MHV amplitudes in N=4 SUSY Yang-Mills theory and quantum geometry of the momentum space

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MHV amplitudes are the simplest objects to discuss within the gauge/string duality

Simplification at large N - MHV amplitudes are described by the single function of the kinematical variables

Properties of the tree amplitudes

- Holomorphy - it depends only on the "'half'' of the momentum variables $p_{\alpha,\dot{\alpha}} = \lambda_\alpha \bar{\lambda}_{\dot{\alpha}}$
- Fermionic representation (Nair,88) - tree amplitudes are the correlators of the chiral fermions of the sphere

- Tree amplitudes admit the twistor representation(Witten,04). Tree MHV amplitudes are localized on the curves in the twistor space. Twistor space in the B model - $CP(3\|4)$
- Localization follows from the holomorphic property of the tree MHV amplitude. Possible link to integrability via fermionic representation
- Stringy interpretation - fermions are the degrees of freedom on the D1-D5 open strings ended on the Euclidean D1 instanton.

- The MHV has very simple form

$$A(1^-, 2^-, 3^+ \dots, n^+) = g^{n-2} \frac{< 12 >^4}{< 12 >< 23 > \cdots < n1 >}$$

- The on-shell momentum of massless particle in the standard spinor notations reads as $p_{a\dot{a}} = \lambda_a \tilde{\lambda}_{\dot{a}}$, λ_a and $\tilde{\lambda}_{\dot{a}}$ are positive and negative helicity spinors.
- Inner products in spinor notations $< \lambda_1, \lambda_2 >= \epsilon_{ab} \lambda_1^a \lambda_2^b$ and $[\tilde{\lambda}_1 \tilde{\lambda}_2] = \epsilon_{\dot{a}\dot{b}} \tilde{\lambda}_1^{\dot{a}} \tilde{\lambda}_2^{\dot{b}}$.

- The generating function for the tree MHV amplitudes - solution to the self-duality equation with the particular boundary conditions (Bardeen 96, Rosly-Selivanov 97).

$$A_{\dot\alpha} = g^{-1}\partial_{\dot\alpha} g$$

$$g_{ptb}(\rho) = 1 + \sum_J g_J(\rho) E_J + \dots + \sum_{J_1 \dots J_L} g_{J_1 \dots J_L}(\rho) E_{J_1} \dots E_{J_L} + \dots$$

- The E_{J_1} is the solution to the free equation of motion
- The coefficients are derived from self-duality condition

$$g_{J_1 \dots J_L}(\rho) = \frac{< \rho, q^{j_1} >< j_1, q^{j_2} >< j_2, q^{j_3} > \cdots < j_{L-1}, q^{j_L} >}{< \rho, j_1 >< j_1, j_2 >< j_2, j_3 > \cdots < j_{L-1}, j_L >}$$

- The resummation of the tree amplitudes can be done resulting into the so-called MHV Lagrangian (Cachazo-Scwrchk-Witten). The tree MHV amplitude corresponds to the vertex in this Lagrangian
- The same solution to the self-duality equation provides the canonical transformation from the tree light-cone YM lagrangian to the MHV Lagrangian (Rosly-A.G., 04). There is still problem concerning the derivation of the non -vanishing all-plus one-loop amplitude from the Jacobian of this transformation
- The proper analogy: instanton solution to the selfduality equation generates t'Hooft vertex in QCD. Here the different solution to the selfduality equation (perturbiner) generates the infinite set of MHV vertexes

Properties of the loop MHV amplitudes

- Exponentiation of the ratio $\frac{M_{all-loop}}{M_{tree}}$ which contains the IR divergent and finite parts.
- BDS conjecture for the all loop answer

$$\log \frac{M_{all-loop}}{M_{tree}} = (IR_{div} + \Gamma_{cusp}(\lambda) M_{one-loop})$$

- It involves only two main ingredients - one-loop amplitude and all-loop $\Gamma_{cusp}(\lambda)$
- $\Gamma_{cusp}(\lambda)$ obeys the integral equation (Beisert-Eden-Staudacher) and can be derived recursively
- It fails starting from six external legs at two loops (Bern -Dixon-Kosower, Drummond-Henn-Korchemsky-Sokachev, Lipatov-Kotikov) and at large number of legs at strong coupling(Alday-Maldacena)

- There is conjecture that $\frac{M_{all-loop}}{M_{tree}}$ coincides with the Abelian Wilson polygon built from the external light-like momenta p_i.
- The conjecture was formulated at strong coupling (Alday-Maldacena, 06) upon the T-duality at the worldsheet of the string in the AdS_5 geometry
- Checked at weak coupling (one and two loops) as well (Drummond- Henn- Korchemsky- Sokachev, Bern-Dixon-Kosover, Brandhuber-Heslop-Travagnini 07).
- Important role of Ward identities with respect to the special conformal transformation in determination of the Wilson polygon (Drummond-Henn-Korchemsky-Sokachev)
- There is no satisfactory stringy twistor explanation of the loop MHV amplitudes and this duality. Expectation - closed string modes contribute (Cachazo-Swrchk-Witten)

Main Questions

- Is there fermionic representation of the loop MHV amplitudes similar to the tree case?
- Is there link with integrability at generic kinematics ? The integrability behind the amplitudes is known at low-loop Regge limit (Lipatov 93, Faddeev-Korchemsky 94) only
- What is the stringy geometrical origin of the BDS conjecture, if any?
- What is the physical origin of MHV amplitude-Wilson polygon duality?

c=1 example

- Consider c=1 string (1d-target space + Liouville direction). The only degrees of freedom - massless tachyons with the discrete momenta
- Exact answer for the tachyonic amplitudes (Dijkgraaf,Plesser,Moore 94)
- Generating function for the amplitude - τ function for the Toda integrable systems. "'Times"'- generating parameters for the tachyon operators with the different momenta

- Generating function admits representation via chiral fermions on the Riemann surface in the B model

$$x^2 - y^2 = 1$$

in the background of the particular abelian gauge field A(z) which provides the "S-matrix"

- This Riemann surface parameterizes the particular moduli space.
- The fermions in the B model represent the Lagrangian branes in the A model - ZZ branes. They are not literally fermions - better to think of as Wigner functions. Two types of branes FZZT branes - localized in the Lioville direction but extended on the Riemann surface. ZZ branes- extended(semiinfinitely) in the Lioville direction and localized on the Riemann surface

- The generating function for the amplitude

$$\tau(t_k) =< 0|exp(\sum t_k V_k)exp\int(\bar{\psi}A\psi)exp(\sum t_{-k}V_{-k})|0>$$

- There are two sets of times t_k parametrizing the asymptotics and positions of the LL branes=fermions on the curve

- The amplitude can be represented in terms of the "'Wilson polygon"' for the auxiliary abelian gauge field! This gauge field has nothing to do with the initial tachyonic scalar degrees of freedom. The auxiliary abelian gauge field A(z) corresponds to the "point of Grassmanian" and yields the choice of the vacuum state in the theory.
- Riemann surface reflects the hidden moduli space of the theory (chiral ring) and it is quantized. Equation of the Riemann surface becomes the operator acting on the wave function.The following commutation relation is implied

$$[x, y] = i\hbar$$

- This procedure of the quantization of the Riemann surface is familiar in the theory of integrable systems. Quantum Riemann surface =Baxter eqution
- Solution to the Baxter equation - wave function of the single separated variable - Lagrangian brane(Nekrasov-Rubtsov-A.G. 2000)
- Polynomial solution to the Baxter equation - Bethe equations

- Consider the moduli space of the complex structures for genus zero surface with n marked points, $M_{0,n}$. Inequivalent triangulations of the surface can be mapped into set of geodesics on the upper half-plane
- This manifold has the Poisson structure and can be quantized in the different coordinates (Kashaev-Fock-Chekhov, 97-01). The generating function of the special canonical transformations (flip) on this symplectic manifold is provided by $Li_2(z)$ where z- is so-called shear coordinate related to the conformal cross-ratio of four points on the real axe

$$exp(z) = \frac{(x_1 - x_2)(x_3 - x_4)}{(x_1 - x_3)(x_2 - x_4)}$$

- The natural objects geodesics can be determined in terms of shear variables z_a
- The simplectic structure in terms of these variables is simple $\sum_a dz_a \wedge dz_b$ where a corresponds to oriented edge and b is edge next to the right
- Upon quantization

$$[Z_a, Z_b] = 2\pi\hbar\{z_a, z_b\}$$

- Five-term Roger's relation for $Li_2(x)$ classically and its quantization for quantum dilogariphm

$$\Psi(V)\Psi(U) = \Psi(U)\Psi(-UV)\Psi(V) \qquad UV = qVU$$

- Quantum mechanically there is flip-generating operator of the "'duality"' K acting on this phase space with the property $\hat{K}^5 = 1$. It is the analogue of the Q-operator in the theory of the integrable systems since it is build from the eigenfunction of the "'quantum spectral curve operator"'

$$e^u + e^v + 1 = 0$$

gets transformed into the Baxter equation

$$(e^{i\partial_v} + e^v + 1)Q(v) = 0$$

with the Poisson bracket

$$[v, u] = i\hbar$$

- Let us use the representation for the finite part of the one-loop amplitude as the sum of the following dilogariphms

$$\sum_i \sum_r Li_2\left(1 - \frac{x^2_{i,i+r} x^2_{i+1,i+r+1}}{x^2_{i,i+r+1} x^2_{i-1,i+r}}\right)$$

$$x_{i,k} = p_i - p_k$$

where p_i are the external on-shell momenta

- Conjecture One-loop amplitude with n-gluons is described in terms of the "fermions" living on the spectral curve which is embedded into the mirror of the topological vertex. MHV amplitude - fermion correlator on the spectral curve. Spectral curve parameterizes the moduli space $M_{0,n}$
- BDS conjecture for all-loop answer=quasiclassics of the fermionic correlator with the identification

$$\hbar^{-1} = \Gamma_{cusp}(\lambda)$$

- Fermions represent the D1 instantons with open strings. In the mirror dual geometry fermions represent Lagrangian branes. Fermions live on the moduli space of the complex structures. They are transformed nontrivially under the change of patches on the surface because of its quantum nature
- The spectral curve is embedded as the holomorphic surface in the internal 3-dimensional complex space

$$xy = e^{u} + e^{v} + 1$$

- Quantization of the spectral curve involves the coupling constant
$$\frac{1}{g_{YM}^2} = \frac{\int B_{NS-NS}}{g_s}$$
Usually it is assumed that g_s yields the "Planck constant" for the quantization of the moduli space of the complex structures. However equally some function of Yang-Mills coupling can be considered as the quantization parameter.

- Is our "Planck konstant" for moduli space reasonable? Some "evidences"
- Γ_{cusp} enters into the r.h.s. of loop equations (Drukker-Gross-Ooguri)
- Γ_{cusp} enters into the quantum anomaly term in the Ward identity for Wilson polygon
- There are moduli of the solution at strong coupling if $n > 4$ so their quantization is the natural origin for dilogariphms at strong coupling. Quantization parameter of sigma model is inverse of Γ_{cusp} indeed.
- More generically; selfintersection of worldlines yields the quantization of geometry?

- The expression for the Γ_{cusp} involves the wave function depending on the eigenvalue of the length operator on the moduli phase space.

$$\frac{d < W(\theta) >}{dlogm^2} = \Gamma_{cusp}(\alpha, \theta)$$

where $cosh(\theta) = Trg_1 g_2^{-1}$ with SL(2,R) group elements. Γ_{cusp} at one loop coincides with the wave function on the quantized moduli space

- The variable θ corresponds to the length of the geodesics which the wave function depends on. It is related to the variables x_i introduced above

- Quasiclassics for the solution to the Baxter equation

$$\Psi(z,\hbar) = \int \frac{e^{ipz}}{p.sinh(\pi p) sinh(\pi \hbar p)} dp$$

reduces to

$$\Psi(x) \to exp(\hbar^{-1} Li_2(x) + ...)$$

Arguments of the Li_2 in the expression for the amplitudes correspond to the shear coordinates on the moduli space.

- The one-loop MHV amplitude can be presented in the following form

$$M_{one-loop} \propto < 0|\Psi(z_1)...\Psi(z_n) exp(\psi_k A_{nk} \psi_k)|0>$$

- The variables ψ_k are the modes of the fermion on the spectral curve. The matrix $A_{n,k}$ for the corresponding spectral curve is known (Aganagic-Vafa-Klemm-Marino 03)

- Solution to the Baxter equation is the operator for the Backlund transformation
- In the Regge limit the Baxter operator plays the similar role it provides the Lipatov,s duality transformation. In the thermodynamical limit of the one-loop spin chain one gets the worldsheet T-duality (Korchemsky- A.G. to appear)
- From the worldsheet viewpoint one considers the discretization of the Liuville mode and the Faddeev-Volkov model yields the good candidate for the correct S-matrix.

Introduction for the Conclusion

- The representation of the loop MHV amplitude as the chiral fermion correlator on the spectral curve is suggested. Nontrivial effect of closed string degrees of freedom(Kodaira-Spencer gravity)
- Link to the integrability behind generic MHV amplitudes via fermionic representation. 3-KP integrable system known as the integrable system behind the topological vertex is the good candidate
- BDS conjecture can be reformulated in terms of the quantum geometry of the momentum space with $\Gamma_{cusp}(\lambda)$ as the quantization parameter
- Wilson polygon - MHV amplitude duality is based on the fermionic representation of the amplitude and the gauge field is the "Berry connection"