Gluon scattering amplitudes/Wilson loops duality in gauge theories

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Based on work in collaboration with

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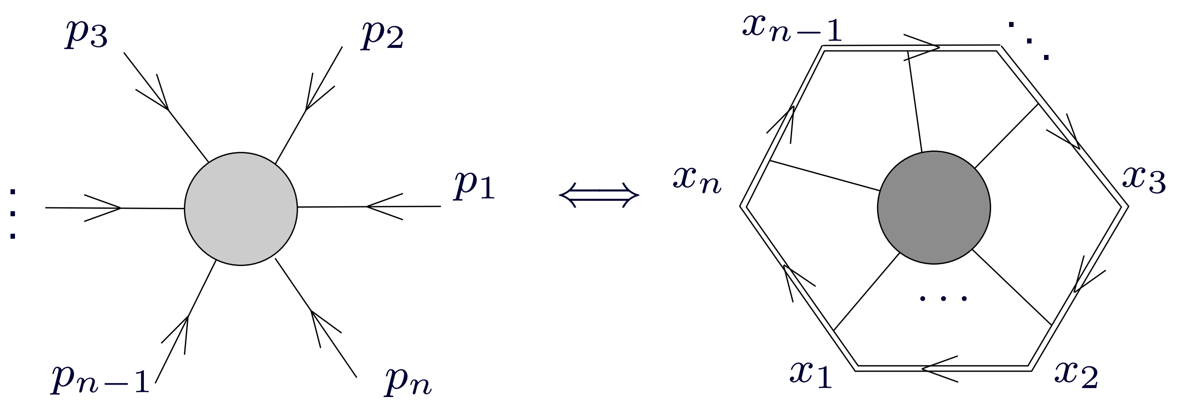
Outline

- On-shell gluon scattering amplitudes
- Iterative structure at weak/strong coupling in $\mathcal{N}=4$ SYM
- Dual conformal invariance – hidden symmetry of planar amplitudes
- Scattering amplitude/Wilson loop duality in $\mathcal{N}=4$ SYM



$$\langle 0|S|1^-2^-3^+\dots n^+\rangle \qquad \langle 0|\,\mathrm{tr}\,\mathrm{P}\exp\left(i\oint_C dx\cdot A(x)\right)|0\rangle$$

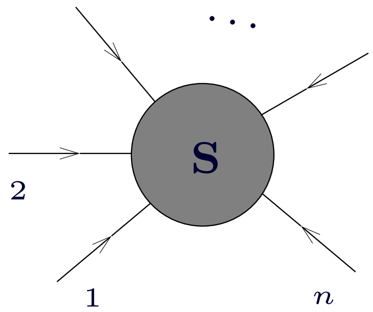
- Scattering amplitude/Wilson loop duality in QCD

On-shell gluon scattering amplitudes in $\mathcal{N}=4$ SYM

- ✔ $\mathcal{N}=4$ SYM – (super)conformal gauge theory with the $SU(N_c)$ gauge group

  Inherits all symmetries of the classical Lagrangian ... but are there some 'hidden' symmetries?

- ✔ Gluon scattering amplitudes in $\mathcal{N}=4$ SYM

$\mathcal{A}_n =$ 

  - ✗ Quantum numbers of on-shell gluons $|i\rangle = |p_i, h_i, a_i\rangle$: momentum ($(p_i^\mu)^2 = 0$), helicity $(h=\pm 1)$, color (a)
  - ✗ On-shell matrix elements of $S-$matrix
  - ✗ Suffer from IR divergences $\mapsto$ require IR regularization
  - ✗ Close cousin to QCD gluon amplitudes

- ✔ Color-ordered planar partial amplitudes

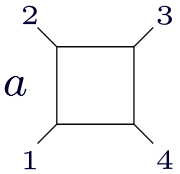
$$\mathcal{A}_n = \mathrm{tr}\left[T^{a_1}T^{a_2}\dots T^{a_n}\right] A_n^{h_1,h_2,\dots,h_n}(p_1,p_2,\dots,p_n) + \text{[Bose symmetry]}$$

- ✔ Recent activity is inspired by two findings
  - ✗ The amplitude $\mathcal{A}_4$ reveals interesting iterative structure at weak coupling [Bern,Dixon,Kosower,Smirnov]
  - ✗ The same structure emerges at strong coupling via AdS/CFT [Alday,Maldacena]

  Where does this structure come from? Dual conformal symmetry! [Drummond,Henn,GK,Smirnov,Sokatchev]

Four-gluon amplitude in $\mathcal{N}=4$ SYM at weak coupling

$$\mathcal{A}_4/\mathcal{A}_4^{(\text{tree})} = 1 + a$$



$$+ O(a^2)\,, \qquad a = \frac{g^2_{\text{YM}} N_c}{8\pi^2}$$

[Green,Schwarz,Brink'82]

All-loop planar amplitude can be split into a IR divergent and a finite part

$$\mathcal{A}_4(s,t) = \text{Div}(s,t,\epsilon_{\text{IR}})\,\text{Fin}(s/t)$$

- ✔ IR divergences appear to all loops as poles in ϵ_{IR} (in dim.reg. with $D = 4 - 2\epsilon_{\text{IR}}$)
- ✔ IR divergences exponentiate (in any gauge theory!) [Mueller],[Sen],[Collins],[Sterman],[GK]'78-86

$$\text{Div}(s,t,\epsilon_{\text{IR}}) = \exp\left\{-\frac{1}{2}\sum_{l=1}^{\infty} a^l \left(\frac{\Gamma^{(l)}_{\text{cusp}}}{(l\epsilon_{\text{IR}})^2} + \frac{G^{(l)}}{l\epsilon_{\text{IR}}}\right)\left[(-s)^{l\epsilon_{\text{IR}}} + (-t)^{l\epsilon_{\text{IR}}}\right]\right\}$$

- ✔ *IR divergences* are in the one-to-one correspondence with *UV divergences* of Wilson loops

[Ivanov,GK,Radyushkin'86]

$\Gamma_{\text{cusp}}(a) = \sum_l a^l \Gamma^{(l)}_{\text{cusp}} =$ *cusp anomalous dimension of Wilson loops*

$G(a) = \sum_l a^l G^{(l)}_{\text{cusp}} =$ *collinear anomalous dimension*

- ✔ *What about finite part of the amplitude* $\text{Fin}(s/t)$*? Does it have a simple structure?*

$$\text{Fin}_{\text{QCD}}(s/t) = [\text{4 pages long mess}]\,, \qquad \text{Fin}_{\mathcal{N}=4}(s/t) = \text{BDS conjecture}$$

Four-gluon amplitude in $\mathcal{N}=4$ SYM at weak coupling II

- ✔ Bern-Dixon-Smirnov (BDS) conjecture:

$$\text{Fin}(s/t) = 1 + a\left[\tfrac{1}{2}\ln^2(s/t) + 4\zeta_2\right] + O(a^2) \stackrel{\text{all loops}}{\Longrightarrow} \exp\left[\frac{\Gamma_{\text{cusp}}(a)}{4}\ln^2(s/t) + \text{const}\right]$$

  - ✗ Compared to QCD,
    - (i) the complicated functions of s/t are replaced by the elementary function $\ln^2(s/t)$;
    - (ii) no higher powers of logs appear in $\ln(\text{Fin}(s/t))$ at higher loops;
    - (iii) the coefficient of $\ln^2(s/t)$ is determined by the cusp anomalous dimension $\Gamma_{\text{cusp}}(a)$ just like the coefficient of the double IR pole.
  - ✗ The conjecture has been verified up to three loops [Anastasiou,Bern,Dixon,Kosower'03],[Bern,Dixon,Smirnov'05]
  - ✗ A similar conjecture exists for n-gluon MHV amplitudes [Bern,Dixon,Smirnov'05]
  - ✗ It has been confirmed for $n=5$ at two loops [Cachazo,Spradlin,Volovich'04], [Bern,Czakon,Kosower,Roiban,Smirnov'06]

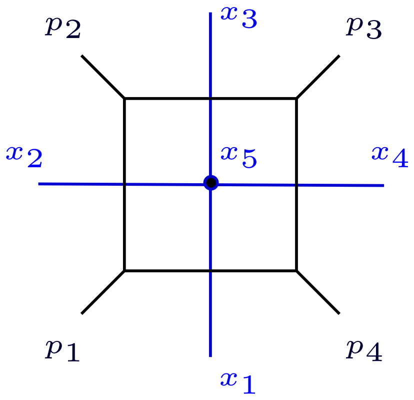
- ✔ *Surprising features of the finite part of the MHV amplitudes in planar $\mathcal{N}=4$ SYM:*
  - ☞ Why should finite corrections exponentiate?
  - ☞ Why should they be related to the cusp anomaly of Wilson loop?

Dual conformal symmetry

Examine one-loop ‘scalar box’ diagram

- ✔ Change variables to go to a *dual ‘coordinate space’* picture (not a Fourier transform!)

$$p_1 = x_1 - x_2 \equiv x_{12}\,, \quad p_2 = x_{23}\,, \quad p_3 = x_{34}\,, \quad p_4 = x_{41}\,, \quad k = x_{15}$$



$$= \int \frac{d^4k\,(p_1+p_2)^2(p_2+p_3)^2}{k^2(k-p_1)^2(k-p_1-p_2)^2(k+p_4)^2} = \int \frac{d^4x_5\, x_{13}^2 x_{24}^2}{x_{15}^2 x_{25}^2 x_{35}^2 x_{45}^2}$$

Check conformal invariance by inversion $x_i^\mu \to x_i^\mu / x_i^2$

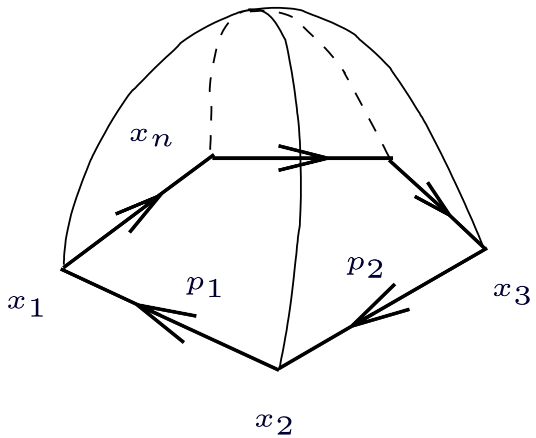
[Broadhurst],[Drummond,Henn,Smirnov,Sokatchev]

- ✔ The integral is invariant under conformal $SO(2,4)$ transformations in the dual space!
- ✔ The symmetry *is not related* to conformal $SO(2,4)$ symmetry of $\mathcal{N}=4$ SYM
- ✔ All scalar integrals contributing to A_4 up to four loops possess the dual conformal invariance!
- ✔ If the dual conformal symmetry survives to all loops, it allows us to determine four- and five-gluon planar scattering amplitudes to all loops! [Drummond,Henn,GK,Sokatchev],[Alday,Maldacena]
- ✔ Dual conformality is slightly broken by the infrared regulator
- ✔ For *planar* integrals only!

Four-gluon amplitude from AdS/CFT

Alday-Maldacena proposal:

✔ On-shell scattering amplitude is described by a classical string world-sheet in AdS$_5$



✗ On-shell gluon momenta $p_1^\mu, \ldots, p_n^\mu$ define sequence of light-like segments on the boundary

✗ The closed contour has n cusps with the *dual coordinates* x_i^μ (the same as at weak coupling!)

$$x^\mu_{i,i+1} \equiv x_i^\mu - x_{i+1}^\mu := p_i^\mu$$

The dual conformal symmetry also exists at strong coupling!

✔ Is in agreement with the Bern-Dixon-Smirnov (BDS) ansatz for $n = 4$ amplitudes

✔ Admits generalization to arbitrary $n-$gluon amplitudes but it is difficult to construct explicit solutions for 'minimal surface' in AdS

✔ Agreement with the BDS ansatz is also observed for $n = 5$ gluon amplitudes [Komargodski] but disagreement is found for $n \to \infty$ $\mapsto$ *the BDS ansatz needs to be modified* [Alday,Maldacena]

The same questions to answer as at weak coupling:

☞ *Why should finite corrections exponentiate?*

☞ *Why should they be related to the cusp anomaly of Wilson loop?*

From gluon amplitudes to Wilson loops

Common properties of gluon scattering amplitudes at both weak and strong coupling:

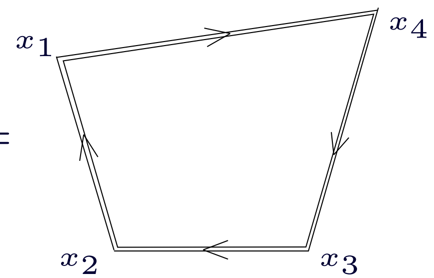
(1) IR divergences of $\mathcal{A}_4$ are in one-to-one correspondence with UV div. of *cusped Wilson loops*

(2) The gluons scattering amplitudes possess a hidden *dual conformal symmetry*

☞ *Is it possible to identify the object in $\mathcal{N}=4$ SYM for which both properties are manifest ?*

Yes! The expectation value of light-like Wilson loop in $\mathcal{N}=4$ SYM [Drummond-Henn-GK-Sokatchev]

$$W(C_4) = \frac{1}{N_c}\langle 0|\mathrm{Tr}\,\mathrm{P}\exp\left(ig\oint_{C_4} dx^\mu A_\mu(x)\right)|0\rangle\,, \qquad C_4 =$$



✔ Gauge invariant functional of the integration contour C_4 in Minkowski space-time

✔ The contour is made out of 4 light-like segments $C_4 = \ell_1 \cup \ell_2 \cup \ell_3 \cup \ell_4$ joining the cusp points x_i^μ

$$x_i^\mu - x_{i+1}^\mu = p_i^\mu = \text{on-shell gluon momenta}$$

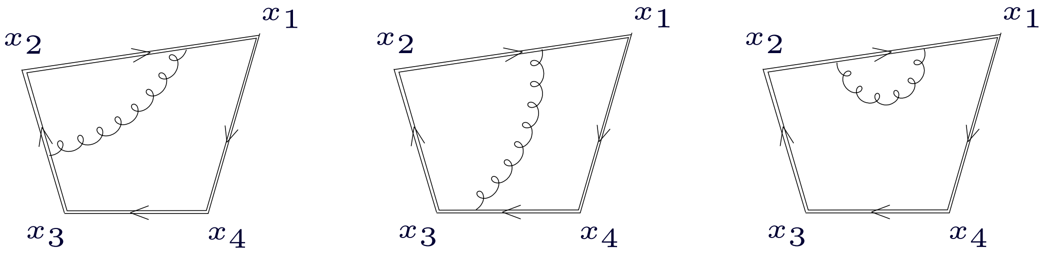
✔ The contour C_4 has four light-like cusps $\mapsto$ $W(C_4)$ has UV divergencies

✔ Conformal symmetry of $\mathcal{N}=4$ SYM $\mapsto$ conformal invariance of $W(C_4)$ in dual coordinates x^μ

Gluon scattering amplitudes/Wilson loop duality I

The one-loop expression for the light-like Wilson loop (with $x_{jk}^2 = (x_j - x_k)^2$) [Drummond,GK,Sokatchev]

$\ln W(C_4) =$



$$= \frac{g^2}{4\pi^2} C_F \left\{ -\frac{1}{{\epsilon_{\rm UV}}^2} \left[(-x_{13}^2 \mu^2)^{\epsilon_{\rm UV}} + (-x_{24}^2 \mu^2)^{\epsilon_{\rm UV}} \right] + \frac{1}{2} \ln^2 \left(\frac{x_{13}^2}{x_{24}^2} \right) + \text{const} \right\} + O(g^4)$$

The one-loop expression for the gluon scattering amplitude

$$\ln \mathcal{A}_4(s,t) = \frac{g^2}{4\pi^2} C_F \left\{ -\frac{1}{{\epsilon_{\rm IR}}^2} \left[(-s/\mu_{\rm IR}^2)^{\epsilon_{\rm IR}} + (-t/\mu_{\rm IR}^2)^{\epsilon_{\rm IR}} \right] + \frac{1}{2} \ln^2 \left(\frac{s}{t} \right) + \text{const} \right\} + O(g^4)$$

✔ Identity the light-like segments with the on-shell gluon momenta $x_{i,i+1}^\mu \equiv x_i^\mu - x_{i+1}^\mu := p_i^\mu$:

$$x_{13}^2 \, \mu^2 := s/\mu_{\rm IR}^2 \,, \qquad x_{24}^2 \, \mu^2 := t/\mu_{\rm IR}^2 \,, \qquad x_{13}^2/x_{24}^2 := s/t$$

- ☞ UV divergencies of the light-like Wilson loop match IR divergences of the gluon amplitude
- ☞ the finite $\sim \ln^2(s/t)$ corrections coincide to one loop!

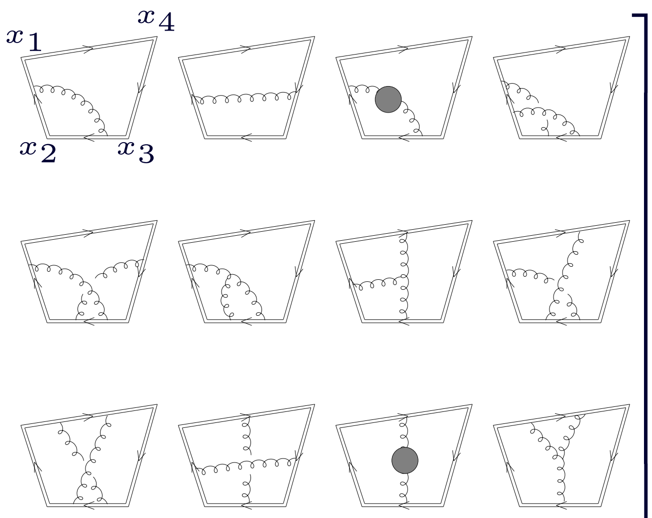
Gluon scattering amplitudes/Wilson loop duality II

Drummond-(Henn)-GK-Sokatchev proposal: *gluon amplitudes are dual to light-like Wilson loops*

$$\ln \mathcal{A}_4 = \ln W(C_4) + O(1/N_c^2, \epsilon_{\rm IR})\,.$$

- ✔ At strong coupling, the relation holds to leading order in $1/\sqrt{\lambda}$ [Alday,Maldacena]
- ✔ At weak coupling, the relation was verified to two loops [Drummond,Henn,GK,Sokatchev]

$$\ln \mathcal{A}_4 = \ln W(C_4) = \left[\quad\right] = \frac{1}{4}\Gamma_{\rm cusp}(g)\ln^2(s/t) + \text{Div}$$



- ✔ Generalization to $n \geq 5$ gluon MHV amplitudes

$$\ln \mathcal{A}_n^{(\rm MHV)} = \ln W(C_n) + O(1/N_c^2)\,, \quad C_n = \text{light-like } n-\text{(poly)gon}$$

  - ✗ At weak coupling, matches the BDS ansatz to one loop [Brandhuber,Heslop,Travaglini]
  - ✗ The duality relation for $n = 5$ (pentagon) was verified to two loops [Drummond,Henn,GK,Sokatchev]

Conformal Ward identities for light-like Wilson loop

Main idea: *make use of conformal invariance of light-like Wilson loops in $\mathcal{N}=4$ SYM + duality relation to fix the finite part of $n-$gluon amplitudes*

- ✔ Conformal $SO(2,4)$ transformations map light-like polygon C_n into another light-like polygon C'_n
- ✔ If the Wilson loop $W(C_n)$ were well-defined (=finite) in $D=4$ dimensions then

$$W(C_n)=W(C'_n)$$

- ✔ ... but $W(C_n)$ has cusp UV singularities $\mapsto$ dim.reg. breaks conformal invariance

$$W(C_n) = W(C'_n) \times \text{[cusp anomaly]}$$

- ✔ *All-loop* anomalous conformal Ward identities for the *finite part* of the Wilson loop

$$W(C_n) = \exp(F_n) \times \text{[UV divergencies]}$$

under dilatations, $\mathbb{D}$, and special conformal transformations, $\mathbb{K}^\mu$, [Drummond,Henn,GK,Sokatchev]

$$\mathbb{D}\, F_n \equiv \sum_{i=1}^{n} (x_i \cdot \partial_{x_i}) F_n = 0$$

$$\mathbb{K}^\mu\, F_n \equiv \sum_{i=1}^{n} \left[2x_i^\mu (x_i \cdot \partial_{x_i}) - x_i^2 \partial_{x_i}^\mu\right] F_n = \frac{1}{2}\Gamma_{\text{cusp}}(a) \sum_{i=1}^{n} x_{i,i+1}^\mu \ln\Big(\frac{x_{i,i+2}^2}{x_{i-1,i+1}^2}\Big)$$

The same relations also hold at strong coupling [Alday,Maldacena],[Komargodski]

Finite part of light-like Wilson loops

The consequences of the conformal Ward identity for the finite part of the Wilson loop W_n

- ✔ $n = 4, 5$ are special: there are no conformal invariants (too few distances due to $x^2_{i,i+1} = 0$)
  $\Longrightarrow$ the Ward identity has a *unique all-loop solution* (up to an additive constant)

$$F_4 = \frac{1}{4}\Gamma_{\text{cusp}}(a)\ln^2\left(\frac{x_{13}^2}{x_{24}^2}\right) + \text{ const },$$

$$F_5 = -\frac{1}{8}\Gamma_{\text{cusp}}(a)\sum_{i=1}^{5}\ln\left(\frac{x^2_{i,i+2}}{x^2_{i,i+3}}\right)\ln\left(\frac{x^2_{i+1,i+3}}{x^2_{i+2,i+4}}\right) + \text{ const}$$

  Exactly the functional forms of the BDS ansatz for the 4- and 5-point MHV amplitudes!

- ✔ Starting from $n = 6$ there are conformal invariants in the form of cross-ratios

$$u_1 = \frac{x_{13}^2 x_{46}^2}{x_{14}^2 x_{36}^2}, \qquad u_2 = \frac{x_{24}^2 x_{15}^2}{x_{25}^2 x_{14}^2}, \qquad u_3 = \frac{x_{35}^2 x_{26}^2}{x_{36}^2 x_{25}^2}$$

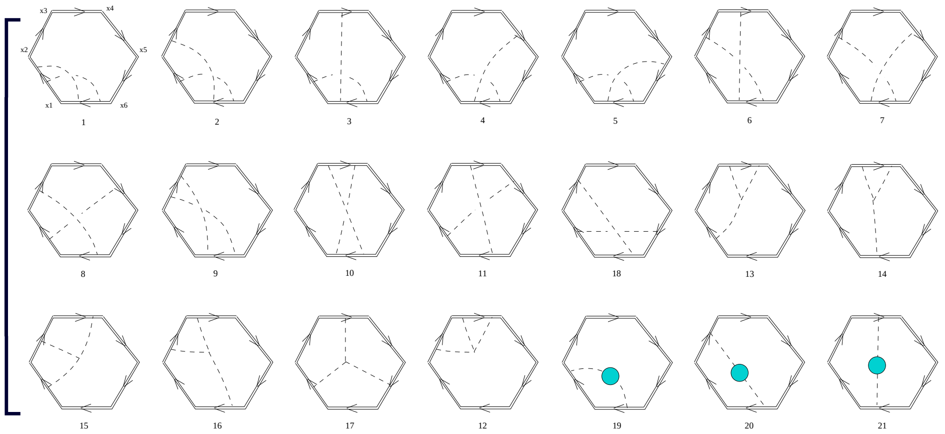
  Hence the general solution of the Ward identity for $W(C_n)$ with $n \geq 6$ contains *an arbitrary function* of the conformal cross-ratios.

- ✔ The BDS ansatz is a solution of the conformal Ward identity for *arbitrary* n but the ansatz should be modified for $n \geq 6$ starting from two loops... *what is a missing function of u_1, u_2 and u_3?*

Discrepancy function

✔ We computed the two-loop hexagon Wilson loop $W(C_6)$ ... [Drummond, Henn, GK, Sokatchev'07]

$$\ln W(C_6) = \Bigg[$$

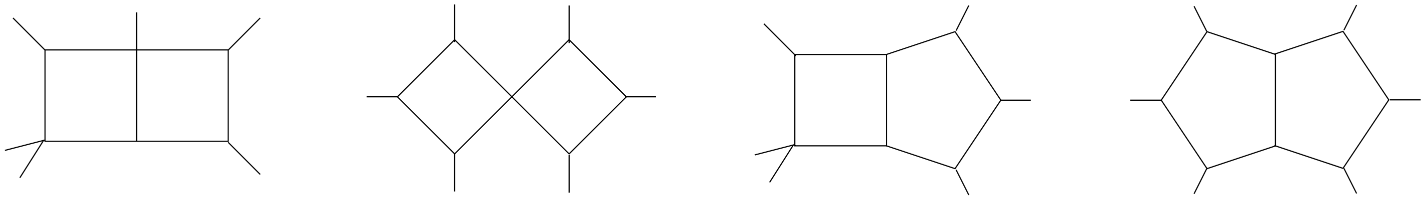


$$\Bigg]$$

... and found a **discrepancy** $\ln W(C_6) \neq \ln \mathcal{M}_6^{(\mathrm{BDS})}$

✔ Bern-Dixon-Kosower-Roiban-Spradlin-Vergu-Volovich computed 6-gluon amplitude to 2 loops

$$\mathcal{M}_6^{(\mathrm{MHV})} =$$



$$+ \ldots$$

... and found a **discrepancy** $\ln \mathcal{M}_6^{(\mathrm{MHV})} \neq \ln \mathcal{M}_6^{(\mathrm{BDS})}$

☞ The BDS ansatz **fails** for $n = 6$ starting from two loops.

☞ *What about Wilson loop duality?* $\ln \mathcal{M}_6^{(\mathrm{MHV})} \stackrel{?}{=} \ln W(C_6)$

6-gluon amplitude/hexagon Wilson loop duality

- Comparison between the DHKS discrepancy function $\Delta_{\rm WL}$ and the BDKRSVV results for the six-gluon amplitude $\Delta_{\rm MHV}$:

Kinematical point	(u_1, u_2, u_3)	$\Delta_{\rm WL} - \Delta_{\rm WL}^{(0)}$	$\Delta_{\rm MHV} - \Delta_{\rm MHV}^{(0)}$
$K^{(1)}$	$(1/4, 1/4, 1/4)$	$< 10^{-5}$	-0.018 ± 0.023
$K^{(2)}$	$(0.547253, 0.203822, 0.88127)$	-2.75533	-2.753 ± 0.015
$K^{(3)}$	$(28/17, 16/5, 112/85)$	-4.74460	-4.7445 ± 0.0075
$K^{(4)}$	$(1/9, 1/9, 1/9)$	4.09138	4.12 ± 0.10
$K^{(5)}$	$(4/81, 4/81, 4/81)$	9.72553	10.00 ± 0.50

evaluated for different kinematical configurations, e.g.

$$K^{(1)}: \quad x_{13}^2 = -0.7236200\,, \quad x_{24}^2 = -0.9213500\,, \quad x_{35}^2 = -0.2723200\,, \quad x_{46}^2 = -0.3582300\,, \quad x_{36}^2 = -0.4825841\,,$$
$$x_{15}^2 = -0.4235500\,, \quad x_{26}^2 = -0.3218573\,, \quad x_{14}^2 = -2.1486192\,, \quad x_{25}^2 = -0.7264904\,.$$

- Two nontrivial functions coincide with an accuracy $< 10^{-4}$!

✌ *The Wilson loop/gluon scattering amplitude duality holds at $n = 6$ to two loops!!*

✌ *There are now little doubts that the duality relation also holds for arbitrary n to all loops!!!*

Four-gluon amplitude/Wilson loop duality in QCD

Finite part of four-gluon amplitude in QCD at two loops

$$\text{Fin}_{\color{red}{\text{QCD}}}{}^{(2)}(s,t,u) = A(x,y,z) + O(1/N_c^2, n_f/N_c)$$

[Glover,Oleari,Tejeda-Yeomans'01]

with notations $x = -\frac{t}{s}$, $y = -\frac{u}{s}$, $z = -\frac{u}{t}$, $X = \log x$, $Y = \log y$, $S = \log z$

$$\begin{aligned}
A \quad = \quad & \Big\{ \Big(48\,\text{Li}_4(x) - 48\,\text{Li}_4(y) - 128\,\text{Li}_4(z) + 40\,\text{Li}_3(x)\,X - 64\,\text{Li}_3(x)\,Y - \tfrac{98}{3}\,\text{Li}_3(x) + 64\,\text{Li}_3(y)\,X - 40\,\text{Li}_3(y)\,Y + 18\,\text{Li}_3(y) \\
& + \tfrac{98}{3}\,\text{Li}_2(x)\,X - \tfrac{16}{3}\,\text{Li}_2(x)\,\pi^2 - 18\,\text{Li}_2(y)\,Y - \tfrac{37}{6}\,X^4 + 28\,X^3\,Y - \tfrac{23}{3}\,X^3 - 16\,X^2\,Y^2 + \tfrac{49}{3}\,X^2\,Y - \tfrac{35}{3}\,X^2\,\pi^2 - \tfrac{38}{3}\,X^2 \\
& - \tfrac{22}{3}\,S\,X^2 - \tfrac{20}{3}\,X\,Y^3 - 9\,X\,Y^2 + 8\,X\,Y\,\pi^2 + 10\,X\,Y - \tfrac{31}{12}\,X\,\pi^2 - 22\,\zeta_3\,X + \tfrac{22}{3}\,S\,X + \tfrac{37}{27}\,X + \tfrac{11}{6}\,Y^4 - \tfrac{41}{9}\,Y^3 - \tfrac{11}{3}\,Y^2\,\pi^2 \\
& - \tfrac{22}{3}\,S\,Y^2 + \tfrac{266}{9}\,Y^2 - \tfrac{35}{12}\,Y\,\pi^2 + \tfrac{418}{9}\,S\,Y + \tfrac{257}{9}\,Y + 18\,\zeta_3\,Y - \tfrac{31}{30}\,\pi^4 - \tfrac{11}{9}\,S\,\pi^2 + \tfrac{31}{9}\,\pi^2 + \tfrac{242}{9}\,S^2 + \tfrac{418}{9}\,\zeta_3 + \tfrac{2156}{27}\,S \\
& - \tfrac{11093}{81} - 8\,S\,\zeta_3 \Big)\,\frac{t^2}{s^2} + \Big(-256\,\text{Li}_4(x) - 96\,\text{Li}_4(y) + 96\,\text{Li}_4(z) + 80\,\text{Li}_3(x)\,X + 48\,\text{Li}_3(x)\,Y - \tfrac{64}{3}\,\text{Li}_3(x) - 48\,\text{Li}_3(y)\,X \\
& + 96\,\text{Li}_3(y)\,Y - \tfrac{304}{3}\,\text{Li}_3(y) + \tfrac{64}{3}\,\text{Li}_2(x)\,X - \tfrac{32}{3}\,\text{Li}_2(x)\,\pi^2 + \tfrac{304}{3}\,\text{Li}_2(y)\,Y + \tfrac{26}{3}\,X^4 - \tfrac{64}{3}\,X^3\,Y - \tfrac{64}{3}\,X^3 + 20\,X^2\,Y^2 \\
& + \tfrac{136}{3}\,X^2\,Y + 24\,X^2\,\pi^2 + 76\,X^2 - \tfrac{88}{3}\,S\,X^2 + \tfrac{8}{3}\,X\,Y^3 + \tfrac{104}{3}\,X\,Y^2 - \tfrac{16}{3}\,X\,Y\,\pi^2 + \tfrac{176}{3}\,S\,X\,Y - \tfrac{136}{3}\,X\,Y - \tfrac{50}{3}\,X\,\pi^2 \\
& - 48\,\zeta_3\,X + \tfrac{2350}{27}\,X + \tfrac{440}{3}\,S\,X + 4\,Y^4 - \tfrac{176}{9}\,Y^3 + \tfrac{4}{3}\,Y^2\,\pi^2 - \tfrac{176}{3}\,S\,Y^2 - \tfrac{494}{9}\,Y\,\pi^2 + \tfrac{5392}{27}\,Y - 64\,\zeta_3\,Y + \tfrac{496}{45}\,\pi^4 \\
& - \tfrac{308}{9}\,S\,\pi^2 + \tfrac{200}{9}\,\pi^2 + \tfrac{968}{9}\,S^2 + \tfrac{8624}{27}\,S - \tfrac{44372}{81} + \tfrac{1864}{9}\,\zeta_3 - 32\,S\,\zeta_3 \Big)\,\frac{t}{u} + \Big(\tfrac{88}{3}\,\text{Li}_3(x) - \tfrac{88}{3}\,\text{Li}_2(x)\,X + 2\,X^4 - 8\,X^3\,Y \\
& - \tfrac{220}{9}\,X^3 + 12\,X^2\,Y^2 + \tfrac{88}{3}\,X^2\,Y + \tfrac{8}{3}\,X^2\,\pi^2 - \tfrac{88}{3}\,S\,X^2 + \tfrac{304}{9}\,X^2 - 8\,X\,Y^3 - \tfrac{16}{3}\,X\,Y\,\pi^2 + \tfrac{176}{3}\,S\,X\,Y - \tfrac{77}{3}\,X\,\pi^2 \\
& + \tfrac{1616}{27}\,X + \tfrac{968}{9}\,S\,X - 8\,\zeta_3\,X + 4\,Y^4 - \tfrac{176}{9}\,Y^3 - \tfrac{20}{3}\,Y^2\,\pi^2 - \tfrac{176}{3}\,S\,Y^2 - \tfrac{638}{9}\,Y\,\pi^2 - 16\,\zeta_3\,Y + \tfrac{5392}{27}\,Y - \tfrac{4}{15}\,\pi^4 - \tfrac{308}{9}\,S\,\pi^2 \\
& - 20\,\pi^2 - 32\,S\,\zeta_3 + \tfrac{1408}{9}\,\zeta_3 + \tfrac{968}{9}\,S^2 - \tfrac{44372}{81} + \tfrac{8624}{27}\,S \Big)\,\frac{t^2}{u^2} + \Big(\tfrac{44}{3}\,\text{Li}_3(x) - \tfrac{44}{3}\,\text{Li}_2(x)\,X - X^4 + \tfrac{110}{9}\,X^3 - \tfrac{22}{3}\,X^2\,Y \\
& + \tfrac{14}{3}\,X^2\,\pi^2 + \tfrac{44}{3}\,S\,X^2 - \tfrac{152}{9}\,X^2 - 10\,X\,Y + \tfrac{11}{2}\,X\,\pi^2 + 4\,\zeta_3\,X - \tfrac{484}{9}\,S\,X - \tfrac{808}{27}\,X + \tfrac{7}{30}\,\pi^4 - \tfrac{31}{9}\,\pi^2 \\
& + \tfrac{11}{9}\,S\,\pi^2 - \tfrac{418}{9}\,\zeta_3 - \tfrac{242}{9}\,S^2 - \tfrac{2156}{27}\,S + 8\,S\,\zeta_3 + \tfrac{11093}{81} \Big)\,\frac{u\,t}{s^2} + \Big(-176\,\text{Li}_4(x) + 88\,\text{Li}_3(x)\,X - 168\,\text{Li}_3(x)\,Y - \ldots
\end{aligned}$$

Four-gluon amplitude/Wilson loop duality in QCD II

✔ Planar four-gluon QCD scattering amplitude in the Regge limit $s \gg -t$ [Schnitzer'76],[Fadin,Kuraev,Lipatov'76]

$$\mathcal{M}_4^{(\mathrm{QCD})}(s,t) \sim (s/(-t))^{\omega_R(-t)} + \dots$$

The Regge trajectory $\omega_R(-t)$ is known to two loops [Fadin,Fiore,Kotsky'96]

✔ The all-loop gluon Regge trajectory in QCD [GK'96]

$$\omega_R^{(\mathrm{QCD})}(-t) = \frac{1}{2}\int_{(-t)}^{\mu_{\mathrm{IR}}^2} \frac{dk_\perp^2}{k_\perp^2} \Gamma_{\mathrm{cusp}}(a(k_\perp^2)) + \Gamma_R(a(-t)) + [\text{poles in } 1/\epsilon_{\mathrm{IR}}]\,,$$

✔ Rectangular Wilson loop in QCD in the Regge limit $|x_{13}^2| \gg |x_{24}^2|$

$$W^{(\mathrm{QCD})}(C_4) \sim \left(x_{13}^2/(-x_{24}^2)\right)^{\omega_{\mathrm{W}}(-x_{24}^2)} + \dots$$

✔ The all-loop Wilson loop ‘trajectory’ in QCD

$$\omega_{\mathrm{W}}^{(\mathrm{QCD})}(-t) = \frac{1}{2}\int_{(-t)}^{\mu_{\mathrm{UV}}^2} \frac{dk_\perp^2}{k_\perp^2} \Gamma_{\mathrm{cusp}}(a(k_\perp^2)) + \Gamma_{\mathrm{W}}(a(-t)) + [\text{poles in } 1/\epsilon_{\mathrm{UV}}]\,,$$

✔ *The duality relation holds in QCD in the Regge limit only!* [GK'96]

$$\ln \mathcal{M}_4^{(\mathrm{QCD})}(s,t) = \ln W^{(\mathrm{QCD})}(C_4) + O(t/s)$$

while in $\mathcal{N}=4$ SYM it is exact for arbitrary t/s

Conclusions and open questions

- ✔ Planar gluon scattering amplitudes possess the dual conformal symmetry at both weak and strong coupling (is not a symmetry of the full $\mathcal{N}=4$ SYM!)
- ✔ This symmetry becomes manifest within the gauge scattering amplitude/Wilson loop duality
- ✔ We do not understand the origin of this symmetry but we do know how to make use of it:
  - ✗ The anomalous conformal Ward identities uniquely fix the form of the finite part of $n=4$ and $n=5$ gluon amplitudes, in complete agreement with the BDS conjecture
  - ✗ Starting from $n=6$, the conformal symmetry is *not* sufficient to fix the finite part of the Wilson loop (=discrepancy function)
  - ✗ Remarkably enough, the DHKS discrepancy function for the $n=6$ Wilson loop coincides with the BDKRSVV discrepancy function for the six-gluon amplitude
- ✔ We have now good reasons to believe that the Wilson loop/gluon amplitude duality holds for any n to all loops... but
  - ✗ What is the origin of the dual conformal symmetry?
  - ✗ Who controls a nontrivial discrepancy function of conformal ratios?

Should be related to integrability of planar $\mathcal{N}=4$ SYM. More work is needed!

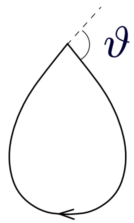
Back-up slides

What is the cusp anomalous dimension

✔ Cusp anomaly is a very ‘unfortunate’ feature of Wilson loops evaluated over an *Euclidean* closed contour with a cusp – generates the anomalous dimension [Polyakov’80]

$$\langle \mathrm{tr}\,\mathsf{P}\exp\left(i\oint_C dx\cdot A(x)\right)\rangle \sim (\Lambda_{\rm UV})^{\Gamma_{\rm cusp}(g,\vartheta)}\,,$$

$C =$



✔ A very ‘fortunate’ property of Wilson loop – the cusp anomaly controls the *infrared* asymptotics of scattering amplitudes in gauge theories [GK, Radyushkin’86]

- ✗ The integration contour C is defined by the particle momenta
- ✗ The cusp angle ϑ is related to the scattering angles in *Minkowski* space-time, $|\vartheta| \gg 1$

$$\Gamma_{\rm cusp}(g,\vartheta) = \vartheta\,\Gamma_{\rm cusp}(g) + O(\vartheta^0)\,,$$

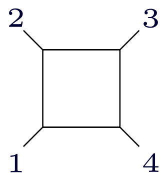
✔ *The cusp anomalous dimension* $\Gamma_{\rm cusp}(g)$ is an ubiquitous observable in gauge theories: [GK’89]

- ✗ Logarithmic scaling of anomalous dimensions of high-spin Wilson operators;
- ✗ IR singularities of on-shell gluon scattering amplitudes;
- ✗ Gluon Regge trajectory;
- ✗ Sudakov asymptotics of elastic form factors;
- ✗ ...

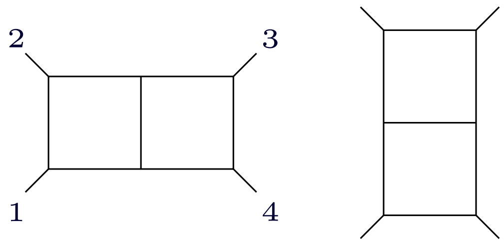
Four-gluon planar amplitude at weak coupling

Weak coupling corrections to $A_4/A_4^{(0)}$ can be expressed in terms of scalar integrals:

- ✔ One loop: [Green,Schwarz,Brink'82]

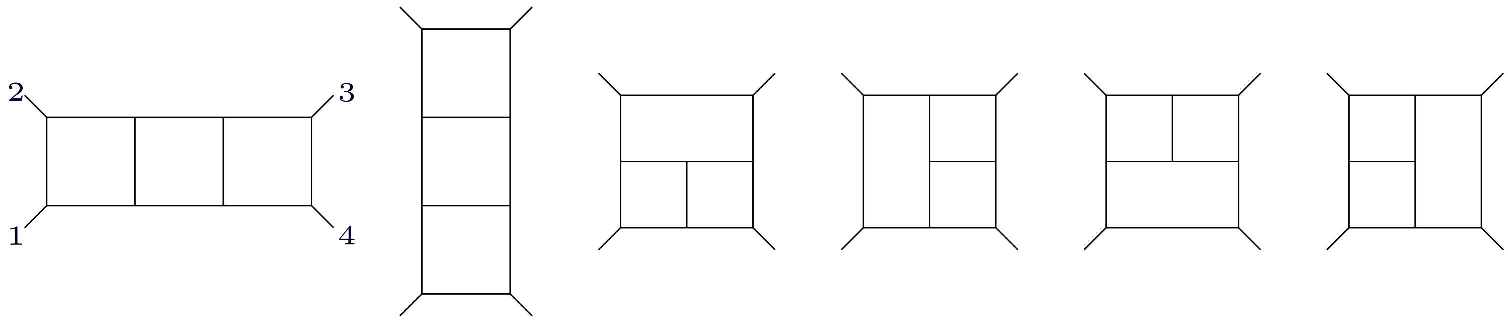


- ✔ Two loops: [Bern,Rozowsky,Yan'97]



all-loop iteration structure conjectured [Anastasiou,Bern,Dixon,Kosower'03]

- ✔ Three loops: [Bern,Dixon,Smirnov'05]



iteration structure confirmed!

- ✔ Four loops: scalar integrals of 8 different topologies are identified [Bern,Czakov,Dixon,Kosower,Smirnov'06]

Light-like Wilson loops

To lowest order in the coupling constant,

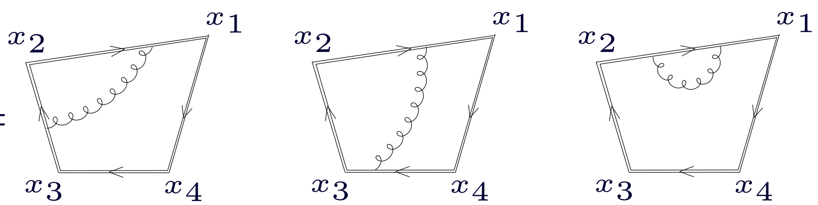
$$W(C_4) = 1 + \frac{1}{2}(ig)^2 C_F \sum_{1\leq j,\, k\leq 4} \int_{\ell_j} dx^\mu \int_{\ell_k} dy^\nu \, G_{\mu\nu}(x-y) + O(g^4)\,, \qquad (1)$$

- The gluon propagator in the coordinate representation (the Feynman gauge + dimensional regularization, $D = 4 - 2\epsilon$)

$$G_{\mu\nu}(x) = -g_{\mu\nu} \frac{\Gamma(1-\epsilon)}{4\pi^2} (-x^2 + i0)^{-1+\epsilon} (\mu^2\pi)^\epsilon \,.$$

- Feynman diagram representation

$$\ln W(C_4) =$$



- The light-like Wilson loop is IR finite but has UV divergences due to cusps on the integration contour C_4

$$\ln W(C_4) = \frac{g^2}{4\pi^2} C_F \left\{ -\frac{1}{2\epsilon^2} \sum_{i=1}^{4} \left(-x_{i-1,i+1}^2 \, \mu^2\right)^\epsilon + O(\epsilon^0) \right\} + O(g^4)\,.$$