# Pushing dimensional reduction of QCD to lower temperatures

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arXiv:0801.1566 with A. Kurkela and A. Vuorinen

Really: hep-ph/0604100, A. Vuorinen and L. Yaffe

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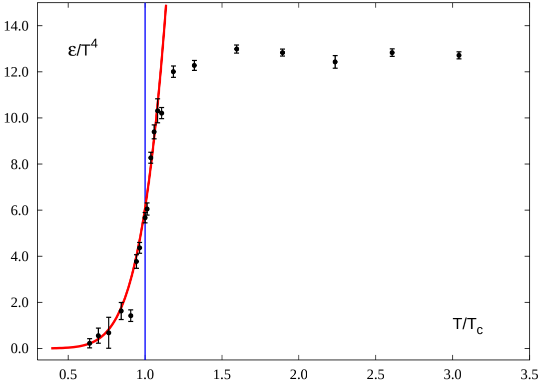
GGI, Florence, June 2008

# Motivation

QCD thermodynamics well understood at low $T$:

hadron resonance gas

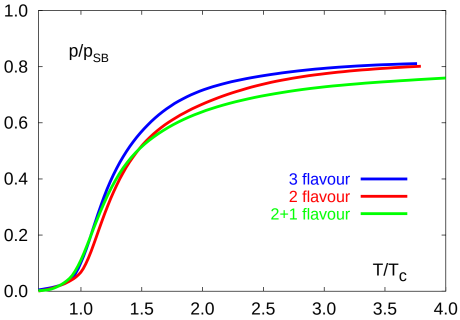



Karsch et al., hep-ph/0303108

# Motivation

... and at asymptotically high $T$: gas of free quarks and gluons

What about $T \sim$ **a few $T_c$, ie. experimental range?**



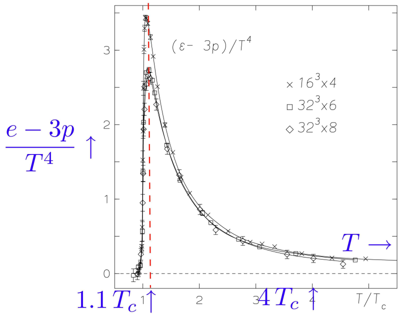

Karsch et al., hep-lat/9602007

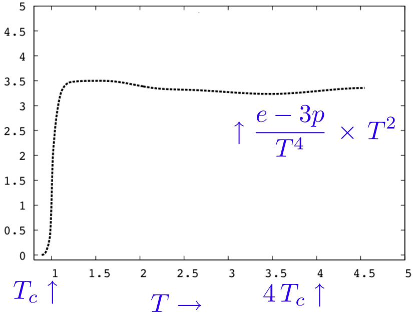
Still far from non-interacting gas

# Motivation

... and at asymptotically high $T$: gas of free quarks and gluons

What about $T \sim$ **a few $T_c$, ie. experimental range?**



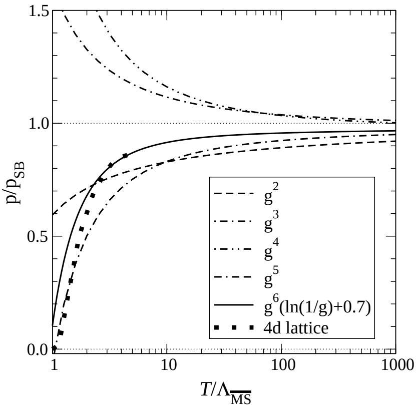




Pisarski, hep-ph/0612191

Far from leading order perturbation theory $(e - 3p)/T^4 \sim \log T$

# Perturbative expansion

IR divergences $\longrightarrow$ non-perturbative (Linde)




Kajantie et al., hep-ph/0211321

Spatial area law $\leftrightarrow$ non-perturbative $\forall\, T$

# Brute force

Solution: $(3+1)d$ lattice simulations

However:

- $N_\tau$ must be large ($\mathcal{O}(10)$) to control $a \longrightarrow 0$ limit

  $T_c$ ? Fodor et al. $\leftrightarrow$ Karsch et al.

- Finite density ??

**Alternative approach?**

# Dimensional reduction

- dim $(d+1)$ system with one compact dimension:
  looks like dim $d$ at distances $\gg \quad \beta = \frac{1}{T}$

$\beta$

- degrees of freedom are static modes $\phi_0(\vec{x})$
  $$\phi(\vec{x},\tau) = T\sum_{n=-\infty}^{+\infty} \exp(i\omega_n\tau)\phi_n(\vec{x})$$
- Effective action: integrate out non-static modes

$$Z = \int \mathcal{D}\phi_0 \mathcal{D}\phi_n \exp(-S_0(\phi_0) - S_n(\phi_0,\phi_n))$$
$$= \int \mathcal{D}\phi_0 \exp(-S_0(\phi_0) - S_{\rm eff}(\phi_0))$$

with

$$\exp(-S_{\rm eff}(\phi_0)) \equiv \int \mathcal{D}\phi_n \exp(-S_n(\phi_0,\phi_n))$$

- In practice?

Goal is to reproduce Green's fncts $\langle\phi_0(\vec{0})\phi_0(\vec{x})\rangle$ for $|\vec{x}| \gg \beta$

$T$ is UV cutoff for $S_{\rm eff}$

# Dimensional reduction for QCD

Asymptotic freedom:
$g(T) \sim 1/\log T$ causes separation of scales at high $T$:

- hard modes, energy $\mathcal{O}(T)$: non-static, esp. fermions (odd Matsubara)
- soft modes, $\mathcal{O}(gT)$: Debye mass $\langle A_0(0)A_0(x)\rangle$
- ultrasoft modes, $\mathcal{O}(g^2T)$: magnetic masses $\langle A_i(0)A_i(x)\rangle$

# Perturbative approach

Asymptotic freedom allows/enforces evaluation of $S_{\text{eff}}$ by perturbation theory

- Degrees of freedom are static $A_i, A_0$, ie. $3d$ YM with adjoint Higgs
- Adjust couplings of $S_{\text{eff}}$ to match Green's fncts *in perturbation theory*

$\longrightarrow$ after integrating out hard modes: EQCD

$$S_{\text{EQCD}} = \int d^3x \left( \frac{1}{2} F_{ij}^2 + \mathrm{Tr} D_i A_0 D_i A_0 + m_E^2 A_0^2 + \lambda_E A_0^4 \right)$$

with $F_{ij} = \partial_i A_j - \partial_j A_i - i g_3 [A_i, A_j]$, and $g_3^2 = g^2 T$

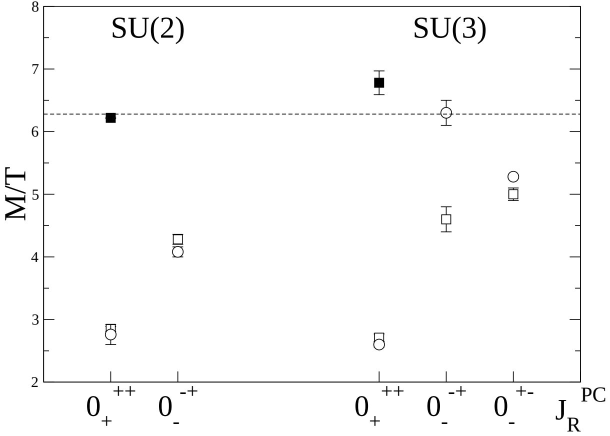
$m_E(T), \lambda_E(T)$ fixed by perturbative matching

$\longrightarrow$ after integrating out soft modes: MQCD

$$S_{\text{MQCD}} = \int d^3x \, \frac{1}{2} F_{ij}^2$$

# Successes and limitations of EQCD

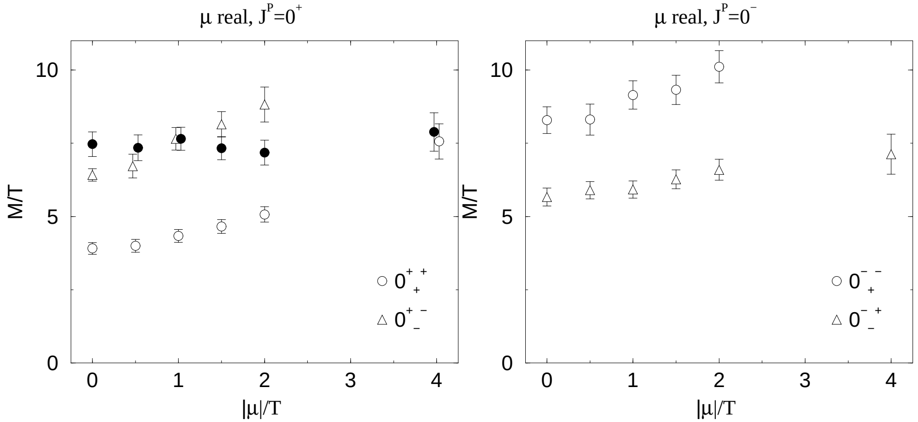
Screening masses down to $\sim 2T_c$




Laermann & Philipsen, hep-ph/0303042

# Successes and limitations of EQCD

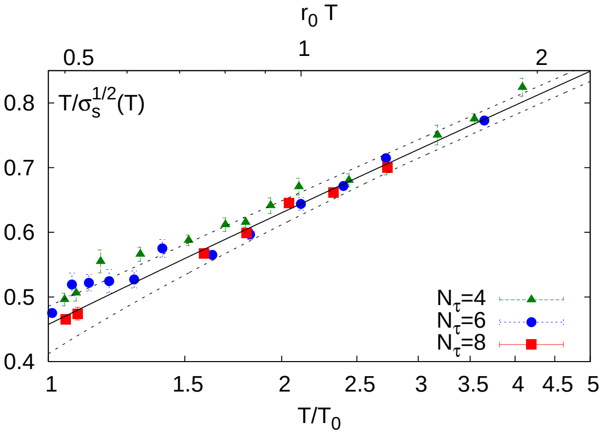
Screening masses at finite density ($T = 2T_c$)




Hart et al., hep-ph/0004060

# Successes and limitations of EQCD

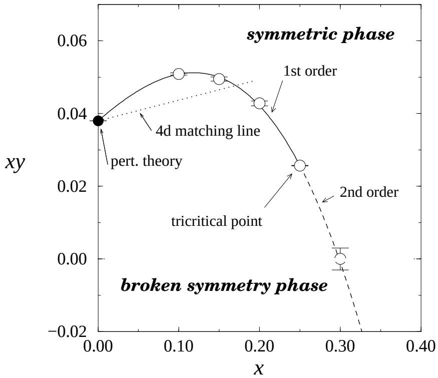
Spatial string tension down to $\sim T_c$ ?




Karsch et al., 0806.3264

# Successes and limitations of EQCD

Wrong phase diagram $\Rightarrow$ must fail near $T_c$




$S_{\mathrm{EQCD}} = \int d^3x \left(\frac{1}{2}F_{ij}^2 + \mathrm{Tr} D_i A_0 D_i A_0 + m_E^2 A_0^2 + \lambda_E A_0^4\right)$

Symmetry is $A_0 \leftrightarrow -A_0$, ie. $Z_2$

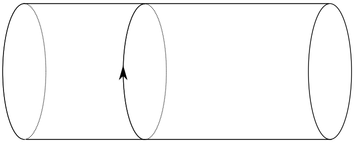
Matching line is in the wrong phase

# Root of the problem

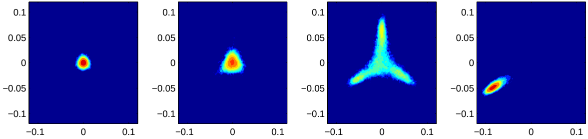
- Perturbation theory $\Rightarrow$ small fluctuations around one vacuum $A_0 = 0$
- YM vacuum is $N_c$-degenerate:

center symmetry (spontaneously broken for $T > T_c$)

$A_\mu(x) \rightarrow s(x)(A_\mu(x) + i\partial_\mu)s(x)^\dagger$, with $s(x + \beta\hat{e}_\tau) = \exp(i\frac{2\pi}{N_c}k)s(x)$



$P(x) \equiv: \exp(i\int_0^\beta d\tau A_0(x,\tau))$ :




**Effective action should respect symmetries of original action**

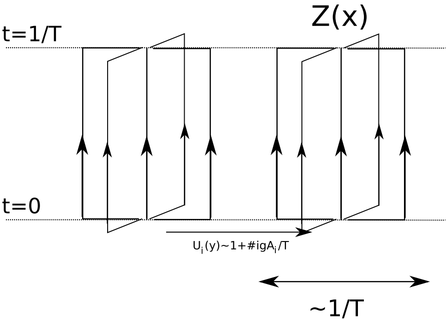
# Polyakov loop vs coarse-grained Polyakov loop

- Use Polyakov loop $P(x)$ instead of $A_0$ in $S_{\text{eff}}$ Pisarski

But: $P(x) \in SU(N)$ $\longrightarrow$ non-renormalizable (non-linear $\sigma$-model)
cf. PNJL

- Here: degree of freedom is **coarse-grained** Polyakov loop Yaffe

$$\mathcal{Z}(x) \equiv \frac{T}{V_{\text{block}}} \int d^3y \; U(x,y)P(y)U(y,x)$$




renormalizability preserved: easy $T \longrightarrow \infty$ matching with perturbation theory

# Simplest: $SU(2)$ Yang-Mills

- sum of $SU(2)$ matrices is multiple of $SU(2)$ matrix

$$\mathcal{Z} = \lambda\Omega, \quad \Omega \in SU(2), \ \lambda > 0$$

- Parametrization: $\mathcal{Z} = \frac{1}{2}(\Sigma\mathbf{1} + i\Pi_a\sigma_a)$

$$\mathcal{L}_{\rm eff} = g_3^{-2}\left[\tfrac{1}{2}{\rm Tr}F_{ij}^2 + {\rm Tr}(D_i\mathcal{Z}^\dagger D_i\mathcal{Z}) + V(\mathcal{Z})\right]$$

- Include all $Z_2$-symmetric super-renormalizable terms:

$$V(\mathcal{Z}) = b_1\Sigma^2 + b_2\Pi_a^2 + c_1\Sigma^4 + c_2(\Pi_a^2)^2 + c_3\Sigma^2\Pi_a^2$$

Local gauge invariance $\mathcal{Z}(x) \to \Omega(x)\mathcal{Z}(x)\Omega^{-1}(x)$
Global $Z_2$ symmetry: $\mathcal{Z} \to -\mathcal{Z}$ (actually $\Sigma \to -\Sigma, \Pi \to -\Pi$ indep.)

# Perturbative matching

- Determine [almost] parameters $\{b_1, b_2, c_1, c_2, c_3\}$ using perturbation theory

- Split potential into hard and soft pieces: $V(\mathcal{Z}) = V_h + g_3^2 V_s$

- Hard potential $\rightarrow$ scales $\sim T$ (magnitude of coarse-grained Pol.)
$V_h = h_1 \mathrm{Tr}\mathcal{Z}^\dagger\mathcal{Z} + h_2(\mathrm{Tr}\mathcal{Z}^\dagger\mathcal{Z})^2$ $O(4)$ symmetric

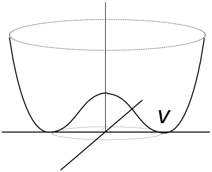
- Soft potential $\rightarrow$ EQCD at high $T$
$V_s = s_1 \mathrm{Tr}\Pi^2 + s_2(\mathrm{Tr}\Pi^2)^2 + s_3\Sigma^4$

# Leading order

- Classical solution: $\Sigma(x) = \bar{\Sigma},\ \Pi(x) = \bar{\Pi}\delta_{a,3}$ which minimize

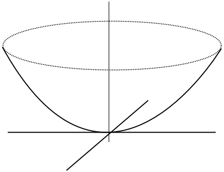
$$V_{\rm class} = \frac{g_3^{-2}}{4}(\bar{\Sigma}^2 + \bar{\Pi}^2)(2h_1 + h_2(\bar{\Sigma}^2 + \bar{\Pi}^2))$$

Two possible cases ($h_2 > 0$ for stability):




$h_1 < 0 \ \longrightarrow$ deconfined

$\bar{\Sigma} = \bar{\Pi} = -\frac{h_1}{h_2}, \mathrm{Tr}\mathcal{Z} \equiv v$



$h_1 > 0 \ \longrightarrow$ confined

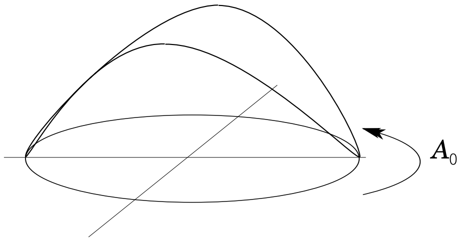
$\bar{\Sigma} = \bar{\Pi} = \mathrm{Tr}\mathcal{Z} = 0$

# 1-loop effective potential

- At 1-loop, $U(1)$ symmetry of potential is broken:

$$\bar{\Sigma} = v\cos(\pi\alpha), \quad \bar{\Pi} = v\sin(\pi\alpha)$$

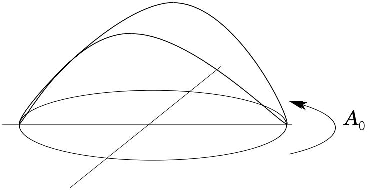
$$V_{\text{eff}} = \frac{s_1 v^2}{2}\sin^2(\pi\alpha) + \frac{s_2 v^4}{4}\sin^4(\pi\alpha) + s_3 v^4\cos^4(\pi\alpha) - \frac{v^3}{3\pi}|\sin(\pi\alpha)|^3 + \mathcal{O}(g_3^2)$$



# Matching to EQCD

Decompose fluctuations into radial + angular:

$$\mathcal{Z} = \pm\left[\tfrac{1}{2}v\mathbf{1} + g_3\left(\tfrac{1}{2}\phi\mathbf{1} + i\chi\right)\right]$$




$$\mathcal{L} = \tfrac{1}{2}\mathrm{Tr}F_{ij}^2 + \tfrac{1}{2}\left[(\partial_i\phi)^2 + m_\phi^2\phi^2\right] + \mathrm{Tr}\left[(D_i\chi)^2 + m_\chi^2\chi^2\right] + V_s(\phi,\chi)$$

$$m_\phi^2 = 8v^2c_1 = -2h_1, \text{ heavy}$$

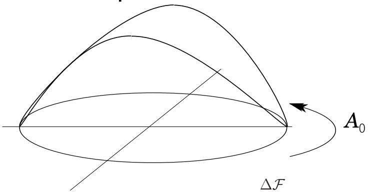
$$m_\chi^2 = 2(b_2 + v^2c_3) = g_3^2(s_1 - 4v^2s_3), \text{ light}$$

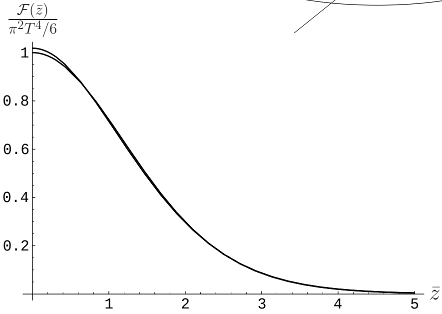
$$\mathcal{L}_{\mathrm{EQCD}} = \tfrac{1}{2}F_{ij}^2 + \mathrm{Tr}D_iA_0D_iA_0 + m_E^2A_0^2 + \lambda_E A_0^4$$

$\chi$ is $A_0$ $\rightarrow$ 2 equations $m_\chi^2 = m_E^2$, $\tilde{\lambda}\chi^4 = \lambda_E A_0^4$

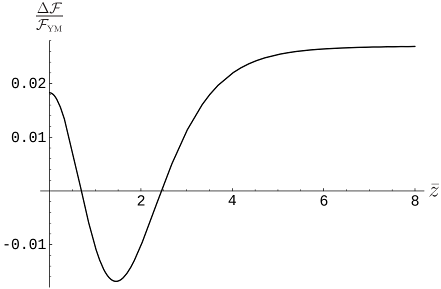
# Domain wall

- Two more equations from matching domain wall $\mathcal{Z}(x=\pm\infty)=\pm\frac{v}{2}\mathbf{1}$ tension and width at 1-loop:




$\mathcal{L}_{SU(2)}$ and $\mathcal{L}_{\rm eff}$ wall profiles




relative difference

# Finally...

- 5 couplings, 4 equations $\rightarrow$ 1 free parameter $r$

$$\mathcal{L}_{\rm eff} = g_3^{-2}\left[\tfrac{1}{2}\mathrm{Tr}F_{ij}^2 + \mathrm{Tr}(D_i\mathcal{Z}^\dagger D_i\mathcal{Z}) + V(\mathcal{Z})\right]$$

$$V(\mathcal{Z}) = b_1\Sigma^2 + b_2\Pi_a^2 + c_1\Sigma^4 + c_2(\Pi_a^2)^2 + c_3\Sigma^2\Pi_a^2$$

$$g_3^2 = g^2 T$$
$$b_1 = -\tfrac{1}{4}r^2T^2, \quad b_2 = -\tfrac{1}{4}r^2T^2 + 0.441841g^2T^2$$
$$c_1 = 0.0311994r^2 + 0.0135415g^2$$
$$c_2 = 0.0311994r^2 + 0.008443432g^2$$
$$c_3 = 0.0623987r^2$$

- Couplings determined by $(g, T)$ and $r = m_\phi/T$
- $r$ fixes mass of coarse-grained Pol. loop magnitude
  $\rightarrow$ any $r \sim \mathcal{O}(1)$ should give similar IR physics

# Phase diagram?

- To determine phase diagram, need [non-perturbative] lattice simulations

$$S_{\rm lat} = S_W + S_{\mathcal{Z}} + V(\hat\Sigma, \hat\Pi),$$
$$S_W = \beta \sum_{x,i<j} \left[1 - \tfrac{1}{2}{\rm Tr}[U_{ij}]\right], \text{ Wilson action}$$
$$S_{\mathcal{Z}} = 2\left(\tfrac{4}{\beta}\right) \sum_{x,i} {\rm Tr}\left[\hat\Pi^2 - \hat\Pi(x)U_i(x)\hat\Pi(x+\hat\imath)U_i^\dagger(x)\right]$$
$$+ \left(\tfrac{4}{\beta}\right) \sum_{x,i} \left(\hat\Sigma^2(x) - \hat\Sigma(x)\hat\Sigma(x+\hat\imath)\right), \text{ kinetic term}$$
$$V = \left(\tfrac{4}{\beta}\right)^3 \sum_x \left[\hat b_1 \hat\Sigma^2 + \hat b_2 \hat\Pi_a^2 + \hat c_1 \hat\Sigma^4 + \hat c_2 \left(\hat\Pi_a^2\right)^2 + \hat c_3 \hat\Sigma^2 \hat\Pi_a^2\right]$$

with $\beta = \frac{4}{a g_3^2}$, $a$ lattice spacing

- Explicit $Z(2)$ symmetry $\hat\Pi \to -\hat\Pi, \hat\Sigma \to -\hat\Sigma$
- Lattice couplings?

# Matching lattice and $\overline{MS}$ continuum couplings

Perturbative 2-loop lattice calculation A. Kurkela, 0704.1416

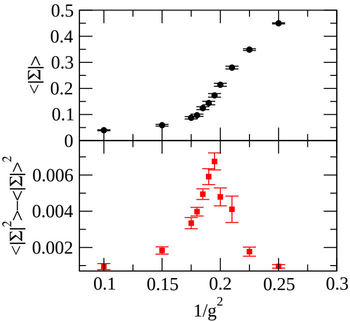
$$\Sigma = g_3\hat{\Sigma} + \mathcal{O}(a), \quad \Pi = g_3\hat{\Pi} + \mathcal{O}(a), \quad c_i = \hat{c}_i + \mathcal{O}(a)$$

$$\hat{b}_1 = b_1/g_3^4 - \frac{2.38193365}{4\pi}(2\hat{c}_1 + \hat{c}_3)\beta + \frac{1}{16\pi^2}\left\{(48\hat{c}_1^2 + 12\hat{c}_3^2 - 12\hat{c}_3)\left[\log 1.5\beta + 0.08849\right] - 6.9537\,\hat{c}_3\right\} + \mathcal{O}(a),$$

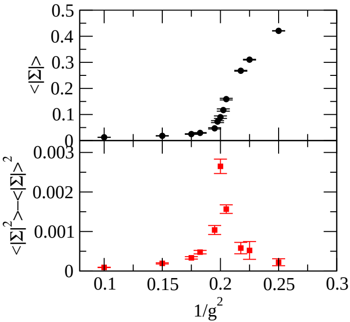
$$\hat{b}_2 = b_2/g_3^4 - \frac{0.7939779}{4\pi}(10\hat{c}_2 + \hat{c}_3 + 2)\beta + \frac{1}{16\pi^2}\Big\{(80\hat{c}_2^2 + 4\hat{c}_3^2 - 40\hat{c}_2)\left[\log 1.5\beta + 0.08849\right] - 23.17895\,\hat{c}_2 - 8.66687\Big\} + \mathcal{O}(a)$$

Matching exact in $g_3$, but $\mathcal{O}(a)$ error $\rightarrow$ continuum extrapolation

# Simulation results



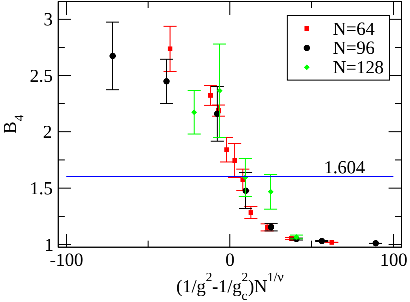

$r^2 = 5, 64^3, \beta = 12$



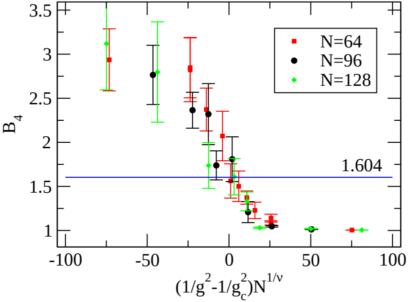

$r^2 = 5, 64^3, \beta = 6$

$Z_2$-restoring phase transition

# Universality class




$r^2 = 5$



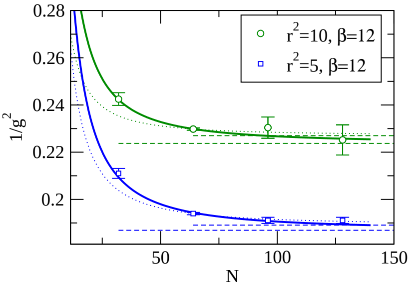

$r^2 = 10$

3*d* Ising universality class:

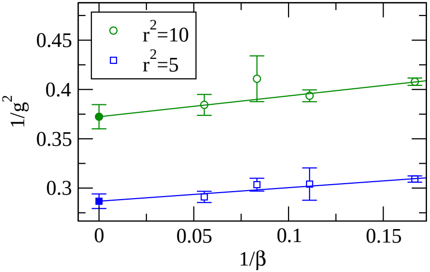
$$B_4 \equiv \frac{\langle \Sigma^4 \rangle}{\langle \Sigma^2 \rangle^2} = 1.604\ldots \text{ at criticality}$$

$$\nu \approx 0.63$$

# Thermodynamic + continuum extrapolations




$V \longrightarrow \infty$



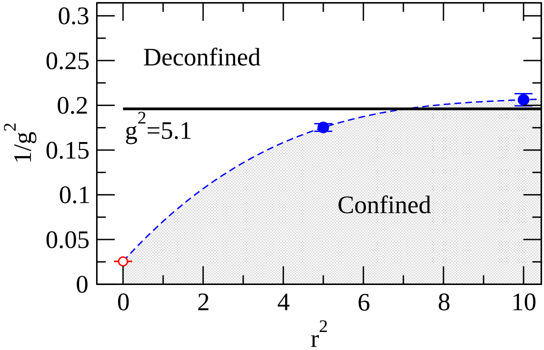

$a \longrightarrow 0$

$r$ small $\longrightarrow$ large correlation length $\longrightarrow$ large volume

$r$ large $\longrightarrow$ large cutoff effects $\longrightarrow$ fine lattice

$r = 0$ : $\Sigma$ decouples $\longrightarrow$ $\lambda\phi^4$ already done X.P. Sun, hep-lat/0209144

# Phase diagram




- Phase transition is robust $\forall\ r$

  $r \longrightarrow \infty$: no radial fluctuations, but still domain wall and transition
- $g^2_{\rm crit}$ depends mildly on $r$ for $r > 1$
- Fixing $g^2(T_c) = 5.1$ gives $r \sim 2.6$:

  dim. red. action completely defined

# Prospects

- To do with $SU(2)$:
  - determine $r(T)$ non-perturbatively
  - check accuracy of effective theory at $T$ near $T_c$
    - domain wall tension
    - spatial string tension
    - screening masses
  - make predictions using effective theory
    - heavy fermions
    - chemical potential

- Also $SU(3)$: $\mathcal{Z} \in GL(3,C)$

$$V(\mathcal{Z}) = V_h(\mathcal{Z}) + g_3^2 V_s(\mathcal{Z}); \quad M \equiv \mathcal{Z} - \frac{1}{3}\mathbf{1}\mathrm{Tr}\mathcal{Z}$$

$$V_h(\mathcal{Z}) = c_1 \mathrm{Tr}\mathcal{Z}^\dagger\mathcal{Z} + c_2(\det\mathcal{Z} + \det\mathcal{Z}^\dagger) + c_3\mathrm{Tr}(\mathcal{Z}^\dagger\mathcal{Z})^2$$

$$V_s(\mathcal{Z}) = d_1 \mathrm{Tr}M^\dagger M + d_2\mathrm{Tr}(M^3 + M^{3\dagger}) + d_3\mathrm{Tr}(M^\dagger M)^2$$

6 couplings, 4 equations $\longrightarrow$ 2 heavy modes to tune

A. Kurkela, 0704.1416