Kitaev Model

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Contents

Anyons and topological quantum computation

'Quantum phenomena do not occur in a Hilbert space. They occur in a laboratory'

Asher Peres

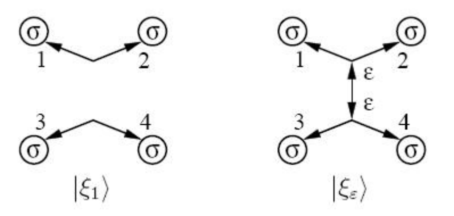
- Local errors, thermic noise and decoherence are considered the main obstacles in the realization of a quantum computer
- Topological properties of physical systems seem to be one of the best answer to overcome those problems
- Qubits encoded in topological states can be insensitive to local perturbations

Different applications of topological states have been proposed to encode qubits:

- **Abelian anyons on a torus** imply a ground state degeneracy and so the possibility to store quantum information (*toric code*)
- **Non–abelian anyons** can be used to implement a universal quantum computer (Kitaev, Freedman,...)

Example of non–abelian fusion rule:

$$\sigma \times \sigma = \mathbb{I} + \varepsilon$$



Kitaev Model

The aim of this talk is to study an example of **anyonic system** realized through a particular honeycomb spin lattice.

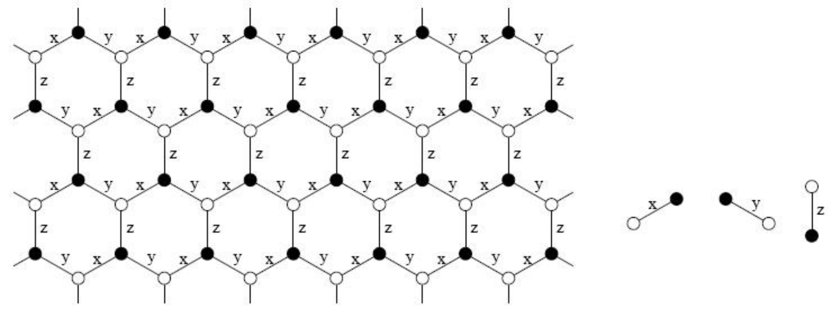
The study of the **Kitaev model** will allow us to understand the main features of **non-abelian anyons** and we'll analyze the interplay between a simple anyonic theory defined by fusion and braiding rules and the conformal field theory of the Ising model ($\mathcal{M}_3$).

Main features of anyonic systems:

- **Energy gaps** which allow the existence of local excitations (exponential decay of correlators)
- **Topological Quantum Numbers** which make such excitations stable (anyons as topological defects: for example vortices)
- **Topological Order**

The Model

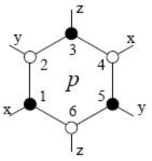
Alexei Kitaev, *Anyons in an exactly solved model and beyond*, arXiv: cond-mat/0506438v3



$$H = -J_x \sum_{x-\text{links}} \sigma_j^x \sigma_k^x - J_y \sum_{y-\text{links}} \sigma_j^y \sigma_k^y - J_z \sum_{z-\text{links}} \sigma_j^z \sigma_k^z$$

$$H = - \sum_{j\ \text{n.n.}\ k} J_{jk} K_{jk}$$

Plaquettes: Integrals of Motion



$$W_p = \sigma_1^x \sigma_2^y \sigma_3^z \sigma_4^x \sigma_5^y \sigma_6^z = K_{12} K_{23} K_{34} K_{45} K_{56} K_{61}$$

Commutation rules:

$$[K_{ij}, W_p] = 0 \quad \forall i, j, p \quad \Rightarrow \quad [H, W_p] = 0\,, \quad [W_q, W_p] = 0 \quad \forall q, p$$

To find the eigenstates of the Hamiltonian it is convenient to divide the total Hilbert space in sectors - eigenspaces of W_p:

$$\mathcal{H} = \bigoplus_{w_1,\ldots,w_m} \mathcal{H}_{w_1,\ldots,w_m}$$

For every n vertices there are $m = n/2$ plaquettes.
There are $2^{n/2}$ sectors of dimension $2^{n/2}$.

To describe the spins one can use annihilation and creation fermionic operators $\left\{a_\uparrow, a_\uparrow^\dagger, a_\downarrow, a_\downarrow^\dagger\right\}$. It is also possible to define their self adjoint linear combinations:

$$c_{2k-1} = a_k + a_k^\dagger \qquad c_{2k} = -i\left(a_k - a_k^\dagger\right)$$

The *Majorana operators* c_j define a Clifford algebra:

$$\{c_i, c_j\} = 2\delta_{ij}$$

Using these operators we are doubling the fermionic Fock space:

$$\{|\uparrow\rangle, |\downarrow\rangle\} \longrightarrow \left\{{\color{red}|00\rangle_{\uparrow\downarrow}}, {\color{red}|11\rangle_{\uparrow\downarrow}}, |01\rangle_{\uparrow\downarrow}, |10\rangle_{\uparrow\downarrow}\right\}$$

We need a projector onto the physical space.

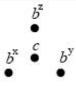
From Majorana to spin operators

For each vertex on the lattice we define:

$$b^x = a_\uparrow + a_\uparrow^\dagger, \quad b^y = -i\left(a_\uparrow - a_\uparrow^\dagger\right), \quad b^z = a_\downarrow + a_\downarrow^\dagger, \quad c = -i\left(a_\downarrow - a_\downarrow^\dagger\right)$$

We can write:

$$\sigma^x = ib^x c, \quad \sigma^y = ib^y c, \quad \sigma^z = ib^z c, \quad D = -i\sigma^x\sigma^y\sigma^z = b^x b^y b^z c$$

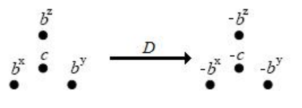


D is the *gauge operator*:

$$[D, \sigma^\alpha] = 0 \quad \forall\alpha$$

Over the physical space $D = 1$ and the projector over the physical space is:

$$P_{phys} = \prod_j \left(\frac{1 + D_j}{2}\right)$$



Kitaev model

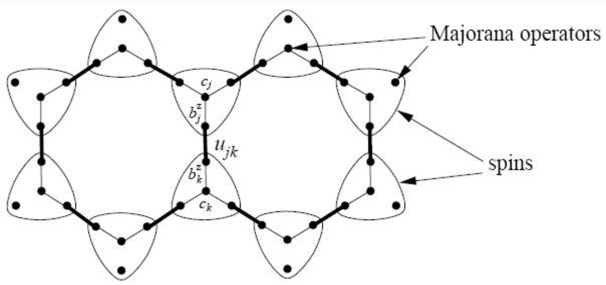
Using the Majorana operators we can rewrite:

$$K_{jk} = \sigma_j^\alpha \sigma_k^\alpha = \left(ib_j^\alpha c_j\right)\left(ib_k^\alpha c_k\right) = -iu_{jk}c_jc_k \quad \text{with} \quad u_{jk} \equiv ib_j^\alpha b_k^\alpha$$

And the Hamiltonian reads:

$$H = \frac{i}{4}\sum_{j,k} A_{jk}c_jc_k \qquad \text{with } A_{jk} \equiv \begin{cases} 2J_{\alpha_{jk}}u_{jk} & \text{if j and k are connected} \\ 0 & \text{otherwise} \end{cases}$$

$$u_{jk} = -u_{kj} \Rightarrow A_{jk} = A_{kj}$$



u_{ij} operators

u_{ij} are hermitian operators such that:

- u_{ij} commute with each other
- u_{ij} commute with H and have eigenvalues $u_{ij} = \pm 1$
- We can study the Hamiltonian in an eigenspace of all the operators u_{jk}
- u_{ij} is not gauge invariant: we need to project onto the physical subspace.
- D_j changes the signs of the three operators u_{jl} linked with j.

Gauge invariant operators

- **Wilson loop** over each plaquette:

$$w_p = \prod_{(j,k)\in p} u_{jk}$$

  Where j is in the even sublattice and k on the odd one.

- **Path operator**:

$$W\left(j_0, ..., j_n\right) = K_{j_n j_{n-1}} ... K_{j_1 j_0} = \left(\prod_{s=1}^{n} -iu_{j_s j_{s-1}} \right) c_n c_0$$

- u_{jk} can be considered a $\mathbb{Z}_2$ gauge field and w_p is the magnetic flux through a plaquette.
- If $w_p = -1$ we have a **vortex** and a Majorana fermion moving around p acquires a -1 phase.

Quadratic Hamiltonian

$$H\left(A\right)=\frac{i}{4}\sum_{j,k}A_{jk}c_jc_k$$

where A is a real skew-symmetric $2m\times 2m$ matrix. Through a transformation $Q\in O\left(2m\right)$ we obtain:

$$H=\frac{i}{2}\sum_{k=1}^{m}\varepsilon_k b'_k b''_k$$

with:

$$\left(b'_1,b''_1,...,b'_m,b''_m\right)=\left(c_1,c_2,...,c_{2m-1},c_{2m}\right)Q$$

and:

$$A=Q\begin{pmatrix} 0 & \varepsilon_1 & & & \\ -\varepsilon_1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & \varepsilon_m \\ & & & -\varepsilon_m & 0 \end{pmatrix}Q^T$$

Quadratic Hamiltonian

H can be diagonalized using creation and annihilation operators:

$$H = \frac{i}{2} \sum_{k=1}^{m} \varepsilon_k b'_k b''_k = \sum_{k=1}^{m} \varepsilon_k \left(a_k^\dagger a_k - \frac{1}{2} \right)$$

with:

$$\begin{pmatrix} a^\dagger \\ a \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} \begin{pmatrix} b' \\ b'' \end{pmatrix}$$

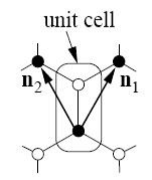
It is possible to define a **spectral projector** P onto the negative eigenvectors of A which identifies the ground state:

$$P = \frac{1}{2} \tilde{Q} \begin{pmatrix} \mathbb{I} & -i\mathbb{I} \\ i\mathbb{I} & \mathbb{I} \end{pmatrix} \tilde{Q}^T \qquad \sum_j P_{kj} c_j \left| \psi_{GS} \right\rangle = 0 \quad \forall k$$

$$a^\dagger a = cPc \qquad \left\langle \psi_{GS} \left| c_j c_k \right| \psi_{GS} \right\rangle = P_{kj}$$

Spectrum

- In the physical space the energy minimum is reached in the vortex free configuration ($w_p = 1 \; \forall p$).
- We can consider the coupling between unit cells:



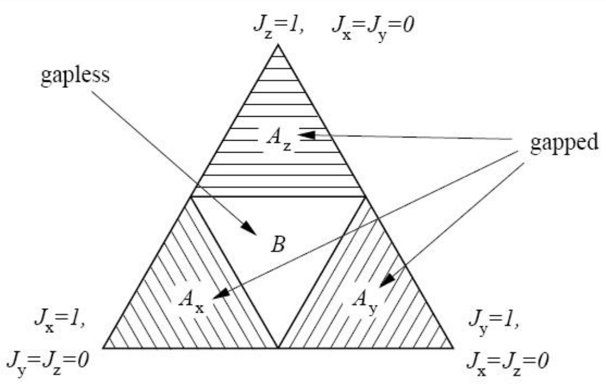
$$H(q) = \frac{i}{2} A(q) = \begin{pmatrix} 0 & if(q) \\ -if^*(q) & 0 \end{pmatrix}$$

$$f(q) = \left(J_x e^{iqn_1} + J_y e^{iqn_2} + J_z \right)$$

- Spectrum: $\varepsilon(q) = \pm |f(q)|$
- $\varepsilon(q)$ vanishes for some q iff the triangle inequalities hold:

$$|J_x| \leq |J_y| + |J_z| \qquad |J_y| \leq |J_z| + |J_x| \qquad |J_z| \leq |J_x| + |J_y|$$

Phase diagram



- Phase B is gapless: there are two values $\pm q_0$ such that $\varepsilon\left(\pm q_0\right)=0$
- B acquires a gap in presence of an external magnetic field
- Phases A are gapped and are related by rotational symmetry

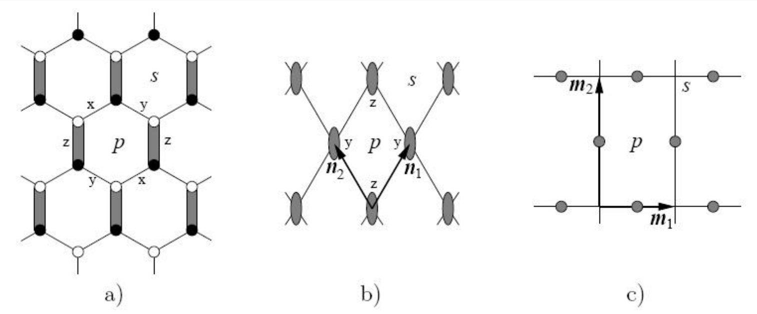
Gapped Phases

- In a gapped phase A correlations decay exponentially. There are no long range interactions.
- Local and distant particles can interact topologically. (*Braiding Rules*)
- We need to identify the right (stable and local) particles (*Superselection Sectors*)
- We will apply a perturbation theory study to reduce the *Kitaev model* to the *Toric model*

Phase A_z: Perturbation Theory

Let us suppose $J_z \gg J_x, J_y$ and $J_z > 0$.

$$H_0 = -J_z \sum_{z-links} \sigma_j^z \sigma_k^z, \qquad V = -J_x \sum_{x-links} \sigma_j^x \sigma_k^x - J_y \sum_{y-links} \sigma_j^y \sigma_k^y$$



The strong $z-$links in the original model (a) become *effective spins* (b) and can be associated with the links of a new lattice (c).

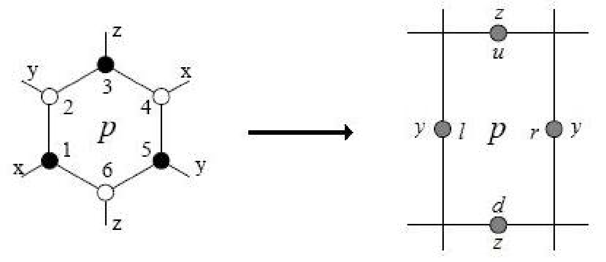
Phase A_z, $J_z \gg J_x, J_y$: Perturbative results

The first 3 orders in the perturbative expansion give just a shift in the spectrum. The fourth order is:

$$H_{eff}^{(4)} = -\frac{J_x^2 J_y^2}{16 J_z^3} \sum_p W_p^{eff}$$

where:

$$W_p = \underbrace{\sigma_1^x \sigma_2^y}_{\sigma_l^y} \sigma_3^z \underbrace{\sigma_4^x \sigma_5^y}_{\sigma_r^y} \sigma_6^z \quad \longrightarrow \quad W_p^{eff} = \sigma_l^y \sigma_u^z \sigma_r^y \sigma_d^z$$



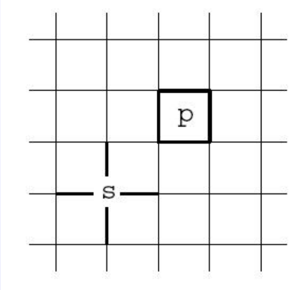
Phase A_z: Toric Code Hamiltonian
A. Kitaev, arXiv: quant-ph/9707021

Through unitary transformations the previous effective Hamiltonian can be mapped onto the *toric code Hamiltonian*:

$$H_{eff} = -J_{eff} \left(\sum_{vertices} A_s + \sum_{plaquettes} B_p \right)$$

with:

$$A_s = \prod_{j \in star(s)} \sigma_j^x, \qquad B_p = \prod_{j \in boundary(p)} \sigma_j^z$$



and:

$$[A_s, B_p] = [B_p, B_q] = [A_s, A_r] = 0$$

and the translational invariance is broken.

Excitations

- Ground State:

$$A_s |\psi\rangle = + |\psi\rangle \qquad B_p |\psi\rangle = + |\psi\rangle$$

- Excitations:
  - **Electric charge *e***: $A_s |e_s\rangle = - |e_s\rangle$
  - **Magnetic vortex *m***: $B_p |m_p\rangle = - |m_p\rangle$
- Superselection sectors: $\mathbb{I}$ (vacuum), e, m, $\varepsilon = e \times m$
- Fusion Rules:

$$e \times e = m \times m = \varepsilon \times \varepsilon = \mathbb{I}$$

$$e \times m = \varepsilon; \quad e \times \varepsilon = m; \quad m \times \varepsilon = e$$

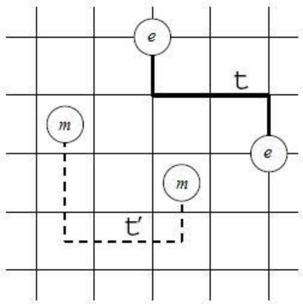
Braiding Rules

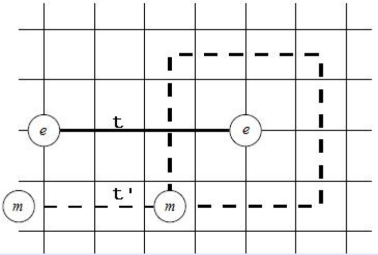
To create a pair of *e*, or move an *e* through a path t we must apply:

$$S^z(t) = \prod_{j \in t} \sigma_j^z$$

To create a pair of *m*, or move an *m* through a path t' we must apply:

$$S^x(t') = \prod_{j \in t'} \sigma_j^x$$

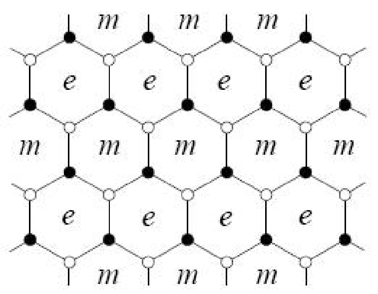




- e and m are bosons;
- Moving an e around an m yields -1;
- ε are fermions.

Gapped Phases

We can translate these results into the original model.
e and m particles correspond to vortices that live in different rows:

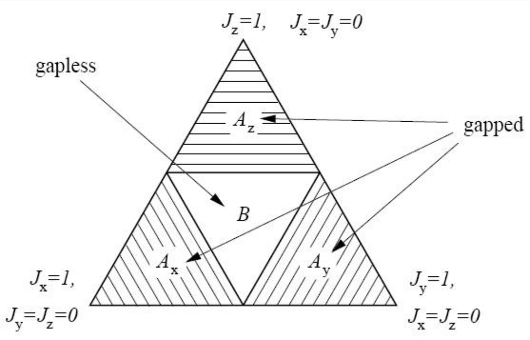


$$e \times m = \varepsilon$$
$$e \times \varepsilon = m$$
$$m \times \varepsilon = e$$

$$e \times e = \mathbb{I}$$
$$m \times m = \mathbb{I}$$
$$\varepsilon \times \varepsilon = \mathbb{I}$$

The Majorana fermions in the original model belong to the superselection sector ε although they are not directly composed of e and m (different energies between c and ε).

Phase B with Magnetic Field: Non–Abelian Sector



- Phase B is characterized by a gapless spectrum
- Due to long range interactions there are no local and stable excitations
- To make phase B acquire a gap we need a perturbation (breaking symmetry T)

Effective Hamiltonian with Magnetic Field

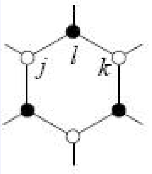
Consider the case $J_x = J_y = J_z = J$ and the following perturbation:

$$V = -\sum_j \left(h_x \sigma_j^x + h_y \sigma_j^y + h_z \sigma_j^z \right)$$

The third perturbative order is:

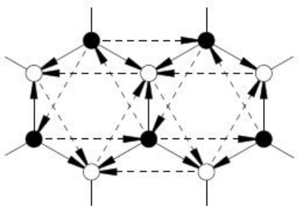
$$H_{eff}^{(3)} \approx -\frac{h_x h_y h_z}{J^2} \sum_{j,k,l} \sigma_j^x \sigma_k^y \sigma_l^z$$

And it contains terms of the following kind:



$$\sigma_j^x \sigma_k^y \sigma_l^z \approx -i c_j c_k$$

Effective Hamiltonian with Magnetic Field



$$H_{eff} = \frac{i}{4} \sum_{j,k} A_{jk} c_j c_k$$

$$A = 2J\left(\longleftarrow\right) + 2\kappa\left(\leftarrow\!\text{--}\,\text{-}\right)$$

$$\kappa \approx \frac{h_x h_y h_z}{J^2}$$

To find the spectrum we consider the cell-coupling in momentum representation:

$$iA(q) = \begin{pmatrix} \Delta(q) & if(q) \\ -if^*(q) & -\Delta(q) \end{pmatrix}, \qquad \varepsilon(q) = \pm\sqrt{\Delta(q)^2 + |f(q)|^2}$$

$$f(q) = 2J\left(e^{iqn_1} + e^{iqn_2} + 1\right), \qquad f(q_0) = 0$$

$$\Delta(q) = 4\kappa\left(sin(qn_1) - sin(qn_2) + sin(q(n_2 - n_1))\right)$$

The spectrum has a gap Δ.

Edge Modes and Chern Number

- If we consider a finite system with a magnetic field, we can show that the Kitaev model has *massless fermionic edge modes*.
- They are chiral Majorana fermions and are similar to the edge modes in a Quantum Hall system.
- Their existence and their spectrum can be deduced from a *truncated Hamiltonian*
- Starting from the projector onto the negative energy states $P(q)$ we can define a Chern Number ν which is linked to the number of Majorana modes:

$$\nu = (\text{n. of left movers } - \text{ n. of right movers}) = \pm 1$$

  the sign depends on the direction of the magnetic field.
- It is possible to show that:

$$\frac{\nu}{2} = c_- \equiv c - \bar{c}$$

Bulk/Edge correspondence

- With a magnetic flux the particles in the system acquire a mass.
- We can study their properties depending on $\nu = \pm 1$.
- Superselection sectors:
  - $\mathbb{I}$: vacuum
  - ε: fermion (massive)
  - σ: vortex (carrying an *unpaired Majorana mode*)
- These particles can be put in correspondence with fields of the kind $\phi\,(\tau + i\nu x)$, acting on the edge
- ϕ are described by holomorphic or antiholomorphic CFTs

Non-Abelian Fusion Rules

- In the bulk the massive fermion ε can be described by two coupled *Majorana modes* (quantum Hall analogy).

$$H = i \sum_{j,k} A_{j,k} a_j^\dagger a_k = \frac{i}{4} \sum_{j,k} A_{j,k} \left(c_j' c_k' + c_j'' c_k'' \right)$$

  with c hermitian (Clifford algebra).
- It is possible to show that, if $\nu = \pm 1$, **every vortex must carry an unpaired Majorana mode**.
- If two vortices σ fuse, they either annihilate completely, or leave a fermion ε behind.
- Fusion rules:

$$\varepsilon \times \varepsilon = \mathbb{I}, \qquad \varepsilon \times \sigma = \sigma, \qquad \sigma \times \sigma = \mathbb{I} + \varepsilon$$

- These are the well known fusion rules of the Ising model $\mathcal{M}_3$!
- We can identify every superselection sector with an edge field:

$$\begin{array}{llll} \nu = +1: & \mathbb{I} = (0,0) & \varepsilon = \left(\frac{1}{2}, 0\right) & \sigma = \left(\frac{1}{16}, 0\right) \\ \nu = -1: & \mathbb{I} = (0,0) & \varepsilon = \left(0, \frac{1}{2}\right) & \sigma = \left(0, \frac{1}{16}\right) \end{array}$$

Non-Abelian Anyons

$$\sigma \times \sigma = \mathbb{I} + \varepsilon$$

- A pair of vortices can be in two different states:

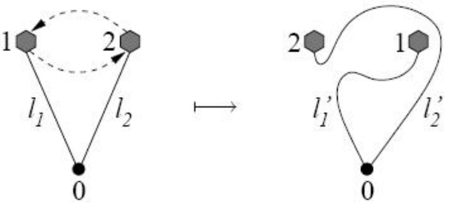
$$d_\sigma = \sqrt{2}$$

- This is the characteristic feature of non–abelian anyons.
- The braiding rule of two σ–particles depends on their state ($\mathbb{I}$ or ε).
- Each vortex σ_p carries an unpaired Majorana mode C_p.
- To study the braiding rule we use a gauge invariant path operator:

$$W\left(l_p\right) = C_p c_0$$

where c_0 is located in a reference point.

$R^{\sigma\sigma}$ Braiding rule



$$RW(l_1)R^\dagger = W(l_1') = W(l_2)$$
$$RW(l_2)R^\dagger = W(l_2') = -W(l_1)$$
$$W(l_1)W(l_2') = -1$$

$$\begin{cases} RC_1R^\dagger = C_2 \\ RC_2R^\dagger = -C_1 \end{cases} \Rightarrow R = \theta e^{-\frac{\pi}{4}C_1C_2}$$

The two possible states of $\sigma \times \sigma$ must be identified with the eigenstates of C_1C_2:

$$C_1C_2\left|\psi_{\mathbb{I}}^{\sigma\sigma}\right\rangle = i\alpha\left|\psi_{\mathbb{I}}^{\sigma\sigma}\right\rangle \qquad C_1C_2\left|\psi_{\varepsilon}^{\sigma\sigma}\right\rangle = -i\alpha\left|\psi_{\varepsilon}^{\sigma\sigma}\right\rangle$$

with $\alpha = \pm 1$

Braiding and Topological spin

From the previous results:

$$R_{\mathbb{I}}^{\sigma\sigma} = \theta e^{-i\alpha\pi/4}$$
$$R_{\varepsilon}^{\sigma\sigma} = \theta e^{i\alpha\pi/4}$$

where θ is a phase.

From CFT and the definition of topological spin we know that:

$$e^{2\pi i\left(h_\sigma - \bar{h}_\sigma\right)} = d_\sigma^{-1}\left(R_{\mathbb{I}}^{\sigma\sigma} + R_{\varepsilon}^{\sigma\sigma}\right)$$

so that a possible solution is:

$$\alpha = \nu \qquad \theta = e^{i\pi\nu/8}$$

- There are 8 possible solutions given by $\theta^8 = -1$.
- They can be classified using nontrivial braiding rules and associativity relations (pentagon and hexagon equations).

Conclusions

- The Kitaev model can be exactly solved through the decomposition in Majorana operators
- We can distinguish two different phases: a gapped spectrum phase and a gapless one
- To study anyons we need an energy gap. We studied the gapped spectrum phase to find e, m and σ particles
- The gapless phase acquires a mass in presence of a magnetic field. In this case we can identify non-abelian anyonic excitations

References

- A. Kitaev, Anyons in an exactly solved model and beyond (*Arxiv: cond-mat/0506438v3*) (2008)
- A. Kitaev, Fault tolerant quantum computation by anyons (*Arxiv: quant-ph/9707021v1*) (1997)

Appendix: Thermal Transport

Cappelli, Huerta, Zemba, *Nucl. Phys. B* **636** (2002) (ArXiv: cond-mat 0111437)

Energy current along the edge:

$$I = \frac{\pi c_-}{12\beta^2}$$

To show it we consider the mapping on the cylinder (periodic in time):

$$z(w) = e^{\frac{2\pi i (v\tau + ix)}{v\beta}} \qquad w = v\tau + ix$$

Stress tensor:

$$\langle T(w) \rangle = \frac{\pi^2 c}{6 v^2 \beta^2}$$

$$I = \langle P \rangle = \frac{v^2}{2\pi} \left\langle T - \bar{T} \right\rangle = \frac{\pi c_-}{12\beta^2}$$

$$I_l = \int n(q)\, \varepsilon(q)\, v(q) \frac{dq}{2\pi} = \frac{1}{2\pi} \int_0^\infty \frac{\varepsilon d\varepsilon}{1 + e^{\beta\varepsilon}} = \frac{\pi}{24\beta^2}$$

$$c_- = \frac{\nu}{2}$$

Appendix: Chern Number

The Chern number is a topological quantity characterizing a 2D system of free fermions with an energy gap:

$$\nu = \frac{1}{2\pi i} \int \mathrm{Tr}\left(P(q) \left(\frac{\partial P}{\partial q_x} \frac{\partial P}{\partial q_y} - \frac{\partial P}{\partial q_y} \frac{\partial P}{\partial q_x} \right) \right) dq_x dq_y$$

This quantity is linked to edge modes on a cylinder: when the energy $\varepsilon(q_x)$ of an edge mode ψ crosses zero, $P(q_x)$ changes by $|\psi\rangle\langle\psi|$. For an edge observable Q we have:

$$\pm 1 \approx \langle\psi| Q |\psi\rangle = \int -\mathrm{Tr}\left(Q \frac{\partial P}{\partial q_x} \right) dq_x$$

For a quantum Hall system the Chern number coincides with the **filling factor**. This can be shown calculating the conductance through Kubo's formula.