Calorons and non-trivial holonomy - an overview

(in: Non-pertubative Methods in Strongly Coupled Gauge Theories)

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$$P_\infty \equiv \lim_{|\vec{x}|\to\infty} P(\vec{x});$$

β = periodicity in Euclidean time $(=T)$

$$P(\vec{x}) = P\exp\left(\int_0^\beta dt\, A_0(\vec{x},t)\right)$$

Non-trivial Holonomy $P_\infty \neq 1 \ (\notin Z_N)$

is like having a Higgs field breaking gauge symmetry spotaneously.

We considered case <u>without net</u> magnetic charge

$\longrightarrow$ topological charge integer

Gross, Pisarski, Yaffe, Rev. Mod. Phys. <u>53</u> ('81) 43.

There is a gauge such that

$$A_\mu(\vec{x}, t+\beta) = P_\infty A_\mu(\vec{x},t) P_\infty^{-1}$$

$$A_0(\vec{x},t) \to 0 \text{ for } |\vec{x}| \to \infty$$

("algebraic" gauge)

In periodic gauge $A_0(\vec{x},t) \to \text{const.} \neq 0$ for $|\vec{x}| \to \infty$

SU(2)

Harrington & Shepard, Phys. Rev. **D17** (1978) 2122.

Periodic (in time) array of (charge 1) instantons: Using 't Hooft ansatz

$$A_\mu(x) = \frac{i}{2}\bar{\eta}^a_{\mu\nu}\tau_a \partial_\nu \ln \phi(x) \qquad \frac{\Box\phi}{\phi}=0$$

$$\phi(x) = 1 + \sum_{n\in\mathbb{Z}} \frac{\rho^2}{(\vec{x}-\vec{a})^2+(t-a_0-n\beta)^2}$$

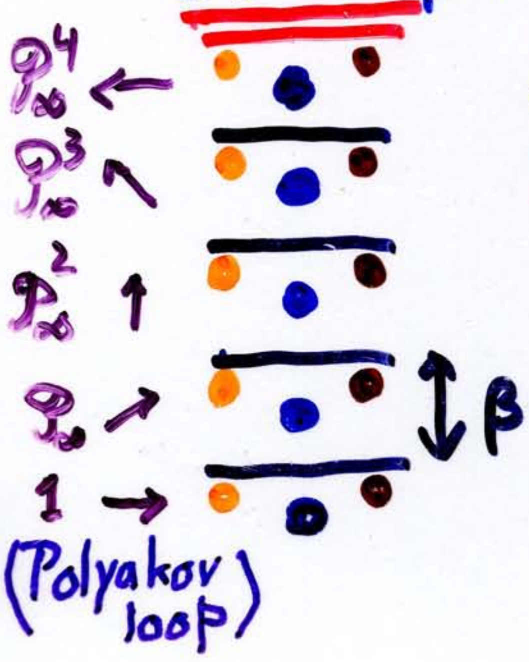
$$= 1 + \frac{\pi\rho^2}{r}\,\frac{\sinh(2\pi r)}{\cosh(2\pi r)-\cos(2\pi t)} \qquad r=|\vec{x}|$$

$(\beta=1)$
$(a_\mu=0)$

For large ρ this approaches a BPS monopole (in singular gauge)

P. Rossi, Nucl. Phys. B149 ('79) 170

Overlap: For large ρ scale is set by β (= distance between periodic copies)



What happens when periodic copies are relative gauge rotated?

$\beta = \rho = 1$

$\pi\rho^2 = |\vec{y}_1 - \vec{y}_2|\beta$

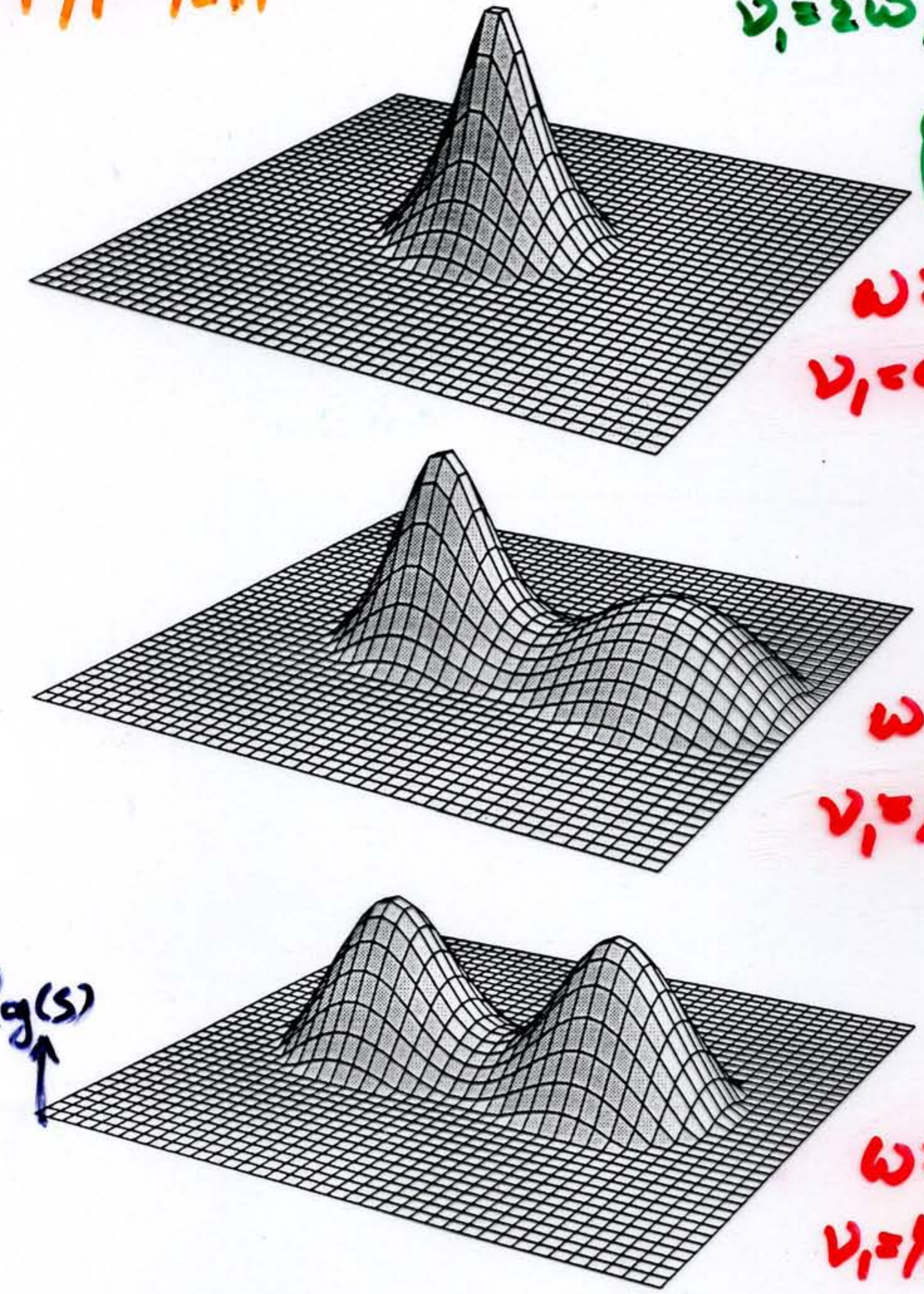
$P_{\vec{\omega}} = \exp(2\pi i \vec{\omega}\cdot\vec{\tau})$

$\omega \equiv |\vec{\omega}|$

$\nu_1 = 2\omega,\ \nu_2 = 1 - 2\omega = 2\bar{\omega}$

($\nu_i \hat{=}$ action fraction)

SU(2)



$\omega = 0$

$\nu_1 = 0,\ \nu_2 = 1$

$\omega = 1/8$

$\nu_1 = 1/4,\ \nu_2 = 3/4$

$\omega = 1/4$

$\nu_1 = 1/2,\ \nu_2 = 1/2$

Figure 1: Profiles for calorons at $\omega = 0,\ 0.125,\ 0.25$ (from top to bottom) with $\rho = 1$. The axis connecting the lumps, separated by a distance π (for $\omega \neq 0$), corresponds to the direction of $\hat{\omega}$. The other direction indicates the distance to this axis, making use of the axial symmetry of the solutions. Vertically is plotted the action density, at the time of its maximal value, on equal logarithmic scales for the three profiles. The profiles were cut off at an action density below $1/e$. The mass ratio of the two lumps is approximately $\omega/\bar{\omega}$, i.e. zero (no second lump), a third and one (equal masses), for the respective values of ω.

Density profiles from:

for $\underline{\underline{\text{all}}}$ ω

$$\boxed{\mathrm{tr}\, F_{\mu\nu}^2(x) = \Box^2 \ln \psi(x)}$$

Interpretation of solutions:

Two constituent BPS monopoles with opposite magnetic charges with masses $8\pi^2\nu_1/T$ and $8\pi^2\nu_2/T$ separated by a distance $\pi\rho^2/T$ along $\hat{\omega}$

Parameters: (Moduli) $\boxed{8}$ $\begin{cases} \zeta = \zeta_\mu \sigma_\mu = \rho g \\ a_\mu \end{cases}$ $\begin{cases} \rho = \text{scale} \\ g = \text{gauge} \end{cases}$

a_μ = position center-mass

For each $\vec{\omega}$ a different moduli space

$$\mathcal{M} = \underbrace{\mathbb{R}^3 \times S^1}_{a_\mu} \times \underbrace{\text{Taub-NUT}}_{\zeta} / \underbrace{Z_2}_{\zeta \leftrightarrow -\zeta} \qquad \boxed{M^2 = 1/16\omega\bar{\omega}}$$

Also understood from Nahm transformation for monopoles: W. Nahm, Lect. Notes. Phys. <u>201</u> ('84) K. Lee, hep-th/9802012; K. Lee & C. Lu hep-th/9802108.

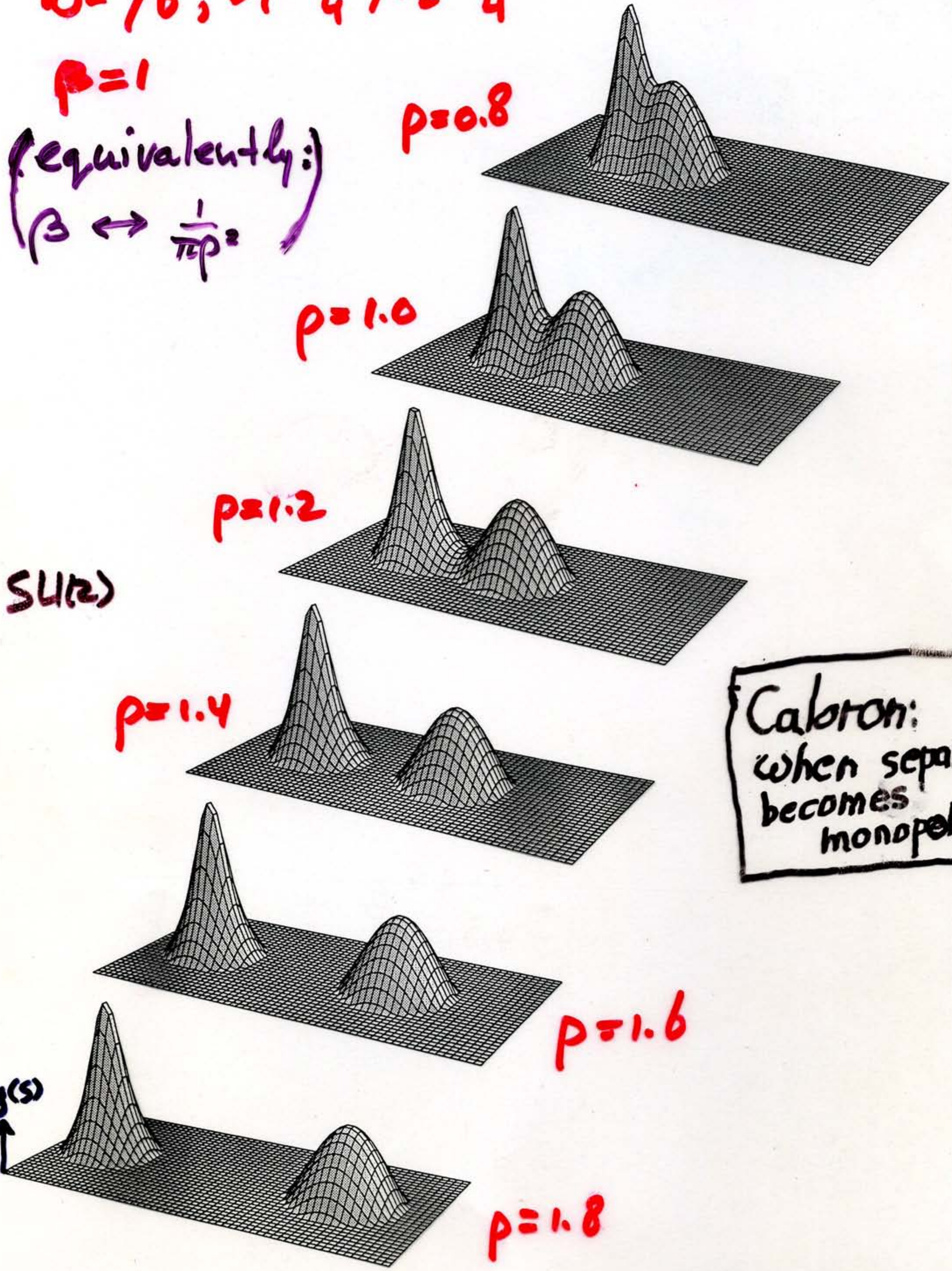
$\omega = 1/8\ ,\ \nu_1 = \frac{1}{4}\ ,\ \nu_2 = \frac{3}{4}$

$P_\infty = \exp(2\pi i \omega \tau_3)$

$\beta = 1$

(equivalently:
$\beta \leftrightarrow \frac{1}{\pi\rho^2}$)

SU(2)



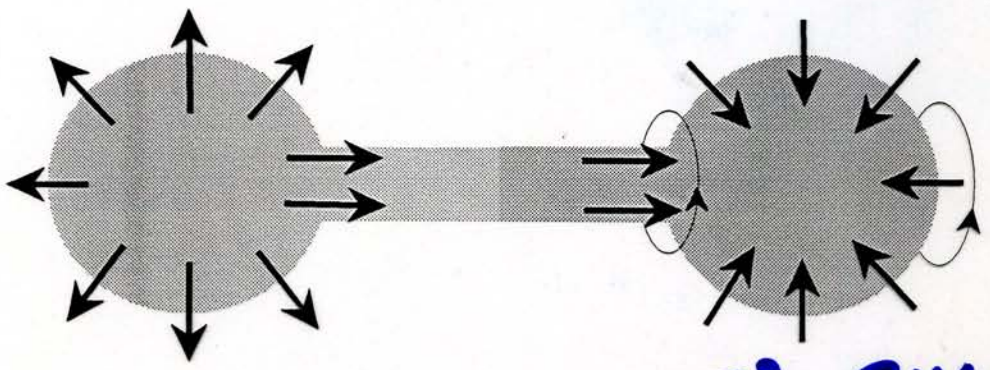
Calorons:
when separ.
becomes
monopoles

$\omega = 1/8$; cutoff $1/e^2$

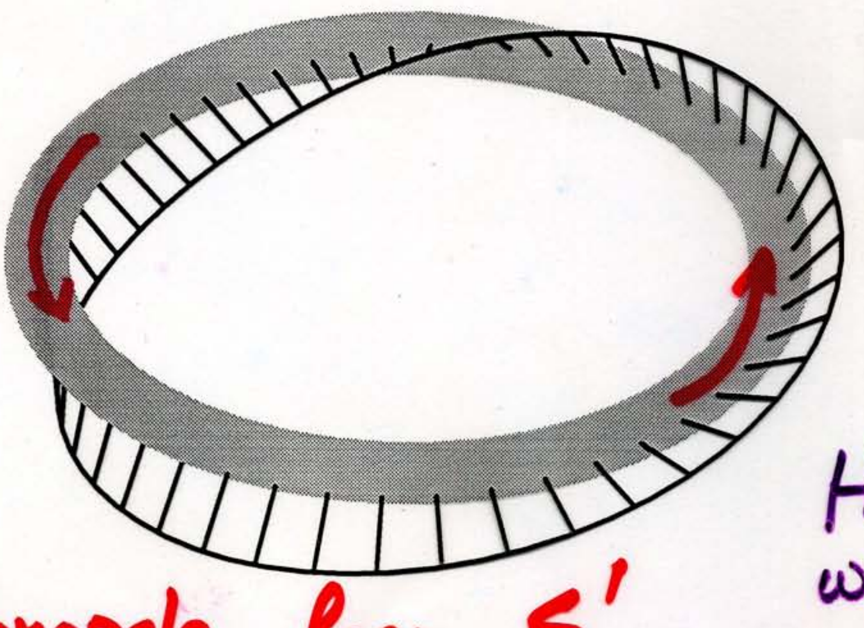
$\rho = 0.8, 1, 1.2, 1.4, 1.6, 1.8$

C. Taubes ('82)

See: Cargèse '83 – Plenum '84, p. 563



$S^2 = SU(2)/U(1)$



Hopf map winding # 1

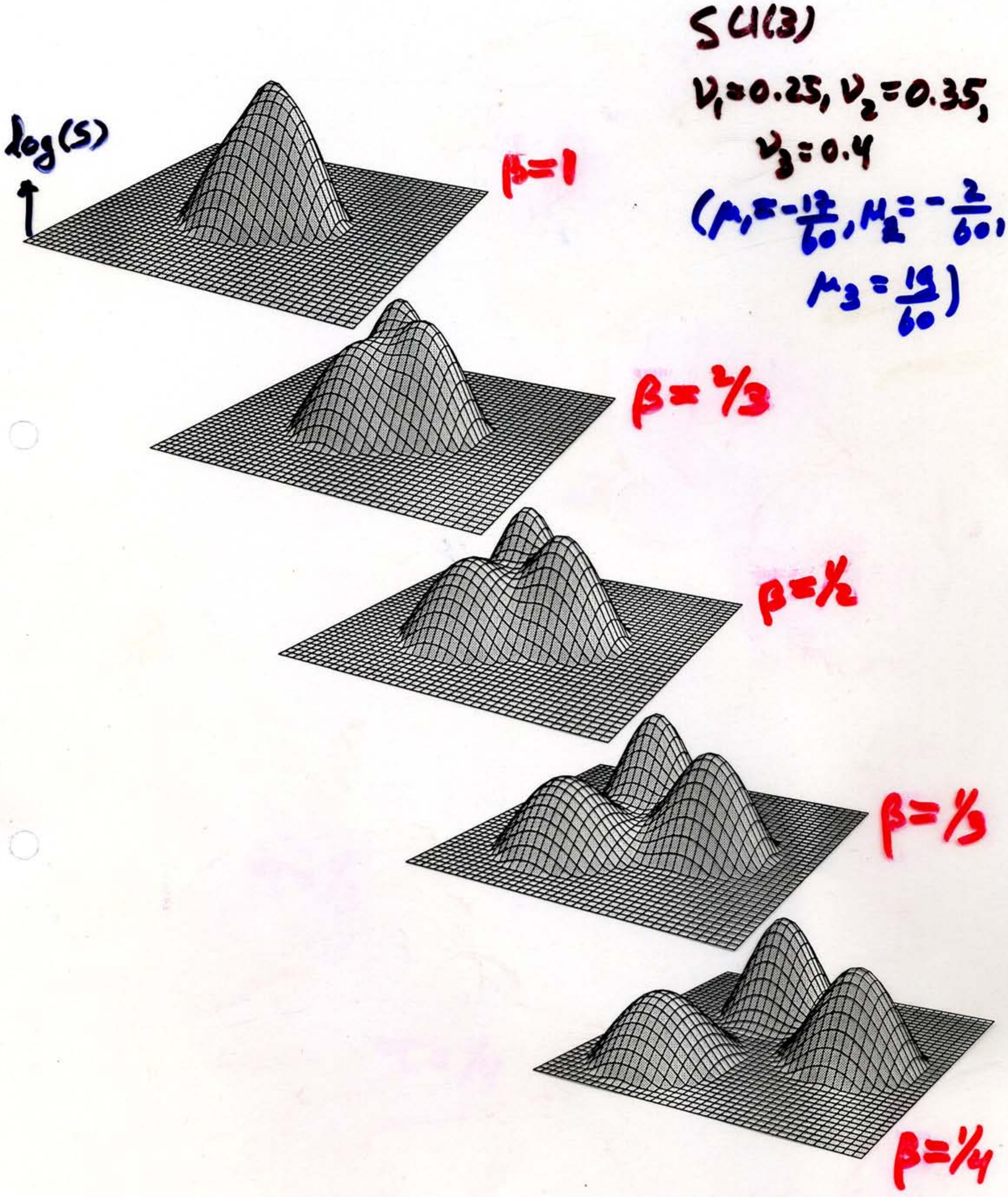
Monopole loop S^1

"Frame" $SU(2)/U(1) = S^2$

$\Rightarrow S^1 \times_{\text{twisted}} S^2 = S^3$

Made more precise by O. Jahn*

*) O. Jahn, J. Phys. A33 (2000) 2997



SU(3)

T = 1, 1.5, 2, 3, 4 | cutoff 1/e |

$\underline{\underline{SU(n)}}$

$\beta = 1$, top. ch. $= 1$

$$\mathrm{tr}\, F_{\mu\nu}^2(x) = \partial_\mu^2 \partial_\nu^2 \log \psi(x)$$

$$\psi(x) = \tfrac{1}{2}\,\mathrm{tr}\left(A_n A_{n-1} \cdots A_1\right) - \cos(2\pi x_0)$$

$$A_m = \begin{pmatrix} r_m & |\vec{y}_m - \vec{y}_{m+1}| \\ 0 & r_{m+1} \end{pmatrix} \begin{pmatrix} c_m & s_m \\ s_m & c_m \end{pmatrix} \frac{1}{r_m}$$

$$r_m = |\vec{x} - \vec{y}_m|$$

$$c_m = \cosh(2\pi \boxed{\nu_m} r_m)$$

$$s_m = \sinh(2\pi \boxed{\nu_m} r_m)$$

$SU(2)$:

$$|\vec{y}_1 - \vec{y}_2| = \pi \rho^2$$

$$\mu_1 = -\omega,\ \mu_2 = \omega$$

$$P(\vec{x}) = \mathrm{P}\exp \int_0^\beta dx_0\, A_0(\vec{x}; x_0)$$

$$\xrightarrow[|\vec{x}| \to \infty]{} P_\infty = \exp\left[2\pi i\, \mathrm{diag}(\mu_1 \cdots \mu_n)\right]$$

$$\sum \mu_i = 0 \qquad \mu_1 \leq \mu_2 \cdots \leq \mu_n \leq \mu_{n+1} \equiv \mu_1 + 1$$

$$\nu_m \equiv \mu_{m+1} - \mu_m \longrightarrow \text{Mass} = 8\pi^2 \nu_m / \beta$$

$$\boxed{\sum_m \nu_m = 1 \longleftrightarrow S = 8\pi^2}$$

Fermion zero-mode $\qquad SU(n)$

$$D\Psi = \bar\sigma_\mu(\partial_\mu + A_\mu)\Psi \,(= 0)$$

$$\bar\sigma_\mu = (1_2; -i\vec\tau) \quad \Psi(t+\beta, \vec x) = e^{2\pi i z}\Psi(t;\vec x) \qquad \boxed{\beta = 1}$$

$z = 1/2$ anti-periodic | $z = 0$ periodic

$$|\Psi(x)|^2 = -\frac{1}{(2\pi)^2}\partial_\mu^2 \hat f_x(z,z)$$

for $z \in [\mu_m, \mu_{m+1}]$

$$\hat f_x(z,z) = \frac{\pi \langle v_m(z)|A_{m-1}\cdots A_1 A_n \cdots A_m|w_m(z)\rangle}{r_m \psi(x)}$$

$$V_m^1(z) = -W_m^2(z) = \sinh(2\pi(z-\mu_m)r_m)$$
$$V_m^2(z) = W_m^1(z) = \cosh(2\pi(z-\mu_m)r_m)$$

When $|\vec y_m - \vec y_{m-1}| \to \infty$ for all m

$$\hat f_x(z,z) \to \frac{\sinh(2\pi(z-\mu_m)r_m)\sinh(2\pi(z-\mu_{m+1})r_m)}{(2\pi)^{-1} r_m \sinh(2\pi \nu_m r_m)}$$

$\nu_m \to \mu_{m+1} - \mu_m$

For $SU(2)$ this gives:

$$\hat f_x(z,z) \to \frac{\pi \tanh(\pi \nu_m r_m)}{r_m}$$

$z = 0 \to m = 1 \; ; \; z = \frac{1}{2} \to m = 2$

$\beta = 1$, $\rho = 1.2$
$\mu_1 = -\omega = -\mu_2 = -1/8$

anti-periodic

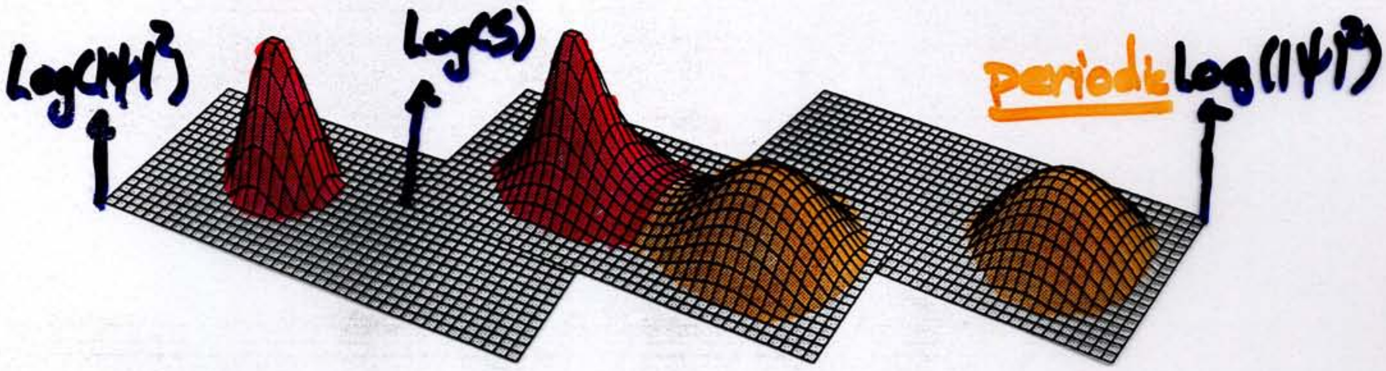


Figure 1: For the two figures on the sides we plot on the same scale the logarithm of the zero-mode densities (cutoff below $1/e^5$) for $\omega = 1/8$ (left Ψ^- / right Ψ^+) and $\omega = 3/8$ (right Ψ^- / left Ψ^+), with $\beta = 1$ and $\rho = 1.2$. In the middle figure we show for the same parameters (both choices of ω give the same action density) the logarithm of the action density (cutoff below $1/2e^2$).

$\nu_1 = 2\omega$, $\nu_2 = 1 - 2\omega$
$\mu_1 = -\omega$, $\mu_2 = \omega$

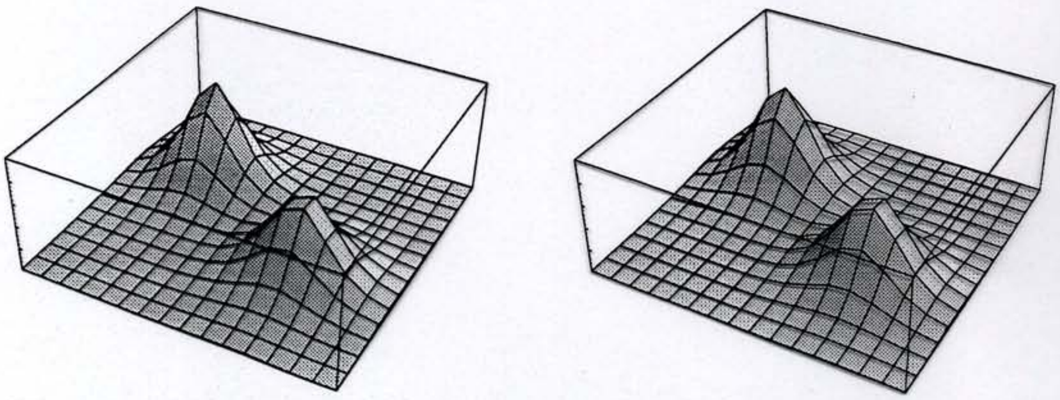


Figure 3: Zero-mode density profiles for the two zero-modes of the lattice caloron (left) on a 4×16^3 lattice for $\vec{k} = (1,1,1)$, created with improved cooling ($\varepsilon = 0$). The profiles fit well to the two zero-modes for the infinite volume analytic caloron solution (shown on the right at $y = t = 0$) with $\omega = \frac{1}{4}$ and constituents at $\vec{y}_1 = (2.50, 0.12, 0.95)$ and $\vec{y}_2 = (1.38, -0.24, 2.67)$, in units where $\beta = l_t = 1$ (or $a = \frac{1}{4}$) and the left most lattice point corresponding to $x = z = 0$. The plots give the added densities of the two zero-modes.

(with M. García Pérez, A. González-Arroyo
and C. Pena)
hep-th/9905016/9905138

adjoint!

For SUSY-YM each constituent monopole carries 2 gluino zero-modes. This saturates $\langle \lambda\lambda \rangle$ and resolves an old problem in computing the Witten index (N.M. Davies, T.J. Hollowood, V.V. Khoze, M.P. Mattis, hep-th/9905015. + NPB 559 (1999) 123)

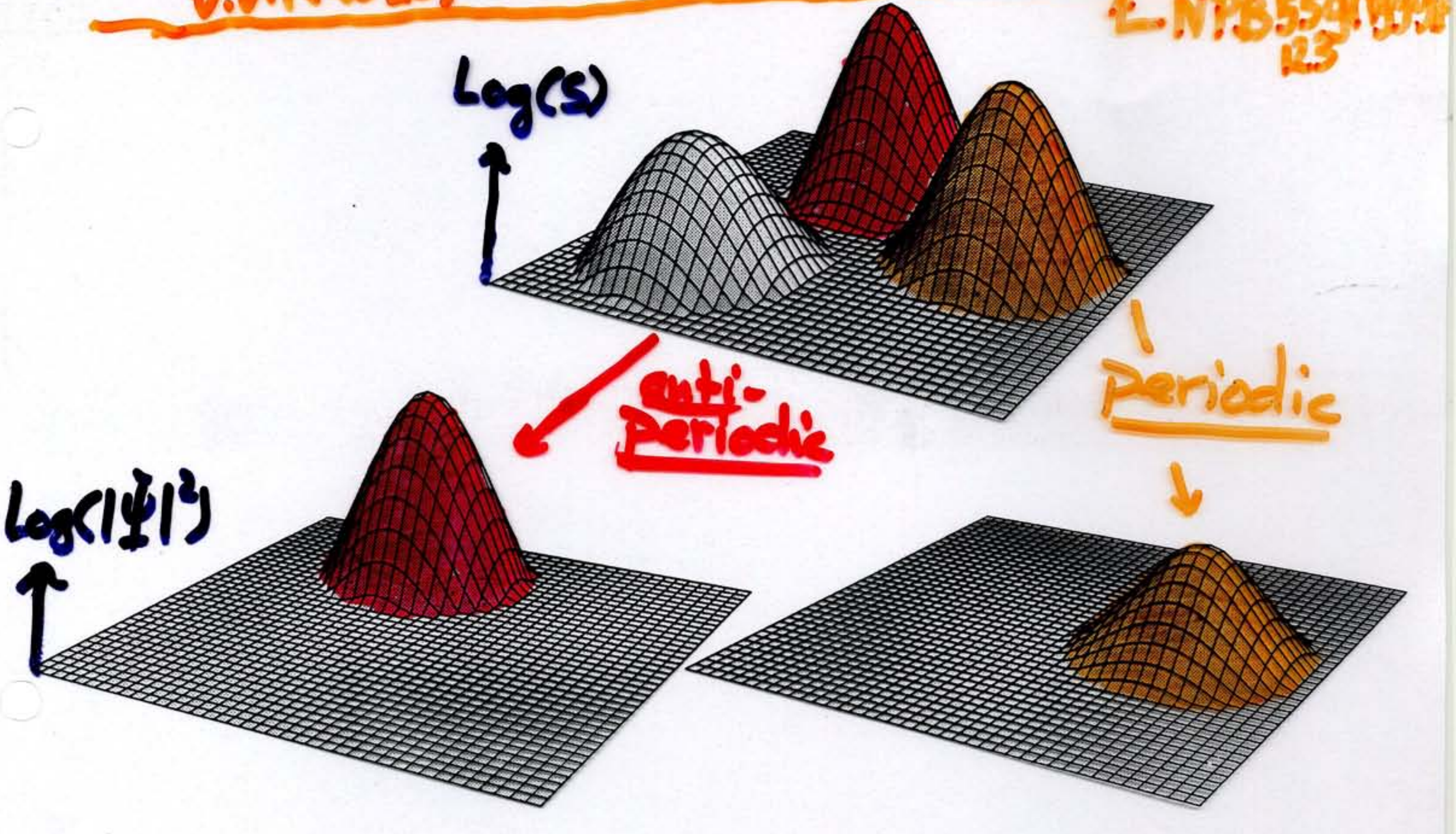


Figure 1: The the action densities (top) for the $SU(3)$ caloron, cut off at $1/(2e)$, on a logarithmic scale, with $(\mu_1, \mu_2, \mu_3) = (-17, -2, 19)/60$ for $t = 0$ in the plane defined by $\vec{y}_1 = (-2, -2, 0)$, $\vec{y}_2 = (0, 2, 0)$ and $\vec{y}_3 = (2, -1, 0)$, for $\beta = 1$, with masses $8\pi^2\nu_i$, $(\nu_1, \nu_2, \nu_3) = (0.25, 0.35, 0.4)$. On the bottom-left is shown the zero-mode density for fermions with anti-periodic boundary conditions in time and on the bottom-right for periodic boundary conditions, at equal logarithmic scales, cut off below $1/e^5$.

(with T. Kraan and M. Chernodub)

NPB (Proc. Suppl) 83-84 (2000) 556

Zero-mode periodic up to $\exp(-2\pi i z)$

$\psi_z(\vec{x}, t+\beta) = e^{-2\pi i z}\psi_z(\vec{x}, t)$

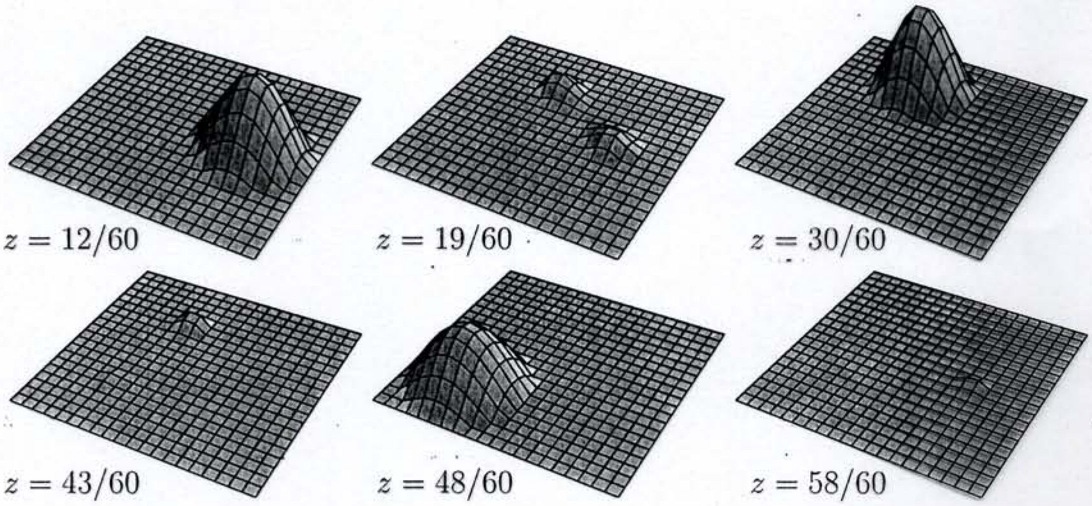


Figure 1: The logarithm of the properly normalized zero-mode density for a typical $SU(3)$ caloron of charge 1, cycling through z. Shown are $z = \mu_j$ (for linear plots see Fig. 2) and three values of z roughly in the middle of each interval $z \in [\mu_j, \mu_{j+1}]$. All plots are on the same scale, cutoff for values of the logarithm below -5. The zero-mode with anti-periodic boundary conditions is found at $z = 30/60$. For the action density of the associated gauge field, see Ref. [13]. See also Ref. [19].

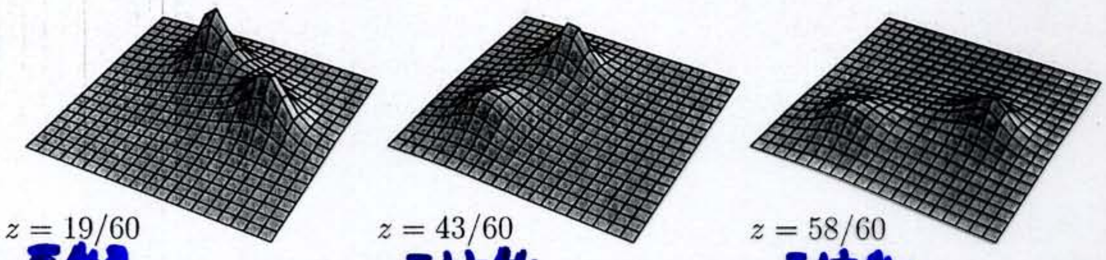
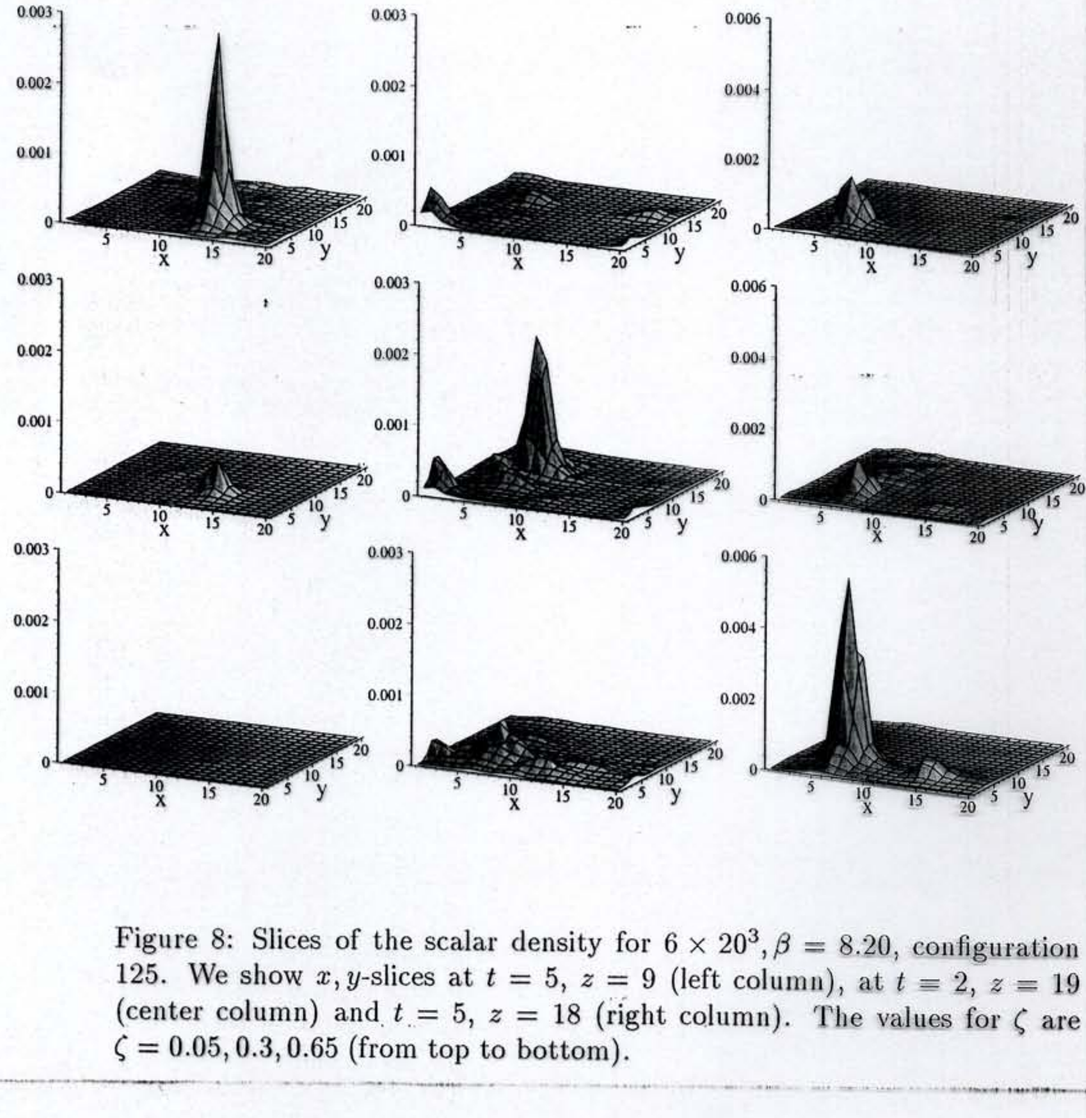


Figure 2: The properly normalized delocalized zero-mode densities for $z = \mu_j$, on the same *linear* scale (cmp. Fig. 1).



ζ=0.05

ζ=0.3

ζ=0.65

Figure 8: Slices of the scalar density for $6 \times 20^3, \beta = 8.20$, configuration 125. We show x, y-slices at $t = 5$, $z = 9$ (left column), at $t = 2$, $z = 19$ (center column) and $t = 5$, $z = 18$ (right column). The values for ζ are $\zeta = 0.05, 0.3, 0.65$ (from top to bottom).

From: C. Gattringer, S. Schaefer,
Nucl. Phys. B654 (2003) 30, hep-lat/0212029

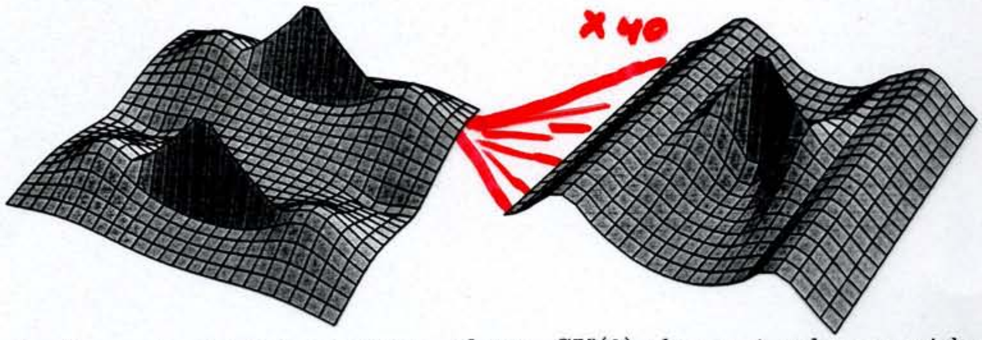


Figure 1: Approximate superposition of two $SU(2)$ charge 1 calorons with its pairs of equal mass constituents at $\vec{x} = (2,0,2),\ (2,0,8)$ and $\vec{x} = (8,0,2),\ (8,0,8)$. The logarithm of the action density is plotted as a function of x and z. The plot on the right shows one of the would-be Dirac strings, zooming in by a factor 40 on the transverse direction.

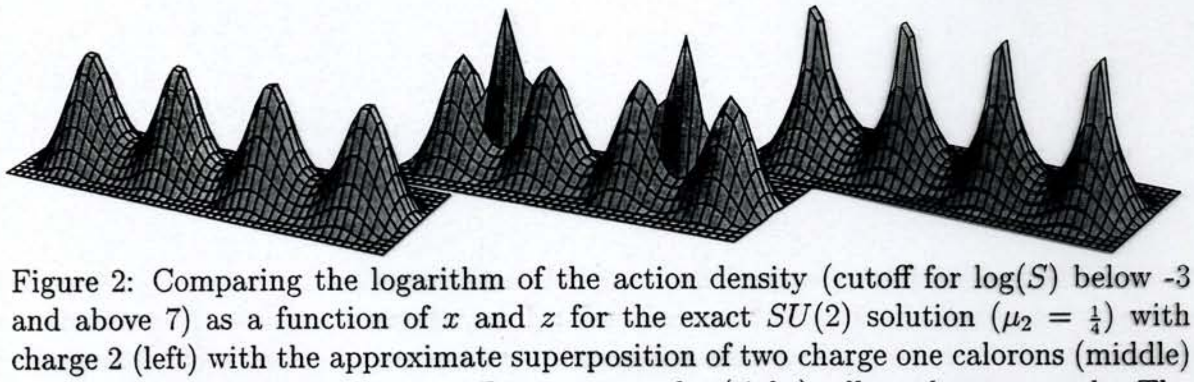


Figure 2: Comparing the logarithm of the action density (cutoff for $\log(S)$ below -3 and above 7) as a function of x and z for the exact $SU(2)$ solution ($\mu_2 = \frac{1}{4}$) with charge 2 (left) with the approximate superposition of two charge one calorons (middle) and the abelian solution based on Dirac monopoles (right), all on the same scale. The two pairs of constituent monopoles are located at $\vec{x} = (0,0,6.031),\ (0,0,2.031)$ and $\vec{x} = (0,0,-2.031),\ (0,0,-6.031)$.

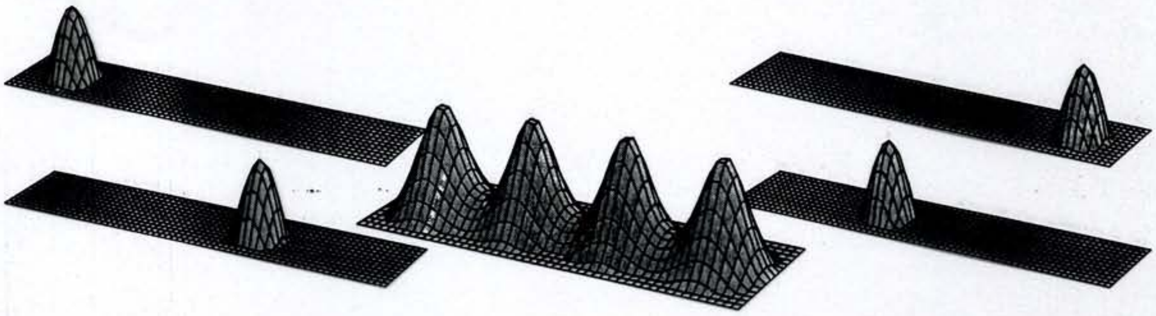
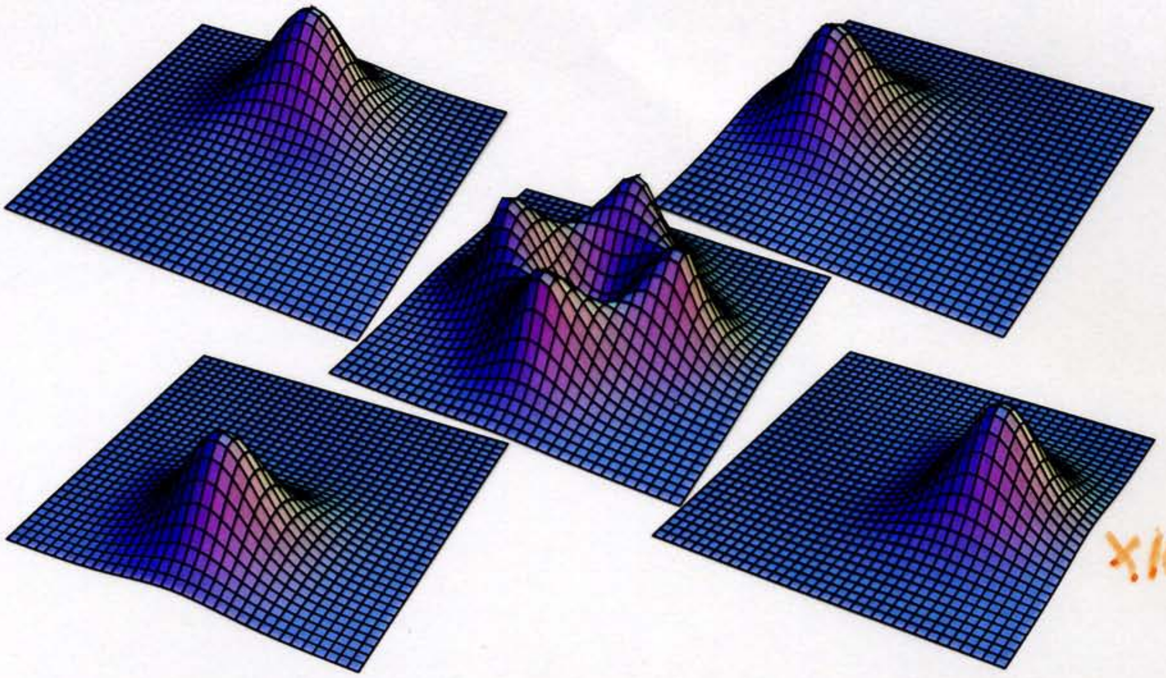
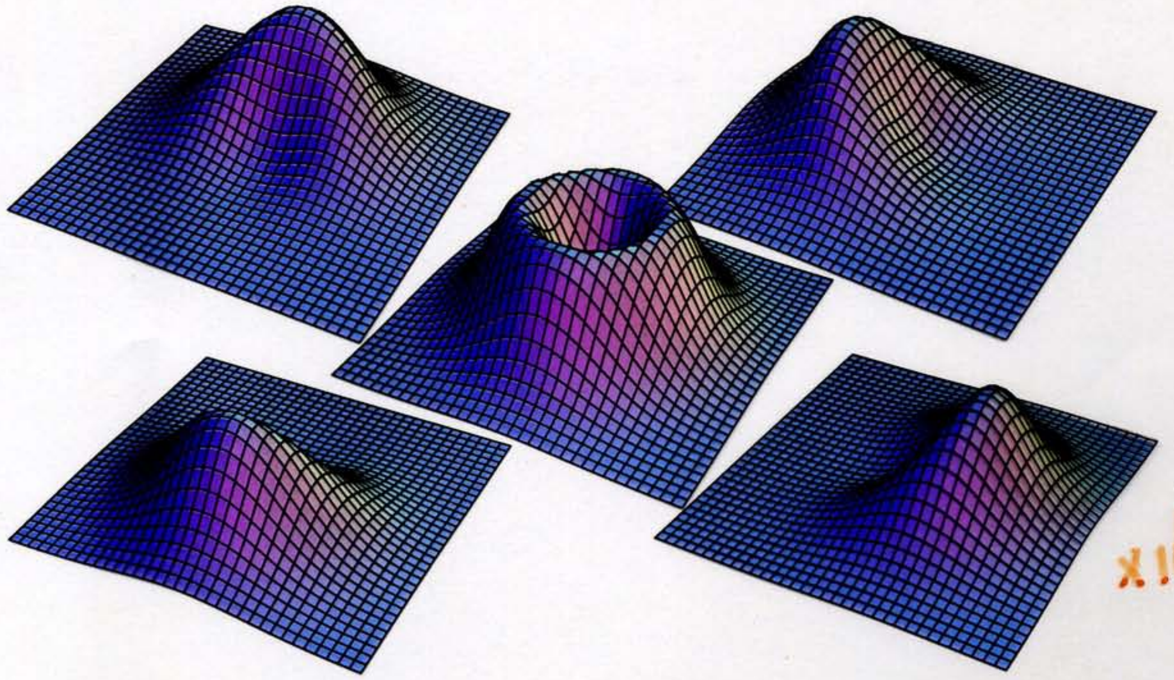


Figure 3: Zero-mode densities for a typical charge 2, $SU(2)$ axially symmetric solution. For comparison the action density (cmp. Fig. 2 of Ref. [11]) is shown in the middle. All are on a logarithmic scale, cutoff below e^{-3}. On the left is shown the two periodic zero-modes ($z = 0$) and on the right the two anti-periodic zero-modes ($z = 1/2$).



$\times 10\pi^2$

Figure 3: The action density in the plane of the constituents at $t = 0$ and the densitities for the two zero-modes, using either periodic (left) or anti-periodic (right) boundary conditions for an SU(2) charge 2 caloron in the so-called "crossed" configuration with $\mathbf{k} = 0.997$, $D = 8.753$ and $\mathrm{tr}\,\mathcal{P}_\infty = 0$.



$\times 16\pi^2$

Figure 4: As above, but for $\mathbf{k} = 0.962$ and $D = 3.894$.

(non-static)

An SU(2) KvBLL caloron gas model and confinement

P. Gerhold, E.-M. Ilgenfritz and M. Müller-Preussker, NPB 760 (2007) 1, 762 (2007) 268.

$$A_\mu^{per}(x) = e^{-2\pi i \vec{\omega}\cdot\vec{\tau}} \sum_i A_\mu^{(i),\,alg}(x)\, e^{2\pi i \vec{\omega}\cdot\vec{\tau}} + 2\pi \vec{\omega}\cdot\vec{\tau}\,\delta_{\mu 4}$$

for a density $n = 1\,\mathrm{fm}^{-4}$ and $\rho = 0.33$ fm good enough (dilute)

$D_1(\rho, T) = A\rho^{b-5}\exp(-c\rho^2)$ ($\bar{\rho}$ fixed)

$\int D_1(\rho,T)\,d\rho = 1$ $T \leq T_c$, $\omega = \frac{1}{4}$ ($\mathcal{V} = \frac{1}{2}$)

$D_2(\rho, T) = A\rho^{b-5}\exp(-\frac{4}{3}(\pi\rho T)^2)$ ($\bar{\rho}$ running)

$\int D_2(\rho,T)\,d\rho = 1$ $T \geq T_c$, $\omega = \omega(\mathcal{V})$

$$\left[\, b = \frac{11}{3}N_c - \frac{2}{3}N_f = \frac{22}{3} \quad (N_c = 2,\ N_f = 0) \,\right]$$

$$\bar{\rho}(T_c)_{conf} = \bar{\rho}(T_c)_{deconf} = 0.37$$

determines c

$T_c = 178$ MeV , $\sigma(T=0) = 318$ MeV/fm

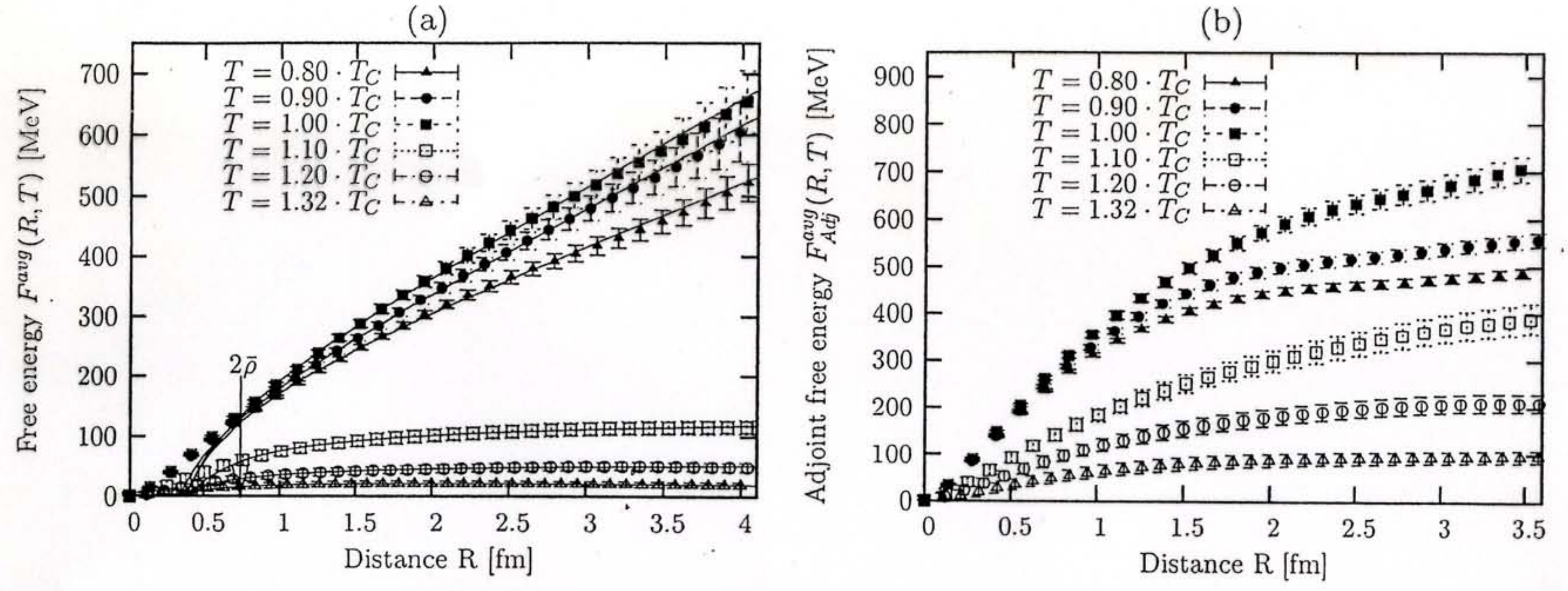


FIG. 11: color averaged free energy versus distance R at different temperatures $T/T_C = 0.8$, 0.9, 1.0 for the confined and at $T/T_C = 1.10$, 1.20, 1.32 for the deconfined phase in the fundamental (a) and in the adjoint (b) representation. The fundamental potentials are fitted to (43) for $R \geq 2\bar{\rho}$. This minimal distance is marked.

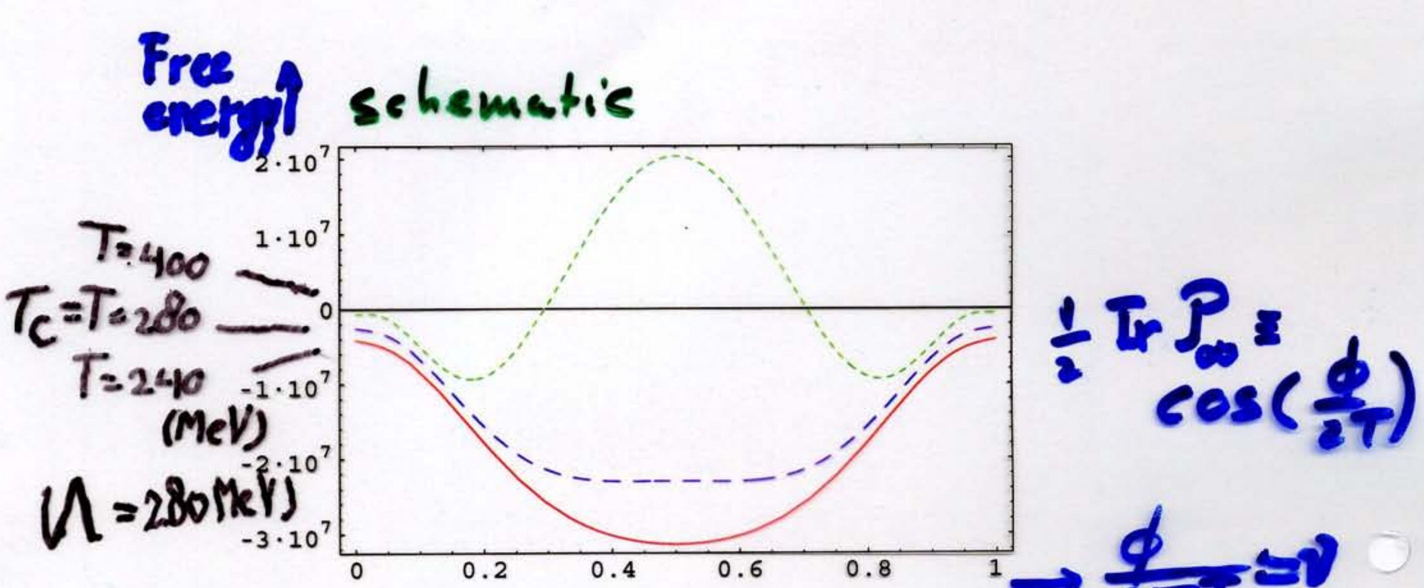


Figure 1: From D. Diakonov, "Instantons at work", Prog.Part.Nucl.Phys. 51 (2003) 173 [hep-ph/0212026].

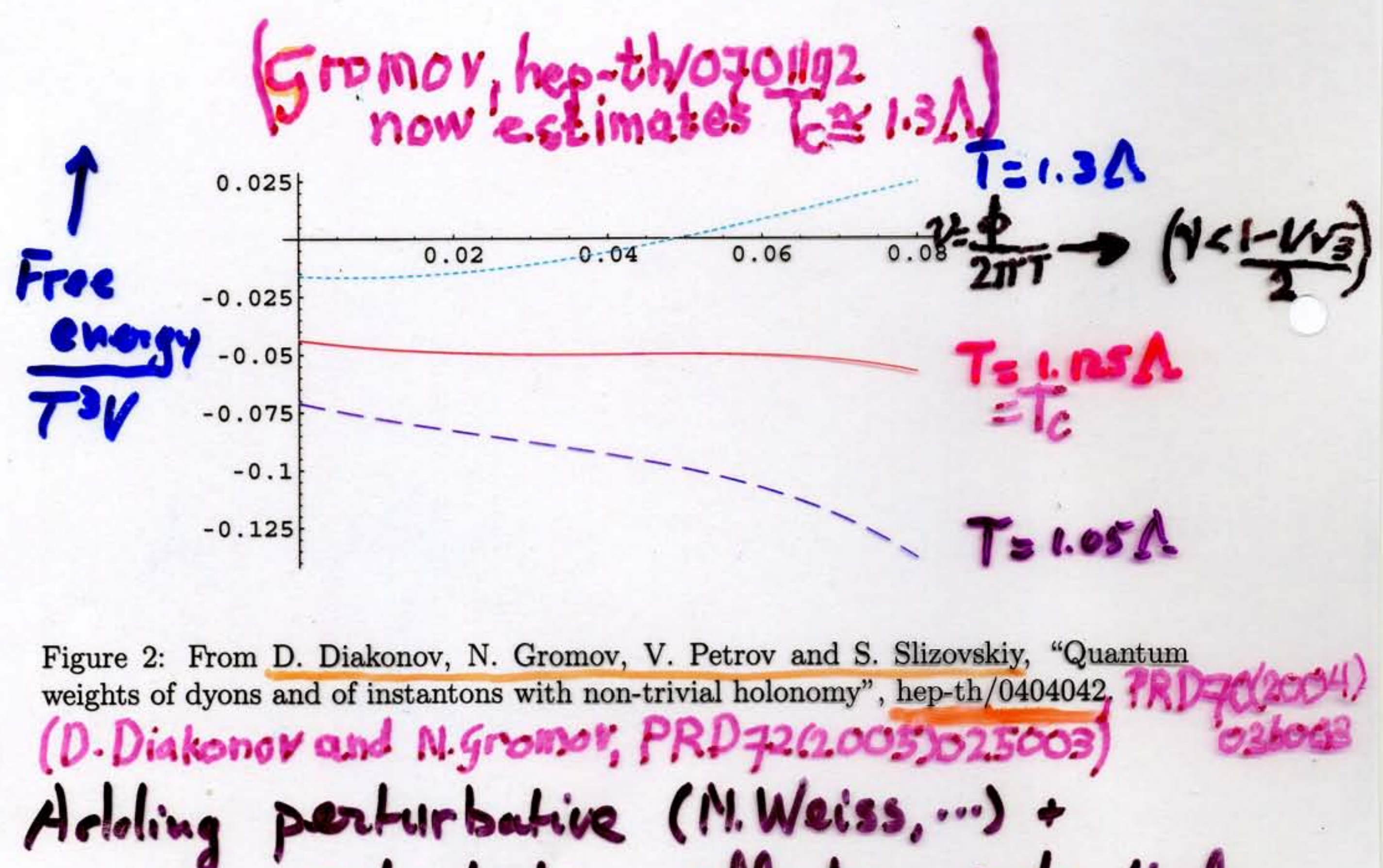


Figure 2: From D. Diakonov, N. Gromov, V. Petrov and S. Slizovskiy, "Quantum weights of dyons and of instantons with non-trivial holonomy", hep-th/0404042, PRD70(2004) 036003

(D. Diakonov and N. Gromov, PRD72(2005)025003)

Adding perturbative (M. Weiss, ...) + non-perturbative effective potentials

(cmp. Susy case, Davies, Hollowood, Khoze & Mattis, Nucl.Phys. B559(1999) 123

(PRD 76 (2007) 056001)

Diakonov and Petrov, hep-th/0704.3181, have written a hyperKähler metric which approximates the metric for arbitrary # of calorons

They generalize for 1 Caloron

$$ds^2 = G_{mn}\, d\vec{x}_m \cdot d\vec{x}_n + (d\psi_m + \vec{W}_{mm'} \cdot d\vec{x}_{m'})\, G^{-1}_{mn} \times (d\psi_n + \vec{W}_{nn'} \cdot d\vec{x}_{n'})$$

(Gibbons-Manton form)

with $G_{mn} = \delta_{mn}\left(4\pi\nu_m + \frac{1}{|\vec{x}_m - \vec{x}_{m+1}|} + \frac{1}{|\vec{x}_m - \vec{x}_{m-1}|}\right) - \frac{\delta_{m,n+1}}{|\vec{x}_m - \vec{x}_{m-1}|} - \frac{\delta_{m,n-1}}{|\vec{x}_m - \vec{x}_{m+1}|}$ and $\vec{W}_{mn} = \delta_{mn}\left(\vec{w}(\vec{x}_m - \vec{x}_{m-1}) + \vec{w}(\vec{x}_m - \vec{x}_{m+1})\right) - \delta_{m,n-1}\, \vec{w}(\vec{x}_m - \vec{x}_{m+1}) - \delta_{m,n+1}\, \vec{w}(\vec{x}_m - \vec{x}_{m-1})$, where $\epsilon^{abc}\partial_b w_c = -\frac{x_a}{|x|^3}$

(Lee, Weinberg, Yi; Kraan; Diakonov, Gromov)

this metric to arbitrary # of calorons by appr. the like-dyon (or monopole) case to

$$G_{mi,nj} = \delta_{mn}\delta_{ij}\Big(4\pi\nu_m + \sum_k \frac{1}{|\vec{x}_{mi} - \vec{x}_{m-1,k}|} + \sum_k \frac{1}{|\vec{x}_{mi} - \vec{x}_{m+1,k}|} - 2\sum_{k \neq i} \frac{1}{|\vec{x}_{mi} - \vec{x}_{mk}|}\Big) - \frac{\delta_{m,n-1}}{|\vec{x}_{mi} - \vec{x}_{m+1,j}|} - \frac{\delta_{m,n+1}}{|\vec{x}_{mi} - \vec{x}_{m-1,j}|} + 2\frac{\delta_{mn}}{|\vec{x}_{mi} - \vec{x}_{nj}|}\Big|_{i \neq j}$$

$$\vec{W}_{mi,nj} = \delta_{mn}\delta_{ij}\Big(\sum_k \vec{w}(\vec{x}_{mi} - \vec{x}_{m-1,k}) + \sum_k \vec{w}(\vec{x}_{mi} - \vec{x}_{m+1,k}) - 2\sum_{k \neq i} \vec{w}(\vec{x}_{mi} - \vec{x}_{mk})\Big) - \delta_{m,n-1}\, \vec{w}(\vec{x}_{mi} - \vec{x}_{m+1,j}) - \delta_{m,n+1}\, \vec{w}(\vec{x}_{mi} - \vec{x}_{m-1,j}) + 2\delta_{mn}\, \vec{w}(\vec{x}_{mi} - \vec{x}_{nj})\Big|_{i \neq j}$$

(it is different than the like-dyon metric of Atiyah, Hitchin when one comes close together – it vanishes already when $r = 1/2\pi\nu_m$ (for $T=1$))

They claim this already gives confinement: $T \lesssim T_c$ the minimum of potential is $\nu_1 = \nu_2 = \cdots = \nu_N = 1/N$ and the trace of the Polyakov loop is 0.

(the corrections due to the small fluctuations are still to be incorporated)