\beta \int dt\, dz\, \left\{ (\partial_k\, n^* \partial_k\, n) + (n^* \partial_k\, n)^2 \right\}$$

where the coupling constant $\beta$ is given by a normalizing integral

$$\beta = \frac{2\pi}{g_2^2} \int_0^\infty dr \left\{ -\frac{d}{dr} f_{NA} + \left( \frac{2}{r}\, f_{NA}^2 + \frac{d}{dr} f_{NA} \right) \frac{\phi_1^2}{\phi_2^2} \right\} = \frac{2\pi}{g_2^2}$$

Gauge theory formulation of $CP(N-1)$ model

$$S_{CP(N-1)} = 2\beta \int d^2x |\nabla_k n^l|^2,$$

where

$$\nabla_k = \partial_k - iA_k$$

$N$ complex fields $\qquad n^l, \qquad l = 1, ..., N$

Constraint:

$$|n^l|^2 = 1.$$

Gauge field can be eliminated:

$$A_k = -\frac{i}{2}\, \bar{n}_l \overset{\leftrightarrow}{\partial_k} n^l$$

Number of degrees of freedom $= 2N - 1 - 1 = 2(N-1)$

## 5 Confined monopoles

Classical picture

$Z_N$ Abelian strings $\Longleftrightarrow$ $N$ classical vacua of $CP(N-1)$ model

$$A_i^{\mathrm{SU}(N)} = \left(n \cdot n^* - \frac{1}{N}\right) \varepsilon_{ij} \frac{x_j}{r^2} f_{NA}(r)$$

$CP(N-1)$ classical vacua: $n^l = \delta^{l l_0}$

Higgs phase for quarks $\Longrightarrow$ confinement of monopoles

Elementary monopoles – junctions of two $Z_N$ strings

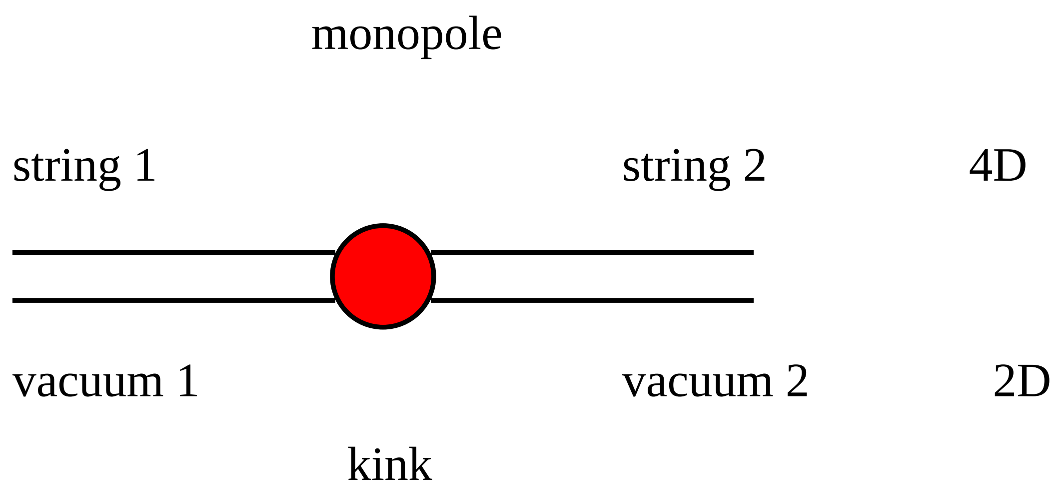
monopole flux $= 4\pi \times \mathrm{diag}\frac{1}{2}\{...0, 1, -1, 0, ...\}$

In 2D $CP(N-1)$ model on the string we have

$N$ vacua $= N$ $Z_N$ strings

and kinks interpolating between these vacua

# Kinks = confined monopoles



## Quantum picture

Non-Abelian limit $m_1 = m_2 = ... = m_N$

$$\langle \Phi \rangle = \langle \frac{1}{2}\, a + T^a\, a^a \rangle = -\frac{1}{\sqrt{2}} \begin{pmatrix} m_1 & \ldots & 0 \\ \ldots & \ldots & \ldots \\ 0 & \ldots & m_N \end{pmatrix},$$

t'Hooft-Polyakov unconfined monopoles:

$$M_{\text{monopole}} = \frac{4\pi |m_{l_0+1} - m_{l_0}|}{g_2^2} \to 0$$

$$\text{monopole size} \sim 1/m_W \sim 1/|m_{l_0+1} - m_{l_0}| \to \infty$$

Classically monopole disappear

Confined monopoles = kinks

are stabilized by quantum (non-perturbative) effects in $CP(N-1)$ model on the string worldsheet

Classically $n^l$ develop VEV ($|n|^2 = 1$)

There are $2(N-1)$ massless Goldstone states.

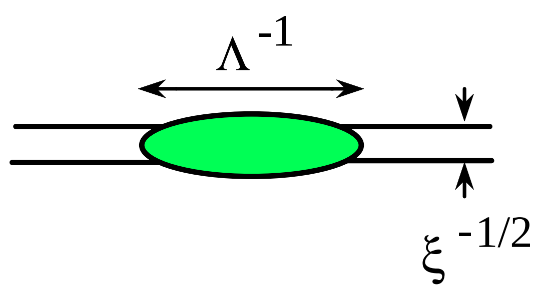
In quantum theory this does not happen

$SU(N)_{C+F}$ global symmetry is unbroken

Mass gap $\sim \Lambda_{CP}$ no massless states ($\langle |n|^2 \rangle = 0$)

$$M_{\text{monopole}} = M_{\text{kink}} \sim \Lambda_{CP}$$

$$\text{monopole size} \sim \Lambda_{CP}^{-1}$$



# 6 Less supersymmetry

- bulk $\mathcal{N} = 2$ SUSY $\Longrightarrow$ $\mathcal{N} = (2, 2)$ $CP(N-1)$ on the string

  *Hanany, Tong 2003*

  *Auzzi, Bolognesi, Evslin, Konishi, Yung 2003*

  *Shifman, Yung 2004*

  *Hanany, Tong 2004*

- bulk $\mathcal{N} = 1$ SUSY $\Longrightarrow$ $\mathcal{N} = (0, 2)$ hetrotic $CP(N-1)$ on the string

  *Edalati, Tong 2007*

  *Tong 2007*

  *Shifman, Yung 2008*

  *Shifman, Yung 2008*

- bulk non-SUSY $\Longrightarrow$ non-SUSY $CP(N-1)$ on the string

  *Gorsky, Shifman, Yung 2004*

  *Gorsky, Shifman, Yung 2005*

# 7 Large N solutions

*Witten 1979*: solved $\mathcal{N}=(2,2)$ and non-SUSY $CP(N-1)$ in large $N$ appreoximation

*Shifman, Yung 2008*: generalized Witten's solution to $\mathcal{N}=(0,2)$ $CP(N-1)$

$\boxed{\mathcal{N}=2\text{ Bulk}} \Longrightarrow \boxed{\mathcal{N}=(2,2)\ CP(N-1)}$

Consider limit $e^2 \to \infty$ in

$$
\begin{aligned}
S_{1+1} &= \int d^2x \left\{ |\nabla_k n^l|^2 + \frac{1}{4e^2}F_{kl}^2 + \frac{1}{e^2}|\partial_k \sigma|^2 + \frac{1}{2e^2}D^2 \right. \\
&+ \left. 2|\sigma|^2|n^l|^2 + iD(|n^l|^2 - 2\beta) + \text{fermions} \right\}
\end{aligned}
$$

Complex scalar $\sigma$, $A_k$ and $D$ form gauge multiplet.

The model has U(1) axial symmetry which is broken by the chiral anomaly down to discrete subgroup $Z_{2N}$ (*Witten 1979*). The field $\sigma$ which is related to the fermion bilinear operator transforms under this symmetry as

$$\sigma \rightarrow e^{\frac{2\pi k}{N}\, i}\sigma, \qquad k = 1, ..., N-1.$$

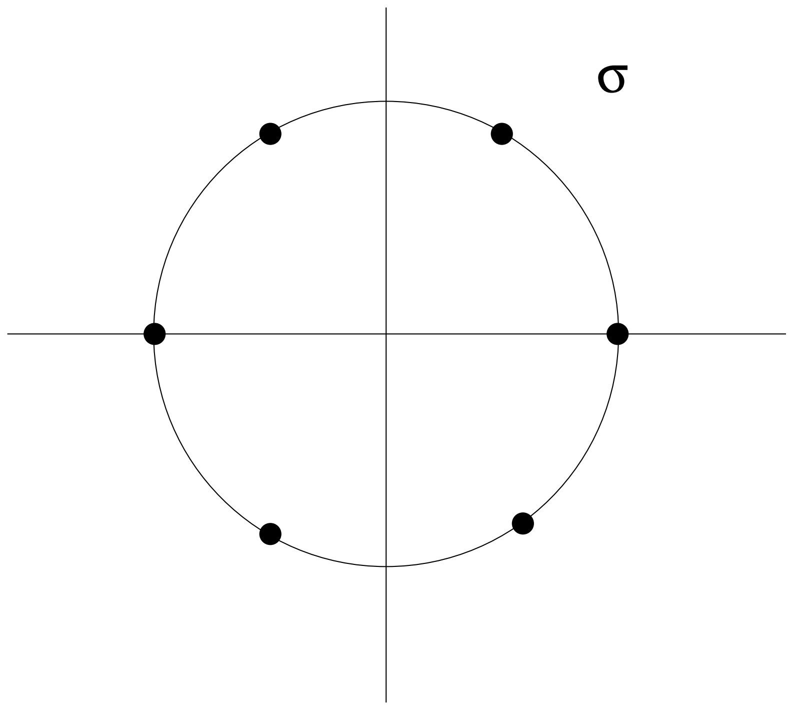
$Z_{2N}$ symmetry is spontaneously broken by the condensation of $\sigma$ down to $Z_2$,

Witten's solution:

$E_{vac} = D = 0 \qquad \Longrightarrow \mathcal{N} = (2,2)$ SUSY is unbroken

$$\sqrt{2}\langle\sigma\rangle = \Lambda_{CP}\, e^{\frac{2\pi k}{N}\, i} \qquad k = 0, ..., N-1.$$

There are $N$ strictly degenerate vacua




$$\sigma \sim \bar{\psi}_L \psi_R$$

$\mathcal{N}=1$ Bulk $\Longrightarrow$ $\mathcal{N}=(0,2)\ CP(N-1)$

Bulk:

$$\mathcal{W}_{3+1}=\frac{\mu}{2}\left[\mathcal{A}^2+(\mathcal{A}^a)^2\right],$$

String:

Consider limit $e^2\to\infty$ in

$$S_{1+1}=\int d^2x\left\{|\nabla_k n^l|^2+\frac{1}{4e^2}F_{kl}^2+\frac{1}{e^2}|\partial_k\sigma|^2+\frac{1}{2e^2}D^2\right.$$
$$\left.+\ 2|\sigma|^2|n^l|^2+iD(|n^l|^2-2\beta)+\frac{N}{2\pi}\,u\,|\sigma|^2+\text{fermions}\right\}$$

$u$ is the deformation parameter $\mathcal{N}=(2,2)\to\mathcal{N}=(0,2)$ ,

$$u\ =\left[\begin{array}{ll}\text{const}\ \beta\ \frac{g_2^4|\mu|^2}{m_W^2}, & \text{small}\quad\mu\\ \text{const}\ \beta\ \log\frac{g_2^2|\mu|}{m_W}, & \text{large}\quad\mu\end{array}\right.$$

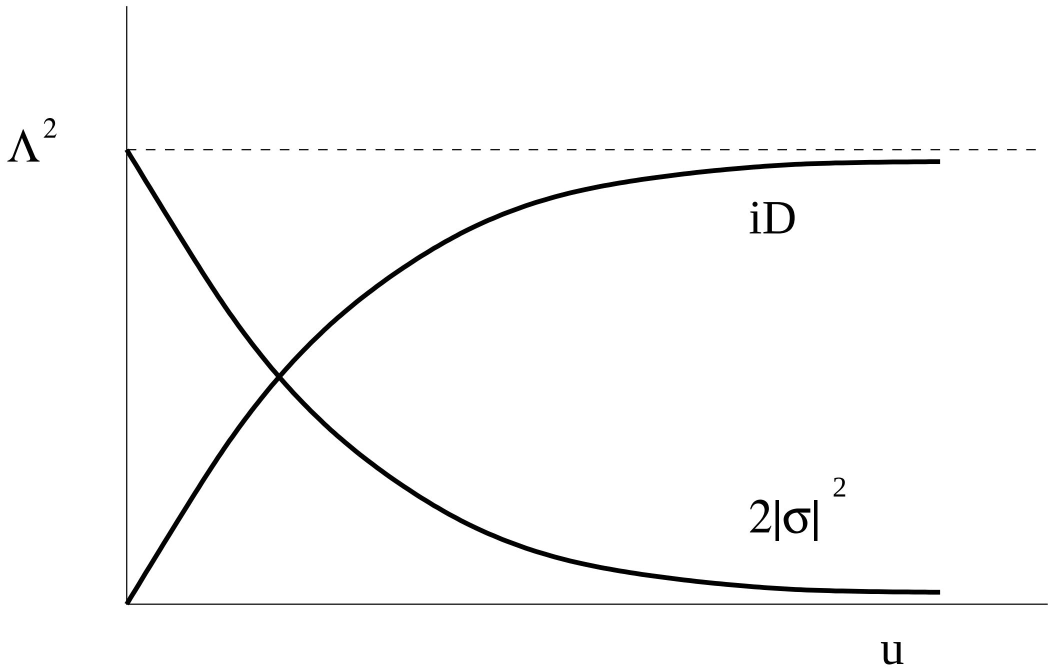
Solution:

$$2|\sigma|^2 = \Lambda^2 \, e^{-u} \, ,$$

$$iD = \Lambda^2 \, \left(1 - e^{-u}\right) \, .$$

Vacuum energy

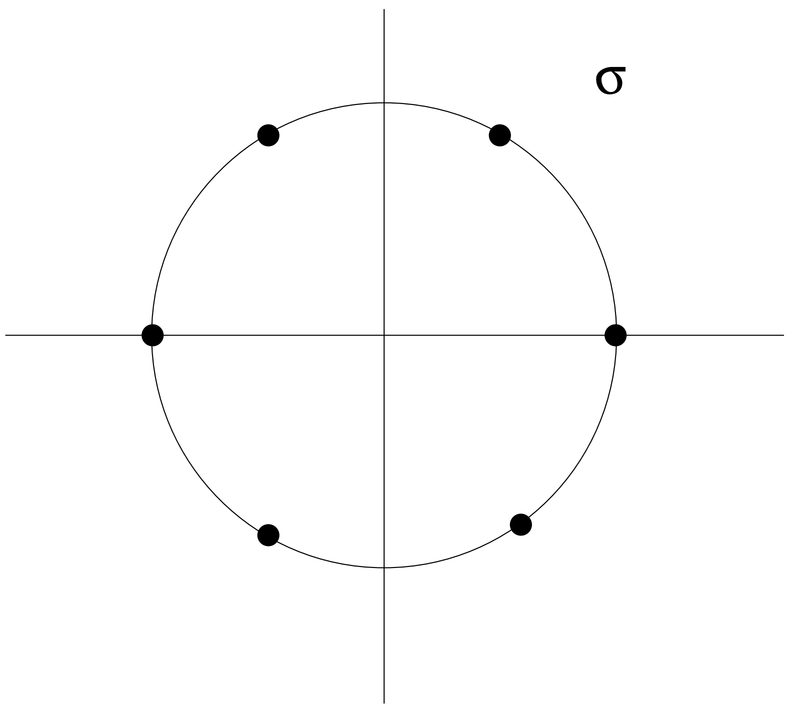
$$E_{vac} = \frac{N}{4\pi} \, iD = \frac{N}{4\pi} \, \Lambda^2 \, \left(1 - e^{-u}\right) \, .$$




SUSY is broken spontaneously

$$\sqrt{2}\langle\sigma\rangle = \Lambda\, e^{\frac{2\pi k}{N}\, i}\, e^{-u/2} \qquad k = 0, ..., N-1.$$

There are $N$ strictly degenerate vacua



non-SUSY $CP(N-1)$

$$\sigma = 0$$

N vacua split

$$E_{vac} = \text{const}\, N\, \Lambda_{CP}^2 \left\{ 1 + \text{const} \left( \frac{2\pi k}{N} \right)^2 \right\}$$

# 8 Kink deconfinement vs confinement

Compare large $N$ solutions of $\mathcal{N} = (2,2)$ , $\mathcal{N} = (0,2)$ and non-SUSY models

**Common features:**

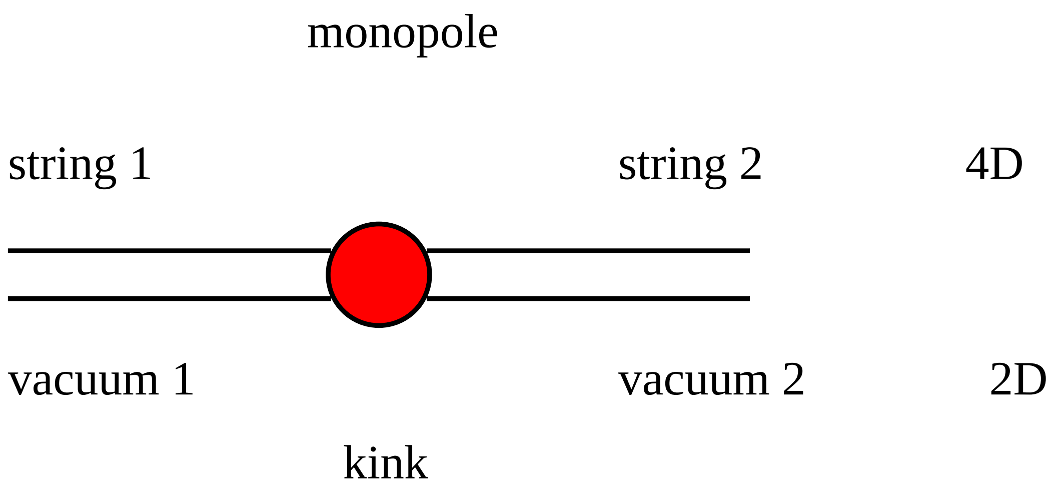
- $n^l$ ( and superpartners) acquire mass $\sim \Lambda$
- $\langle |n^l|^2 \rangle = 0$
- $A_k$, $\sigma$ (+ fermions) become dynamical (kinetic energy is generated)

**Distinctions:**

- $\mathcal{N} = (2,2)$ : SUSY unbroken
- $\mathcal{N} = (0,2)$ : SUSY spontaneously broken
- $\mathcal{N} = (2,2)$ and $\mathcal{N} = (0,2)$ : $N$ degenerate vacua, $Z_{2N} \to Z_2$
  non-SUSY $CP(N-1)$: $N$ vacua are split

# Kinks = confined monopoles

$\mathcal{N} = (2,2)$ and $\mathcal{N} = (0,2)$ models :




*Witten 1979*:

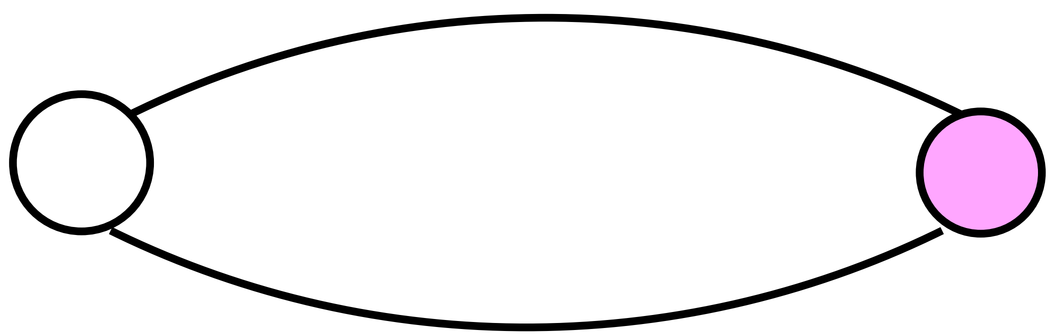
$$n^l = \text{kink}$$

$$n^l = \text{kink}$$

Means that kink acquire global flavor quantum numbers

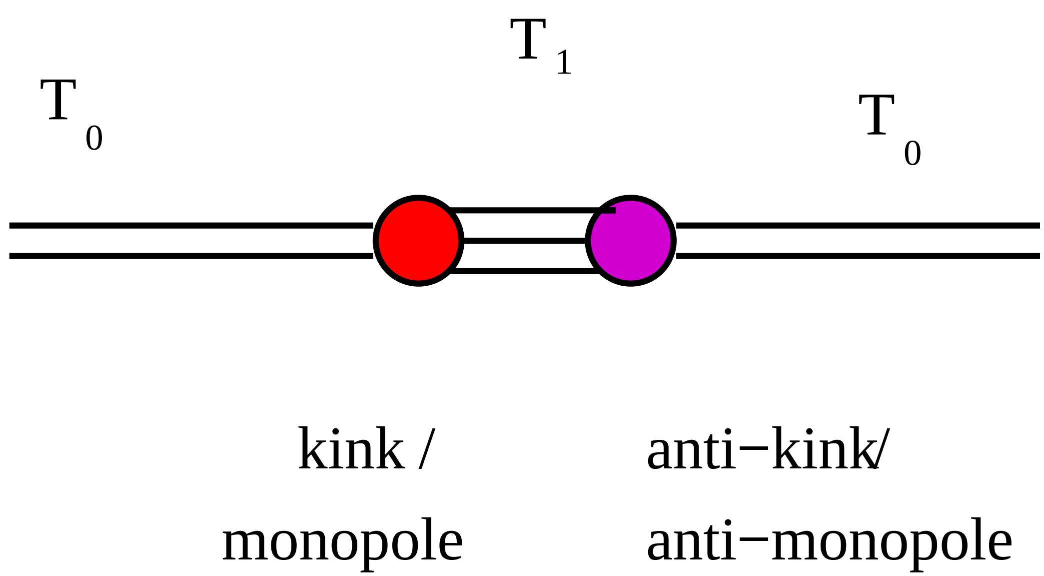
kinks are in fundamental of $\mathrm{SU}(N)_{C+F}$

Therefore monopole-antimonopole 'meson' can be singlet or adjoint of flavor $\mathrm{SU}(N)_{C+F}$



$\mathcal{N} = (2,2)$ and $\mathcal{N} = (0,2)$ models: U(1) gauge field $A_k$ is massive $\longrightarrow$ kink deconfinement

non-SUSY model: U(1) gauge field $A_k$ is massless $\longrightarrow$ kink confinement



**2D kink confinement = splitting of $N$ vacua**

$$T = 2\pi\xi + \text{const}\, N\, \Lambda_{CP}^2 \left\{ 1 + \text{const} \left( \frac{2\pi k}{N} \right)^2 \right\}$$

## 9 Conclusions

- Worldsheet internal dynamics of non-Abelian string in U(N) gauge theory with $N_f = N$ flavors is described by $CP(N-1)$ model
- $\mathcal{N} = 2$ Bulk $\Longrightarrow$ $\mathcal{N} = (2,2)$ $CP(N-1)$

  $\mathcal{N} = 1$ Bulk $\Longrightarrow$ $\mathcal{N} = (0,2)$ heterotic $CP(N-1)$

  non-SUSY Bulk $\Longrightarrow$ non-SUSY $CP(N-1)$
- Non-Abelian confined monopole = $CP(N-1)$ kink

  Stubilized by quantum dynamics on the string at the scale $\Lambda_{CP}$
- $\mathcal{N} = (2,2)$ and $\mathcal{N} = (0,2)$ : 2D deconfinement – monopoles can move freely along the string

  non-SUSY: 2D confinement – monopoles and antimonopoes form a 'meson'-like configuration on the string