Finite Baryon-Number-Induced Chern-Simons Couplings in Holographic QCD

(or: Smectic Phases in Neutron Stars?)

I. Introduction and review of hardwall model

II. Adding CS term and finite baryon density

III. New couplings and dispersion relations

IV. Outlook and Conclusions

I. HARD WALL MODEL

(Erlich, Katz, Son, Stephanov, Phys. Rev. Lett.; hep-ph/0501128)
(da Rold, Pomarol, Nucl. Phys. B; hep-ph/0501218)

"bottom-up approach" to AdS/CFT/QCD

- Ingredients

(i) gravitational background: AdS_5 (forget the S^5)

with IR cutoff to induce confinement

$$ds^2 = \frac{1}{z^2}\left(dx^\mu dx_\mu - dz^2\right) \quad \text{with} \quad \underset{\text{UV}}{0} \le z \le \underset{\text{IR}}{z_m}$$

(set AdS_5 radius $R = 1$)

(ii) field content:

(only include fields relevant to chiral dynamics)

5d masses given by $(\underbrace{\Delta}_{\substack{\text{operator}\\ \text{scaling dimension}}} - p)(\Delta + \underbrace{p}_{\substack{\text{form}\\ \text{degree}}} - 4) = m_5^2$

① $SU(N_f)_L \times SU(N_f)_R$ gauge fields

$$A^a_{L,R\,\mu} \xleftrightarrow{\text{dual to}} \bar{q}_{L,R}\, \gamma^\mu t^a q_{L,R}$$

$(\Delta = 3,\ p = 1,\ m_5^2 = 0)$

② Bifundamental tachyonic field (Higgs-like rôle)

$$\frac{2}{z} X^{\alpha\beta} \longleftrightarrow \bar{q}^{\alpha}_{R}\, q^{\beta}_{L}$$

$(\Delta = 3,\ p = 0,\ m_5^2 = -3)$

($1/z$ factor dictated by dimension of $\bar{q}_R q_L$)

(iii) Action

$$S = \int d^4x\, dz \sqrt{g}\, \mathrm{Tr}\left[|DX|^2 + 3|X|^2 - \frac{1}{4g_5^2}\left(F_L^2 + F_R^2\right)\right] + S_{CS}$$

will discuss this later

with $D_M X = \partial_M X - i A_{L_M} X + i X A_{R_M}$

and generators normalized as $\mathrm{Tr}(t^a_{L,R} t^b_{L,R}) = \frac{1}{2}\delta^{ab}$

• Induce χ_{SB}: by giving $X^{\alpha\beta}$ a z-dependent vev •

(i) Solve linearized e.o.m. for X at zero 4d momentum:

$$X_0^{\alpha\beta} = \frac{1}{2}(m_q z + \sigma z^3)\mathbb{1}^{\alpha\beta} \equiv \frac{v(z)}{2}\mathbb{1}^{\alpha\beta}$$

- z is non-normalizeable mode
  - $\Rightarrow$ its coefficient is dual to source of operator $\bar{q}_R q_L$
- z^3 is normalizeable mode
  - $\Rightarrow$ its coefficient is dual to $\langle \bar{q}_R q_L \rangle$ (chiral condensate)

(ii) Solve for hadron spectrum generated by normalizeable modes of vector $V^a_M = \frac{1}{2}(A^a_{L_M} + A^a_{R_M}) = V_M^a(x)\,\psi_V(z)$

and axial-vector $A^a_M = \frac{1}{2}(A^a_{L_M} - A^a_{R_M})$ gauge fields.

- confining hard wall at $z = z_m$ induces finite spectrum
- vev for $X^{\alpha\beta}$ induces mass splitting between vector and axial vector mesons

(iv) Fix undetermined parameters

$$\underset{\uparrow}{g_5} \quad \underbrace{z_M \quad m_q \quad \sigma}_{\text{by fitting } m_\rho^2,\ f_\pi, \dots \text{ to data}}$$

by matching
to QCD result
for $\langle J^a_{V_\mu}(x)\, J^b_{V_\nu}(0)\rangle$

$$\left(g_5^2 = \frac{12\pi^2}{N_c} \right)$$

RESULTS : Amazingly good agreement with data!

(For global fit to seven observables rms error $\sim 9\%$)

Some shortcomings :

- don't get correct Regge behavior $(m_n \sim n^2)$
- coupling doesn't run

... fixed in more complicated modifications,
e.g. soft wall model
(Karch, Katz, Son, Stephanov,
Phys. Rev D74; hep-ph/0602229)

II. C-S Term and Finite Baryon Density

(SKD ; J. Harvey, Phys. Rev. Lett. 99, arXiV: 0704.1604 [hep-ph])

Add to hard-wall model (with gauge group $U(N_f)_L \times U(N_f)_R$)

(i) Chern-Simons term to reproduce QCD chiral anomaly

$$S_{CS} = \frac{N_c}{24\pi^2} \int_{\mathcal{M}_5} \left[\omega_5(A_L) - \omega_5(A_R) \right]$$

where for hermitian generators

$$\omega_5(A) = \mathrm{Tr}\left[AF^2 + \frac{i}{2} A^3 F - \frac{1}{10} A^5 \right].$$

The variation of this $\omega_5 = d(F^2)$ gives a boundary variation precisely dual to the chiral anomaly:

$$\delta_\Lambda \omega_5^{(0)} = d\,\omega_4^{(1)}$$

where $\omega_4^{(1)} = \mathrm{Tr}\left(\Lambda\, d\left(AdA - \frac{i}{2}A^3\right)\right)$

The coefficient is determined by anomaly – no extra undetermined parameters added.

(ii) Finite baryon density

i.e. turn on normalizeable ($q^2 \neq 0$) mode of dual AdS_5 field

① Recall: we upgraded the gauge group to

$$U(N_f)_L \times U(N_f)_R$$

So we have a $U(1)_V$ gauge field dual to the baryon number current:

$$\underbrace{V_M}_{\text{total}} = \hat{V}_M \hat{t} + V_M^a t^a$$

$\hat{t}$: $U(1)_V$ generator

t^a: $SU(N_f)_V$ generator

$\Rightarrow$ $\hat{V}_0$ is dual to the baryon number operator

② How to normalize $\hat{V}_0$?

(i) Solve linearized equations of motion from action above

$$\hat{V}_0 = A + B\frac{z^2}{2}$$

$\Rightarrow$ A is proportional to μ_q (or μ_B)

$\Rightarrow$ B is proportional to n_B

(ii) Evaluate S for Euclidean time (i.e. finite temp.)

(should give grand canonical potential)

$$S = -\frac{1}{2g_5^2}\int d^4x\, dz \left(-\frac{1}{z}(\hat{F}_{0z})^2\right) = \frac{1}{2g_5^2}\int d^4x\, \hat{V}_0\, \partial_z \hat{V}_0 \Big|_{z=0} =$$

$$= \frac{1}{2g_5^2} AB \int d^4x = \frac{1}{2g_5^2} AB\, (vol_3 \cdot \beta)$$

[redacted]

Compare to grand canonical ensemble

$$S_{\text{Euclid.}} = \mu_B \, (\underbrace{n_B \cdot vol_3}_{N_B}) \, \beta$$

Setting these equal we have

$$\boxed{\frac{1}{2g_5^2} AB \, (vol_3 \cdot \beta) = \mu_B n_B \, (vol_3 \cdot \beta)}$$

(relates A and B)

(iii) Fix A :

- Now use the Chern-Simons term.

$$S_{CS} = \frac{N_c}{64\pi^2} \int d^4x \, dz \, \varepsilon^{MNPQ} \left[\hat{A}_{(L)0} \, \mathrm{Tr}\left(F_{(L)MN} F_{(L)PQ}\right) - \hat{A}_{(R)0} \, \mathrm{Tr}\left(F_{(R)MN} F_{(R)PQ}\right) \right] +$$

trace over $SU(N_f)$

... + other terms ...

$(M, N, P, Q = 1, 2, 3, z)$

- Define $SU(N_f)$ instanton number in the usual way

$$\mathcal{N}_{L,R} = \frac{1}{32\pi^2} \int d^3x \, dz \, \varepsilon^{MNPQ} \, \mathrm{Tr}\left(F_{(L,R)MN} F_{(L,R)PQ}\right)$$

but using z instead of t as fourth dimension.

(Assume F_{MN} is independent of t : static configuration)

This gives a coupling in S_{CS} proportional to $\mathcal{N}_{L,R}$:

$$S_{CS} = \frac{N_c}{2} \int dx^0 \left(\hat{A}_{(L)0} \mathcal{N}_L - \hat{A}_{(R)0} \mathcal{N}_R \right)$$

Relying on the connection between static 5d "instantons" and Skyrmion configurations (i.e. pion field configurations with nontrivial winding number)* we identify states with instanton number

$$\mathcal{N}_L = -\mathcal{N}_R = N_b$$

as baryons of baryon number N_b.

* [For details see Atiyah & Manton Phys Lett B 222
Son & Stephanov Phys Rev. D 69 hep-ph/0304182
Hata, Sakai, Sugimoto & Yamamoto hep-th/0701276]

Now we have the coupling

$$S_{CS} = \frac{N_c}{2} N_B \int \left(\hat{A}_{(L)0} + \hat{A}_{(R)0} \right) dx^0 = N_c N_B \int dx^0 \, \hat{V}_0 = \beta N_c N_B A = \beta \mu_B N_B$$

$$\Rightarrow \quad \boxed{A = \frac{\mu_B}{N_c} = \mu_q}$$

Overall, $A = \dfrac{\mu_B}{N_c}$ and $B = \dfrac{24\pi^2 n_q}{N_c} = 24\pi^2 n_B$:

$$\boxed{\hat{V}_0 = \frac{\mu_B}{N_c} + 24\pi^2 n_B \frac{z^2}{2}}$$

focus on n_B from now on

III. Consequences of $n_B \neq 0$ and S_{CS}

(i) 4d effective action (to quadratic order)

Note: S_{CS} contains a term $\propto \hat{V}_R \partial_M \hat{V}_N \partial_P A_Q \, \varepsilon^{RMNPQ}$

which is quadratic order when $n_B \neq 0$!

- Expand as usual

$$\begin{cases} V_\mu^a(x,z) = g_5 \rho_\mu^a(x)\, \psi_\rho(z) & \text{(rho meson)} \\ A_\mu^a(x,z) = g_5 a_\mu^a(x)\, \psi_a(z) & (a_1 \text{ meson}) \end{cases}$$

where $\psi_{\rho,a}(z)$ satisfies linearized equations of motion with $q_{4d}^2 = m_{\rho,a}^2$.

- Integrate out holographic (z) direction:

(and ignore pions)

$$S_{4d} = \int d^4x \Big[-\tfrac{1}{4}(\rho_{\mu\nu}^a)^2 + \tfrac{1}{2} m_\rho^2 \rho_\mu^a \rho^{a\mu} - \tfrac{1}{4}(a_{\mu\nu}^a)^2 + \tfrac{1}{2} m_a^2 a_\mu^a a^{a\mu}$$
$$+ \underset{\substack{\uparrow \\ \text{not chemical potential}}}{\mu}\, \varepsilon^{ijk}(\rho_i^a \partial_j a_k^a + a_i^a \partial_j \rho_k) \Big]$$

where $i,j,k = 1,2,3$ and

$$\boxed{\mu = 18\pi^2 z_m^2 \, n_B \, I}$$ (dimensions of mass)

here $I = \frac{1}{z_m^2}\int_0^{z_m} dz\, z\, \psi_\rho(z)\psi_{a_1}(z)$ gives overlap between ρ and a_1 wavefunctions and $\partial_z \hat{V}_0$.

For densities typical of neutron stars $n_{B_0} \simeq 0.16\ {}^1/(\text{fermi})^3$

$$\boxed{\mu = (1.05\ \text{GeV})\left(\frac{n_B}{n_{B_0}}\right)}$$ (I is ~ 0.54)

Ⓐ Mixing of transverse ρ and a_1 polarizations

- Consider ρ and a_1 in plane wave state

(without loss of generality, consider propagation along x^3)

$$\rho_\mu(x) = \varepsilon^{(\rho)}_\mu(q)\, e^{-iq\cdot x} \qquad a_\mu(x) = \varepsilon^{(a)}_\mu(q)\, e^{-iq\cdot x}$$

(suppressing gauge indices)

with $q = (q_0, 0, 0, q_3)$

From the equations of motion we have

$$(\Box + m_\rho^2)\, \rho^{a\beta} = 2\mu\, \varepsilon^{ij\beta}\, \partial_i a_j^a$$
$$(\Box + m_a^2)\, a^{a\beta} = 2\mu\, \varepsilon^{ij\beta}\, \partial_i \rho_j^a$$

which gives the usual dispersion relations for $\beta = 0$ and $\beta = 3$, but for transverse polarizations the CS term has an effect

$$\beta = 1, 2: \qquad \boxed{q_0^2 - q_3^2 = \tfrac{1}{2}(m_\rho^2 + m_a^2) \pm \sqrt{(m_a^2 - m_\rho^2) + 16\mu^2 q_3^2}}$$

- for $q_3 \to 0$ "$-$" dispersion relation gives ρ:

$$\varepsilon_1^{a_1} = \frac{i\mathcal{M}^2(q_3)}{2\mu q_3}\,\varepsilon_2^{\rho} \qquad \varepsilon_2^{a_1} = -\frac{i\mathcal{M}^2(q_3)}{2\mu q_3}$$

- for $q_3 \to 0$ "$+$" dispersion gives a_1:

$$\varepsilon_1^{\rho} = -\frac{i\mathcal{M}^2(q_3)}{2\mu q_3}\,\varepsilon_2^{a_1} \qquad \varepsilon_2^{\rho} = \frac{i\mathcal{M}^2(q_3)}{2\mu q_3}\,\varepsilon_1^{a_1}$$

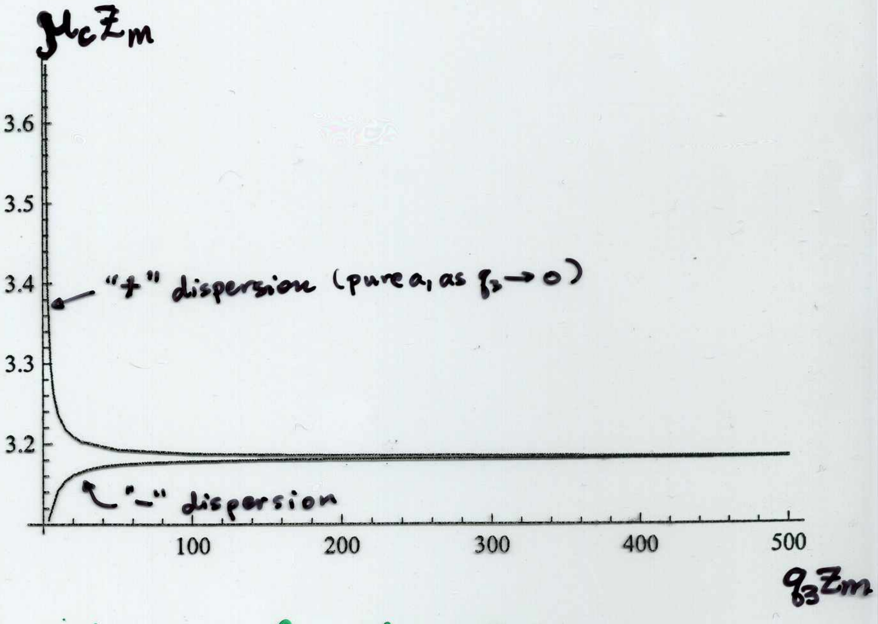
with $\mathcal{M}^2(q_3) = \dfrac{\sqrt{\Delta^4 + 16\mu^2 q_3^2} - \Delta^2}{2}$ for $\Delta^2 = m_a^2 - m_\rho^2$

- For $\mu > \mu_c$ (which depends on q_3) the dispersion relation exhibits tachyonic modes ($dq_0/dq_3 > 1$) indicating a possible instability.

  (i.e. group velocity faster than c)

→ For very large q_3

$$\mu_c \simeq \sqrt{\frac{m_a^2 + m_\rho^2}{2}} \simeq \boxed{1.09 \text{ GeV}}$$



μ_c as a function of q_3

($\mu > \mu_c$ gives $\frac{dq_0}{dq_3} > 1$)

(B) Meson Condensation

Instability in dispersion hints at possibility of forming a condensate!

Rough idea:

(i) Consider the energy density of the system:

$$\mathcal{H} = \frac{1}{2}(\Pi_\rho^2 + \Pi_a^2) + \frac{1}{4}(\rho_{ij}^2 + a_{ij}^2) + \frac{1}{2}m_\rho^2 \rho_\mu \rho^\mu + \frac{1}{2}m_a^2 a_\mu a^\mu$$
$$- \mu \varepsilon^{ijk}(\rho_i \partial_j a_k + a_i \partial_j \rho_k)$$

canonical momenta (forget from now on...)

Complete the square, with $\begin{cases} \vec{B}_\rho = \vec{\nabla} \times \vec{\rho} \\ \vec{B}_a = \vec{\nabla} \times \vec{a} \end{cases}$

$$\mathcal{H} = \ldots + \frac{1}{2}(m_a^2 - \mu^2)\,\vec{a}\cdot\vec{a} + \frac{1}{2}(m_\rho^2 - \mu^2)\,\vec{\rho}\cdot\vec{\rho} +$$
$$+ \frac{1}{2}(\vec{B}_a - \mu\vec{\rho})^2 + \frac{1}{2}(\vec{B}_\rho - \mu\vec{a})^2$$

(ii) Apply the ansatz

$$\vec{a} = v\cos(\mu x_3)\,\hat{x}_2 \qquad \vec{\rho} = v\sin(\mu x_3)\,\hat{x}_1$$

- last two terms in $\mathcal{H}$ vanish
- for $\mu > \sqrt{\frac{m_a^2 + m_\rho^2}{2}}$ we have

$$\mathcal{H} = v^2\left[\frac{m_a^2 + m_\rho^2}{2} - \mu^2\right] < 0\ !$$

- Need to stabilize the condensate with higher order terms!

LOOKS LIKE SMECTIC PHASE IN LIQUID CRYSTALS.

IV. Conclusions and Outlook

- We added a CS term to the hardwall model in order to reproduce the chiral anomaly

- At finite density holography tells us this affects the dispersion relations of transverse polarizations of (axial)vector mesons.
  - → mixed polarization states
  - → (axial) vector meson condensate

Lots left to be done:

- stabilize condensate with higher order terms, and examine its structure in detail
- interactions with pion ?
- add temperature

...

A quick note on 5d instantons vs. Skyrmions:

(nicely explained in e.g. Son & Stephanov Phys Rev D 69
arXiv: hep-ph/0304182)

(also in Hata, Sakai, Sugimoto, Yamamoto
arXiv: hep-th/0701280)

- 5d YM has a conserved topological current

$$\sqrt{g}\, j^M = \frac{1}{32\pi^2} \varepsilon^{MNPQR} F_{NP} F_{QR}$$

With topological charge

$$Q = \int dz\, d^3x \sqrt{g}\, j^0 = \frac{1}{32\pi^2} \underbrace{\int dz\, d^3x\, \varepsilon^{0MNPQ} F_{MN} F_{PQ}}_{\substack{\text{usual instanton \#} \\ \text{(with } z \text{ instead of } t)}}$$

- Recall that $d\omega_3 = \mathrm{Tr}(F^2)$ or define another current

s.t. $$K^{0M} = 4\,\varepsilon^{0MNPQ}\, \mathrm{Tr}\left(A_N \partial_P A_Q - \frac{2i}{3} A_N A_P A_Q\right)$$

Say in this model we can define the pion in terms of a Wilson line

$$U = P e^{\overbrace{\int_0^z dz'\, A_z}^{\text{pion}}}$$

and A_M vanishes on the boundaries. Then, going to $A_z = 0$ gauge requires a gauge transformation by U, so $A_\mu \longrightarrow U A_\mu U^{-1} + i U \partial_\mu U^{-1}$

(or just $iU\partial_\mu U^{-1}$ on the boundaries).

Now, using K^{0M} we can rewrite the top. charge Q:

$$Q = \int dz\, d^3x \frac{1}{32\pi^2} \partial_M K^{0M} = \int d^3x \frac{1}{32\pi^2} K^{0z}\Big|_0^{z_m} =$$

$$= \frac{1}{24\pi^2} \int d^3x\, \varepsilon^{ijk} (U^\dagger \partial_i U)(U^\dagger \partial_j U)(U^\dagger \partial_k U) \Big\} \text{ top. charge in Skyrme model}$$