Gluon scattering amplitudes/Wilson loops
duality in gauge theories

Gregory Korchemsky
Université Paris XI, LPT, Orsay

Based on work in collaboration with
James Drummond, Johannes Henn, and Emery Sokatchev (LAPTH, Annecy)
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Outline

✔ On-shell gluon scattering amplitudes

✔ Iterative structure at weak/strong coupling in $\mathcal{N} = 4$ SYM

✔ Dual conformal invariance – hidden symmetry of planar amplitudes

✔ Scattering amplitude/Wilson loop duality in $\mathcal{N} = 4$ SYM

\[
\langle 0 | S | 1^- 2^- 3^+ \ldots n^+ \rangle \quad \leftrightarrow \quad \langle 0 | \text{tr P exp} \left( i \int_C dx \cdot A(x) \right) | 0 \rangle
\]

✔ Scattering amplitude/Wilson loop duality in QCD
On-shell gluon scattering amplitudes in $\mathcal{N} = 4$ SYM

$\mathcal{N} = 4$ SYM – (super)conformal gauge theory with the $SU(N_c)$ gauge group

Inherits all symmetries of the classical Lagrangian ... but are there some ‘hidden’ symmetries?

Gluon scattering amplitudes in $\mathcal{N} = 4$ SYM

- Quantum numbers of on-shell gluons $|i\rangle = |p_i, h_i, a_i\rangle$:
  - momentum $(p_i^\mu)^2 = 0$,
  - helicity $(h = \pm 1)$,
  - color $(a)$

- On-shell matrix elements of $S$–matrix

- Suffer from IR divergences $\mapsto$ require IR regularization

- Close cousin to QCD gluon amplitudes

Color-ordered planar partial amplitudes

$$A_n = \text{tr} \left[ T^{a_1} T^{a_2} \ldots T^{a_n} \right] A_{h_1, h_2, \ldots, h_n}^{h_1, h_2, \ldots, h_n} (p_1, p_2, \ldots, p_n) + \text{[Bose symmetry]}$$

Recent activity is inspired by two findings

- The amplitude $A_4$ reveals interesting iterative structure at weak coupling $\quad [\text{Bern}, \text{Dixon}, \text{Kosower}, \text{Smirnov}]$

- The same structure emerges at strong coupling via AdS/CFT $\quad [\text{Alday}, \text{Maldacena}]$

Where does this structure come from? Dual conformal symmetry! $\quad [\text{Drummond}, \text{Henn}, \text{GK}, \text{Smirnov}, \text{Sokatchev}]$
Four-gluon amplitude in $\mathcal{N} = 4$ SYM at weak coupling

\[
\mathcal{A}_4/A_4^{\text{tree}} = 1 + a + O(a^2), \quad a = \frac{g_{YM}^2 N_c}{8\pi^2}
\]

[Green, Schwarz, Brink'82]

All-loop planar amplitude can be split into a IR divergent and a finite part

\[
\mathcal{A}_4(s, t) = \text{Div}(s, t, \epsilon_{\text{IR}}) \text{Fin}(s/t)
\]

✓ IR divergences appear to all loops as poles in $\epsilon_{\text{IR}}$ (in dim.reg. with $D = 4 - 2\epsilon_{\text{IR}}$)

✓ IR divergences exponentiate (in any gauge theory!)

\[
\text{Div}(s, t, \epsilon_{\text{IR}}) = \exp \left\{ -\frac{1}{2} \sum_{l=1}^{\infty} a^l \left( \frac{\Gamma_{\text{cusp}}^{(l)}}{(l\epsilon_{\text{IR}})^2} + \frac{G^{(l)}}{l\epsilon_{\text{IR}}} \right) \left[ (s)^{l\epsilon_{\text{IR}}} + (-t)^{l\epsilon_{\text{IR}}} \right] \right\}
\]

✓ IR divergences are in the one-to-one correspondence with UV divergences of Wilson loops

[Mueller], [Sen], [Collins], [Sterman], [GK]'78-86

\[
\Gamma_{\text{cusp}}(a) = \sum_l a^l \Gamma_{\text{cusp}}^{(l)} = \text{cusp anomalous dimension of Wilson loops}
\]

\[
G(a) = \sum_l a^l G_{\text{cusp}}^{(l)} = \text{collinear anomalous dimension}
\]

✓ What about finite part of the amplitude $\text{Fin}(s/t)$? Does it have a simple structure?

\[
\text{Fin}_{\text{QCD}}(s/t) = [4 \text{ pages long mess}], \quad \text{Fin}_{\mathcal{N}=4}(s/t) = \text{BDS conjecture}
\]
Bern-Dixon-Smirnov (BDS) conjecture:

\[
\text{Fin}(s/t) = 1 + a \left[ \frac{1}{2} \ln^2(s/t) + 4\zeta_2 \right] + O(a^2) \quad \text{all loops} \Rightarrow \exp \left[ \frac{\Gamma_{\text{cusp}}(a)}{4} \ln^2(s/t) + \text{const} \right]
\]

Compared to QCD,

(i) the complicated functions of \(s/t\) are replaced by the elementary function \(\ln^2(s/t)\);

(ii) no higher powers of logs appear in \(\ln(\text{Fin}(s/t))\) at higher loops;

(iii) the coefficient of \(\ln^2(s/t)\) is determined by the cusp anomalous dimension \(\Gamma_{\text{cusp}}(a)\) just like the coefficient of the double IR pole.

The conjecture has been verified up to three loops \[\text{[Anastasiou,Bern,Dixon,Kosower'03],[Bern,Dixon,Smirnov'05]}\]

A similar conjecture exists for \(n\)-gluon MHV amplitudes \[\text{[Bern,Dixon,Smirnov'05]}\]

It has been confirmed for \(n = 5\) at two loops \[\text{[Cachazo,Spradlin,Volovich'04], [Bern,Czakon,Kosower,Roiban,Smirnov'06]}\]

Surprising features of the finite part of the MHV amplitudes in planar \(\mathcal{N} = 4\) SYM:

Why should finite corrections exponentiate?

Why should they be related to the cusp anomaly of Wilson loop?
**Dual conformal symmetry**

Examine one-loop ‘scalar box’ diagram

✔ Change variables to go to a dual ‘coordinate space’ picture (not a Fourier transform!)

\[
p_1 = x_1 - x_2 \equiv x_{12}, \quad p_2 = x_{23}, \quad p_3 = x_{34}, \quad p_4 = x_{41}, \quad k = x_{15}
\]

\[
\int \frac{d^4 k (p_1 + p_2)^2 (p_2 + p_3)^2}{k^2 (k - p_1)^2 (k - p_1 - p_2)^2 (k + p_4)^2} = \int \frac{d^4 x_5 x_{13} x_{24}}{x_{15}^2 x_{25}^2 x_{35}^2 x_{45}^2}
\]

*Check conformal invariance by inversion* \( x_i^\mu \rightarrow x_i^\mu / x_i^2 \)

[Broadhurst],[Drummond,Henn,Smirnov,Sokatchev]

✔ The integral is invariant under conformal \( SO(2, 4) \) transformations in the dual space!

✔ The symmetry *is not related* to conformal \( SO(2, 4) \) symmetry of \( \mathcal{N} = 4 \) SYM

✔ All scalar integrals contributing to \( A_4 \) up to four loops possess the dual conformal invariance!

✔ If the dual conformal symmetry survives to all loops, it allows us to determine four- and five-gluon planar scattering amplitudes to all loops! [Drummond,Henn,GK,Sokatchev],[Alday,Maldacena]

✔ Dual conformality is slightly broken by the infrared regulator

✔ For *planar* integrals only!
Four-gluon amplitude from AdS/CFT

Alday-Maldacena proposal:

✓ On-shell scattering amplitude is described by a classical string world-sheet in AdS$_5$

✗ On-shell gluon momenta $p_1^\mu, \ldots, p_n^\mu$ define sequence of light-like segments on the boundary

✗ The closed contour has $n$ cusps with the dual coordinates $x_i^{\mu}$

(1) $x_i^{\mu, i+1} \equiv x_i^{\mu} - x_{i+1}^{\mu} := p_i^{\mu}$

The dual conformal symmetry also exists at strong coupling!

✓ Is in agreement with the Bern-Dixon-Smirnov (BDS) ansatz for $n = 4$ amplitudes

✓ Admits generalization to arbitrary $n$-gluon amplitudes but it is difficult to construct explicit solutions for ‘minimal surface’ in AdS

✓ Agreement with the BDS ansatz is also observed for $n = 5$ gluon amplitudes [Komargodski] but disagreement is found for $n \rightarrow \infty$ $\iff$ the BDS ansatz needs to be modified [Alday,Maldacena]

The same questions to answer as at weak coupling:

☞ Why should finite corrections exponentiate?

☞ Why should they be related to the cusp anomaly of Wilson loop?
From gluon amplitudes to Wilson loops

Common properties of gluon scattering amplitudes at both weak and strong coupling:

1. IR divergences of $A_4$ are in one-to-one correspondence with UV div. of *cusped Wilson loops*

2. The gluons scattering amplitudes possess a hidden *dual conformal symmetry*

☐ *Is it possible to identify the object in $\mathcal{N} = 4$ SYM for which both properties are manifest?*

**Yes! The expectation value of light-like Wilson loop in $\mathcal{N} = 4$ SYM**

$$W(C_4) = \frac{1}{N_c} \langle 0 | \text{Tr } \mathcal{P} \exp \left( ig \int_{C_4} dx^\mu A_\mu(x) \right) | 0 \rangle , \quad C_4 = x_1 \cup x_2 \cup x_3 \cup x_4$$

☑ Gauge invariant functional of the integration contour $C_4$ in Minkowski space-time

☑ The contour is made out of 4 light-like segments $C_4 = \ell_1 \cup \ell_2 \cup \ell_3 \cup \ell_4$ joining the cusp points $x_i^{\mu}$

$$x_i^{\mu} - x_{i+1}^{\mu} = p_i^{\mu} = \text{on-shell gluon momenta}$$

☑ The contour $C_4$ has four light-like cusps $\mapsto W(C_4)$ has UV divergencies

☑ Conformal symmetry of $\mathcal{N} = 4$ SYM $\mapsto$ conformal invariance of $W(C_4)$ in dual coordinates $x^{\mu}$
The one-loop expression for the light-like Wilson loop (with $x_{jk}^2 = (x_j - x_k)^2$) [Drummond,GK,Sokatchev]

$$\ln W(C_4) =$$

$$= \frac{g^2}{4\pi^2} C_F \left\{ -\frac{1}{\epsilon_{UV}^2} \left[ (-x_{13}^2 \mu^2)^{\epsilon_{UV}} + (-x_{24}^2 \mu^2)^{\epsilon_{UV}} \right] + \frac{1}{2} \ln^2 \left( \frac{x_{13}^2}{x_{24}^2} \right) + \text{const} \right\} + O(g^4)$$

The one-loop expression for the gluon scattering amplitude

$$\ln \mathcal{A}_4(s, t) = \frac{g^2}{4\pi^2} C_F \left\{ -\frac{1}{\epsilon_{IR}^2} \left[ (-s/\mu_{IR}^2)^{\epsilon_{IR}} + (-t/\mu_{IR}^2)^{\epsilon_{IR}} \right] + \frac{1}{2} \ln^2 \left( \frac{s}{t} \right) + \text{const} \right\} + O(g^4)$$

- Identity the light-like segments with the on-shell gluon momenta $x_{i,i+1}^{\mu} \equiv x_{i}^{\mu} - x_{i+1}^{\mu} := p_i^{\mu}$:

$$x_{13}^2 \mu^2 := s/\mu_{IR}^2, \quad x_{24}^2 \mu^2 := t/\mu_{IR}^2, \quad x_{13}^2/x_{24}^2 := s/t$$

- UV divergencies of the light-like Wilson loop match IR divergences of the gluon amplitude

- the finite $\sim \ln^2(s/t)$ corrections coincide to one loop!
Gluon scattering amplitudes/Wilson loop duality II

Drummond-(Henn)-GK-Sokatchev proposal: *gluon amplitudes are dual to light-like Wilson loops*

\[ \ln A_4 = \ln W(C_4) + O(1/N_C^2, \epsilon_{IR}). \]

✔ At strong coupling, the relation holds to leading order in \(1/\sqrt{\lambda}\)  
[ Alday, Maldacena]

✔ At weak coupling, the relation was verified to two loops  
[ Drummond, Henn, GK, Sokatchev]

\[ \ln A_4 = \ln W(C_4) = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix} = \frac{1}{4} \Gamma_{cusp}(g) \ln^2(s/t) + \text{Div} \]

✔ Generalization to \(n \geq 5\) gluon MHV amplitudes

\[ \ln A_n^{(MHV)} = \ln W(C_n) + O(1/N_C^2), \quad C_n = \text{light-like } n-(\text{poly})gon \]

✗ At weak coupling, matches the BDS ansatz to one loop  
[ Brandhuber, Heslop, Travaglini]

✗ The duality relation for \(n = 5\) (pentagon) was verified to two loops  
[ Drummond, Henn, GK, Sokatchev]
Conformal Ward identities for light-like Wilson loop

Main idea: make use of conformal invariance of light-like Wilson loops in $\mathcal{N} = 4$ SYM + duality relation to fix the finite part of $n-$gluon amplitudes

- Conformal $SO(2, 4)$ transformations map light-like polygon $C_n$ into another light-like polygon $C'_n$

- If the Wilson loop $W(C_n)$ were well-defined (=finite) in $D = 4$ dimensions then

$$W(C_n) = W(C'_n)$$

- ... but $W(C_n)$ has cusp UV singularities $\Rightarrow$ dim.reg. breaks conformal invariance

$$W(C_n) = W(C'_n) \times \text{[cusp anomaly]}$$

- *All-loop* anomalous conformal Ward identities for the *finite part* of the Wilson loop

$$W(C_n) = \exp(F_n) \times \text{[UV divergencies]}$$

under dilatations, $\mathbb{D}$, and special conformal transformations, $K^\mu$, [Drummond,Henn,GK,Sokatchev]

$$\mathbb{D} F_n \equiv \sum_{i=1}^{n} (x_i \cdot \partial x_i) F_n = 0$$

$$K^\mu F_n \equiv \sum_{i=1}^{n} \left[ 2x_i^\mu (x_i \cdot \partial x_i) - x_i^2 \partial x_i^\mu \right] F_n = \frac{1}{2} \Gamma_{\text{cusp}}(a) \sum_{i=1}^{n} x_i^\mu \ln \left( \frac{x_{i,i+1}^2}{x_{i-1,i+1}^2 + 1} \right)$$

The same relations also hold at strong coupling [Alday,Maldacena],[Komargodski]
Finite part of light-like Wilson loops

The consequences of the conformal Ward identity for the finite part of the Wilson loop $W_n$

✓ $n = 4, 5$ are special: there are no conformal invariants (too few distances due to $x_{i,i+1}^2 = 0$)

⇒ the Ward identity has a unique all-loop solution (up to an additive constant)

$$F_4 = \frac{1}{4} \Gamma_{cusp}(a) \ln^2 \left( \frac{x_{13}^2}{x_{24}^2} \right) + \text{const},$$

$$F_5 = -\frac{1}{8} \Gamma_{cusp}(a) \sum_{i=1}^{5} \ln \left( \frac{x_{i,i+2}^2}{x_{i,i+3}^2} \right) \ln \left( \frac{x_{i+1,i+3}^2}{x_{i+2,i+4}^2} \right) + \text{const}$$

Exactly the functional forms of the BDS ansatz for the 4- and 5-point MHV amplitudes!

✓ Starting from $n = 6$ there are conformal invariants in the form of cross-ratios

$$u_1 = \frac{x_{13}^2 x_{46}^2}{x_{14}^2 x_{36}^2}, \quad u_2 = \frac{x_{24}^2 x_{15}^2}{x_{25}^2 x_{14}^2}, \quad u_3 = \frac{x_{35}^2 x_{26}^2}{x_{36}^2 x_{25}^2}$$

Hence the general solution of the Ward identity for $W(C_n)$ with $n \geq 6$ contains an arbitrary function of the conformal cross-ratios.

✓ The BDS ansatz is a solution of the conformal Ward identity for arbitrary $n$ but the ansatz should be modified for $n \geq 6$ starting from two loops... what is a missing function of $u_1$, $u_2$ and $u_3$?
We computed the two-loop hexagon Wilson loop $W(C_6)$ ... [Drummond, Henn, GK, Sokatchev'07]

\[
\ln W(C_6) = \begin{array}{cccc}
1 & 2 & 3 & 4 \\
5 & 6 & 7 & 8 \\
9 & 10 & 11 & 12 \\
13 & 14 & 15 & 16 \\
17 & 18 & 19 & 20 \\
21 & 22 & 23 & 24 \\
25 & 26 & 27 & 28 \\
29 & 30 & 31 & 32 \\
33 & 34 & 35 & 36 \\
37 & 38 & 39 & 40 \\
\end{array}
\]

... and found a discrepancy $\ln W(C_6) \neq \ln \mathcal{M}_6^{(BDS)}$

Bern-Dixon-Kosower-Roiban-Spradlin-Vergu-Volovich computed 6-gluon amplitude to 2 loops

\[
\mathcal{M}_6^{(MHV)} = \begin{array}{cccc}
\end{array}
\]

... and found a discrepancy $\ln \mathcal{M}_6^{(MHV)} \neq \ln \mathcal{M}_6^{(BDS)}$

☞ The BDS ansatz fails for $n = 6$ starting from two loops.
☞ *What about Wilson loop duality?* $\ln \mathcal{M}_6^{(MHV)} \neq \ln W(C_6)$
6-gluon amplitude/hexagon Wilson loop duality

✔ Comparison between the DHKS discrepancy function $\Delta_{\text{WL}}$ and the BDKRSVV results for the six-gluon amplitude $\Delta_{\text{MHV}}$:

<table>
<thead>
<tr>
<th>Kinematical point</th>
<th>$(u_1, u_2, u_3)$</th>
<th>$\Delta_{\text{WL}} - \Delta^{(0)}_{\text{WL}}$</th>
<th>$\Delta_{\text{MHV}} - \Delta^{(0)}_{\text{MHV}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K^{(1)}$</td>
<td>$(1/4, 1/4, 1/4)$</td>
<td>$&lt; 10^{-5}$</td>
<td>$-0.018 \pm 0.023$</td>
</tr>
<tr>
<td>$K^{(2)}$</td>
<td>$(0.547253, 0.203822, 0.88127)$</td>
<td>$-2.75533$</td>
<td>$-2.753 \pm 0.015$</td>
</tr>
<tr>
<td>$K^{(3)}$</td>
<td>$(28/17, 16/5, 112/85)$</td>
<td>$-4.74460$</td>
<td>$-4.7445 \pm 0.0075$</td>
</tr>
<tr>
<td>$K^{(4)}$</td>
<td>$(1/9, 1/9, 1/9)$</td>
<td>$4.09138$</td>
<td>$4.12 \pm 0.10$</td>
</tr>
<tr>
<td>$K^{(5)}$</td>
<td>$(4/81, 4/81, 4/81)$</td>
<td>$9.72553$</td>
<td>$10.00 \pm 0.50$</td>
</tr>
</tbody>
</table>

evaluated for different kinematical configurations, e.g.

$K^{(1)}$: $x_{13}^2 = -0.7236200$, $x_{24}^2 = -0.9213500$, $x_{35}^2 = -0.2723200$, $x_{46}^2 = -0.3582300$, $x_{36}^2 = -0.4825841$, $x_{15}^2 = -0.4235500$, $x_{26}^2 = -0.3218573$, $x_{14}^2 = -2.1486192$, $x_{25}^2 = -0.7264904$.

✔ Two nontrivial functions coincide with an accuracy $< 10^{-4}$!

✸ The Wilson loop/gluon scattering amplitude duality holds at $n = 6$ to two loops!!

✸ There are now little doubts that the duality relation also holds for arbitrary $n$ to all loops!!
Four-gluon amplitude/Wilson loop duality in QCD

Finite part of four-gluon amplitude in QCD at two loops

\[
\text{Fin}_{\text{QCD}}^{(2)}(s, t, u) = A(x, y, z) + O(1/N_c^2, n_f/N_c)
\]

with notations \( x = -\frac{t}{s}, y = -\frac{u}{s}, z = -\frac{u}{t}, X = \log x, Y = \log y, S = \log z \)

\[
A = \left\{ \begin{array}{l}
(48 \text{Li}_4(x) - 48 \text{Li}_4(y) - 128 \text{Li}_4(z) + 40 \text{Li}_3(x) X - 64 \text{Li}_3(x) Y - \frac{98}{3} \text{Li}_3(x) + 64 \text{Li}_3(y) X - 40 \text{Li}_3(y) Y + 18 \text{Li}_3(y) X \\
+ \frac{98}{3} \text{Li}_2(x) X - \frac{16}{3} \text{Li}_2(x) \pi^2 - 18 \text{Li}_2(y) Y - \frac{37}{6} X^4 + 28 X^3 Y - \frac{23}{3} X^3 - 16 X^2 Y^2 + \frac{49}{3} X^2 Y^2 - \frac{35}{3} X^2 \pi^2 - \frac{38}{3} X^2 Y \\
- \frac{22}{3} S X^2 - \frac{20}{3} X Y^3 - 9 X Y^2 + 8 X Y \pi^2 + 10 X Y - \frac{31}{12} X \pi^2 - 22 \zeta_3 X + \frac{22}{3} S X + \frac{37}{27} X + \frac{11}{6} Y^4 - \frac{41}{9} Y^3 - \frac{11}{3} Y^2 \pi^2 \\
- \frac{22}{3} S Y^2 + \frac{266}{9} Y^2 - \frac{35}{12} Y \pi^2 + \frac{418}{9} S Y + \frac{257}{9} S Y + 18 \zeta_3 Y - \frac{31}{30} \pi^4 - \frac{11}{9} S \pi^2 + \frac{31}{9} \pi^2 + \frac{242}{9} S^2 + \frac{418}{9} \zeta_3 + \frac{2156}{27} S \\
- \frac{11093}{81} - 8 S \zeta_3 \right\} \frac{t^2}{s^2} + \left( -256 \text{Li}_4(x) - 96 \text{Li}_4(y) + 96 \text{Li}_4(z) + 80 \text{Li}_3(x) X + 48 \text{Li}_3(x) Y - \frac{64}{3} \text{Li}_3(x) - 48 \text{Li}_3(y) X \\
+ 96 \text{Li}_3(y) Y - \frac{304}{3} \text{Li}_3(y) + \frac{64}{3} \text{Li}_2(x) X - \frac{32}{3} \text{Li}_2(x) \pi^2 + \frac{304}{3} \text{Li}_2(y) Y + \frac{26}{3} X^4 - \frac{64}{3} X^3 Y - \frac{64}{3} X^3 + 20 X^2 Y^2 \\
+ \frac{136}{3} X^2 Y + 24 X^2 \pi^2 + 76 X^2 - \frac{88}{3} S X^2 + \frac{8}{3} X Y^3 + \frac{104}{3} X Y^2 - \frac{16}{3} X Y \pi^2 + \frac{176}{3} S X Y - \frac{136}{3} X Y - \frac{50}{3} X \pi^2 \\
- 48 \zeta_3 X + \frac{2350}{27} X + \frac{440}{3} S X + 4 Y^4 - \frac{176}{9} Y^3 + \frac{4}{3} Y^2 \pi^2 - \frac{176}{3} S Y^2 - \frac{494}{9} Y \pi^2 + \frac{5392}{27} Y - 64 \zeta_3 Y + \frac{496}{45} \pi^4 \\
- \frac{308}{9} S \pi^2 + \frac{200}{9} \pi^2 + \frac{968}{9} S^2 + \frac{8624}{27} S - \frac{44372}{81} + \frac{1864}{9} \zeta_3 - \frac{32}{3} S \zeta_3 \right\} \frac{t}{u} + \left( \frac{88}{3} \text{Li}_3(x) - \frac{88}{3} \text{Li}_2(x) X + 2 X^4 - 8 X^3 Y \\
- \frac{220}{9} X^3 + 12 X^2 Y^2 + \frac{88}{3} X^2 Y + \frac{8}{3} X^2 \pi^2 - \frac{88}{3} S X^2 + \frac{304}{3} X^2 - 8 X Y^3 - \frac{16}{3} X Y \pi^2 + \frac{176}{3} S X Y - \frac{77}{3} X \pi^2 \\
+ \frac{1616}{27} X + \frac{968}{9} S X - 8 \zeta_3 X + 4 Y^4 - \frac{176}{9} Y^3 - \frac{20}{3} Y^2 \pi^2 - \frac{176}{3} S Y^2 - \frac{638}{9} Y \pi^2 - 16 \zeta_3 Y + \frac{5392}{27} Y - \frac{4}{15} \pi^4 - \frac{308}{9} S \pi^2 \\
- 20 \pi^2 - 32 S \zeta_3 + \frac{1408}{9} \zeta_3 + \frac{968}{9} S^2 - \frac{44372}{81} + \frac{8624}{27} S \right\} \frac{t^2}{u^2} + \left( \frac{44}{3} \text{Li}_3(x) - \frac{44}{3} \text{Li}_2(x) X - X^4 + \frac{110}{3} X^3 - \frac{22}{3} X^2 Y \\
+ \frac{14}{3} X^2 \pi^2 + \frac{44}{3} S X^2 - \frac{152}{9} X^2 - 10 X Y + \frac{11}{2} X \pi^2 + 4 \zeta_3 X - \frac{484}{9} S X - \frac{808}{27} X + \frac{7}{30} \pi^4 - \frac{31}{9} \pi^2 \\
+ \frac{11}{9} S \pi^2 - \frac{418}{9} \zeta_3 - \frac{242}{9} S^2 - \frac{2156}{27} S + 8 S \zeta_3 + \frac{11093}{81} \right\} \frac{u^2}{s^2} + \left( -176 \text{Li}_4(x) + 88 \text{Li}_3(x) X - 168 \text{Li}_3(x) Y - ...ight)
\]
Four-gluon amplitude/Wilson loop duality in QCD II

- Planar four-gluon QCD scattering amplitude in the Regge limit \( s \gg -t \) \cite{Schnitzer'76,Fadin,Kuraev,Lipatov'76}

\[
\mathcal{M}_4^{(QCD)}(s, t) \sim (s/(-t))^{\omega_R(-t)} + \ldots
\]

The Regge trajectory \( \omega_R(-t) \) is known to two loops \cite{Fadin,Fiore,Kotsky'96}

- The all-loop gluon Regge trajectory in QCD \cite{GK'96}

\[
\omega_R^{(QCD)}(-t) = \frac{1}{2} \int_{(-t)}^{\mu_{IR}^2} \frac{dk_\perp^2}{k_\perp^2} \Gamma_{cusp}(a(k_\perp^2)) + \Gamma_R(a(-t)) + \text{poles in } 1/\epsilon_{IR},
\]

- Rectangular Wilson loop in QCD in the Regge limit \(|x_{13}^2| \gg |x_{24}^2|\)

\[
W^{(QCD)}(C_4) \sim (x_{13}^2/(-x_{24}^2))^{\omega_W(-x_{24}^2)} + \ldots
\]

- The all-loop Wilson loop ‘trajectory’ in QCD

\[
\omega_W^{(QCD)}(-t) = \frac{1}{2} \int_{(-t)}^{\mu_{UV}^2} \frac{dk_\perp^2}{k_\perp^2} \Gamma_{cusp}(a(k_\perp^2)) + \Gamma_W(a(-t)) + \text{poles in } 1/\epsilon_{UV},
\]

- The duality relation holds in QCD in the Regge limit only! \cite{GK'96}

\[
\ln \mathcal{M}_4^{(QCD)}(s, t) = \ln W^{(QCD)}(C_4) + O(t/s)
\]

while in \( \mathcal{N} = 4 \) SYM it is exact for arbitrary \( t/s \)
Conclusions and open questions

✔ Planar gluon scattering amplitudes possess the dual conformal symmetry at both weak and strong coupling (is not a symmetry of the full $\mathcal{N} = 4$ SYM!)

✔ This symmetry becomes manifest within the gauge scattering amplitude/Wilson loop duality

✔ We do not understand the origin of this symmetry but we do know how to make use of it:
  ✗ The anomalous conformal Ward identities uniquely fix the form of the finite part of $n = 4$ and $n = 5$ gluon amplitudes, in complete agreement with the BDS conjecture
  ✗ Starting from $n = 6$, the conformal symmetry is not sufficient to fix the finite part of the Wilson loop (=discrepancy function)
  ✗ Remarkably enough, the DHKS discrepancy function for the $n = 6$ Wilson loop coincides with the BDKRSVV discrepancy function for the six-gluon amplitude

✔ We have now good reasons to believe that the Wilson loop/gluon amplitude duality holds for any $n$ to all loops... but
  ✗ What is the origin of the dual conformal symmetry?
  ✗ Who controls a nontrivial discrepancy function of conformal ratios?

Should be related to integrability of planar $\mathcal{N} = 4$ SYM. More work is needed!
Back-up slides
What is the cusp anomalous dimension

✔ Cusp anomaly is a very ‘unfortunate’ feature of Wilson loops evaluated over an Euclidean closed contour with a cusp – generates the anomalous dimension

\[ \langle \text{tr } P \exp \left( i \oint_C dx \cdot A(x) \right) \rangle \sim (\Lambda_{\text{UV}})^{\Gamma_{\text{cusp}}(g, \vartheta)}, \quad C = \]

✔ A very ‘fortunate’ property of Wilson loop – the cusp anomaly controls the infrared asymptotics of scattering amplitudes in gauge theories

✗ The integration contour \( C \) is defined by the particle momenta

✗ The cusp angle \( \vartheta \) is related to the scattering angles in Minkowski space-time, \( |\vartheta| \gg 1 \)

\[ \Gamma_{\text{cusp}}(g, \vartheta) = \vartheta \Gamma_{\text{cusp}}(g) + O(\vartheta^0), \]

✔ The cusp anomalous dimension \( \Gamma_{\text{cusp}}(g) \) is an ubiquitous observable in gauge theories:

✗ Logarithmic scaling of anomalous dimensions of high-spin Wilson operators;

✗ IR singularities of on-shell gluon scattering amplitudes;

✗ Gluon Regge trajectory;

✗ Sudakov asymptotics of elastic form factors;

✗ ...

[Polyakov'80]

[GK, Radyushkin'86]

[GK'89]
Four-gluon planar amplitude at weak coupling

Weak coupling corrections to $A_4/A_4^{(0)}$ can be expressed in terms of scalar integrals:

- **One loop:**
  \[ (1) \]
  \[ 2 \]
  \[ 3 \]
  \[ 4 \]

  [Green, Schwarz, Brink'82]

- **Two loops:**
  [Bern, Rozowsky, Yan'97]
  
  *all-loop iteration structure conjectured*

- **Three loops:**
  [Anastasiou, Bern, Dixon, Kosower'03]
  [Bern, Dixon, Smirnov'05]
  
  *iteration structure confirmed*

- **Four loops:** scalar integrals of 8 different topologies are identified
  [Bern, Czakov, Dixon, Kosower, Smirnov'06]
Light-like Wilson loops

To lowest order in the coupling constant,

\[
W(C_4) = 1 + \frac{1}{2} (ig)^2 C_F \sum_{1 \leq j, k \leq 4} \int_{\ell_j} \int_{\ell_k} dx^\mu dy^\nu \ G_{\mu\nu}(x - y) + O(g^4),
\]

(1)

✔ The gluon propagator in the coordinate representation (the Feynman gauge + dimensional regularization, \(D = 4 - 2\epsilon\))

\[
G_{\mu\nu}(x) = -g_{\mu\nu} \frac{\Gamma(1 - \epsilon)}{4\pi^2} (-x^2 + i0)^{-1+\epsilon} (\mu^2 \pi)^\epsilon.
\]

✔ Feynman diagram representation

✔ The light-like Wilson loop is IR finite but has UV divergences due to cusps on the integration contour \(C_4\)

\[
\ln W(C_4) = \frac{g^2}{4\pi^2} C_F \left\{ -\frac{1}{2\epsilon^2} \sum_{i=1}^{4} (-x_{i-1,i+1}^2, \mu^2)^\epsilon + O(\epsilon^0) \right\} + O(g^4).
\]