Physical combinatorics and TBA:

paths, (m, n) systems and finitized characters

Giovanni Feverati

Laboratoire d'Annecy-le-Vieux de physique theorique

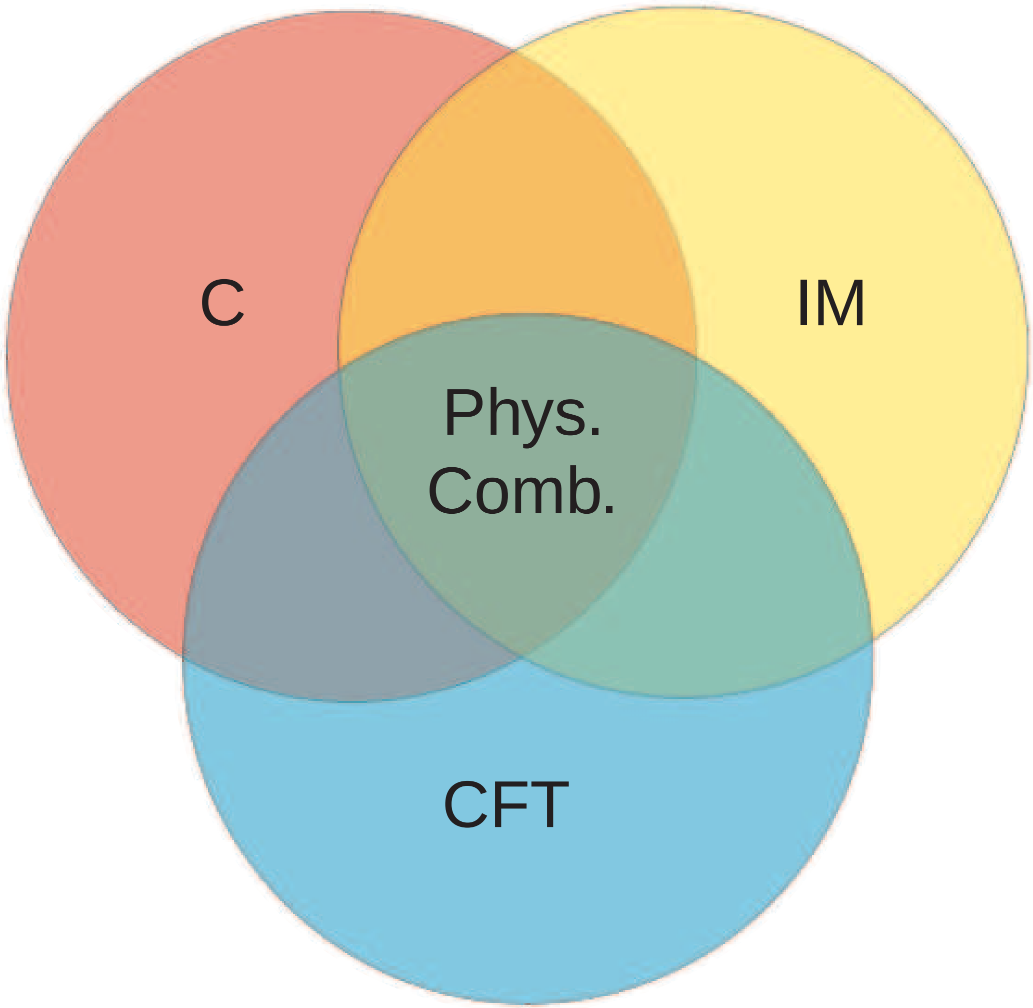
Paul A. Pearce

Department of Mathematics and Statistics
University of Melbourne

Nucl. Phys. B 663, 409-442 (2003), hep-th/0211185, hep-th/0211186
&. . . in preparation

Physical Combinatorics

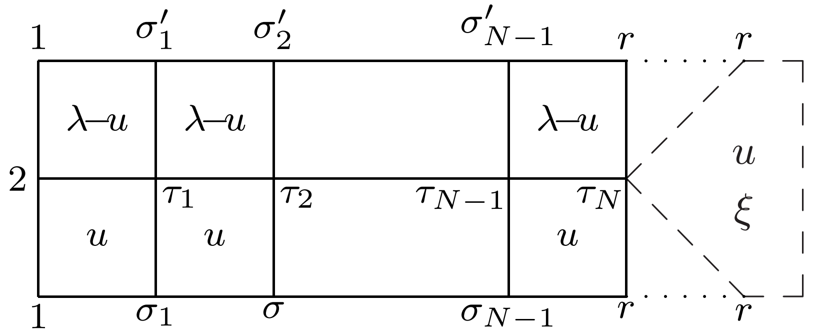
$$C \cap IM \cap CFT = \text{Physical Combinatorics}$$



Integrable RSOS Models with boundaries

Double Row Transfer Matrix:

$$\mathbf{D}(N,u,\xi)_{\sigma,\sigma'} = \sum_{\tau_1,\ldots,\tau_N}$$

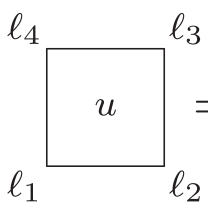


fixed b.c.

boundary interaction

Critical A_L RSOS Face Weights:

(ABF 1984)

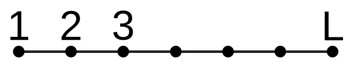


$$= \frac{\sin(\lambda-u)}{\sin\lambda}\,\delta_{\ell_1,\ell_3} + \frac{\sin u}{\sin\lambda}\sqrt{\frac{S_{\ell_1}S_{\ell_3}}{S_{\ell_2}S_{\ell_4}}}\,\delta_{\ell_2,\ell_4}$$

$$\lambda = \frac{\pi}{L+1}\,; \qquad S_\ell = \sin \ell\lambda$$

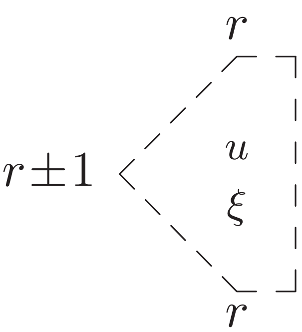
"height" variables: $\ell_j = 1,\ldots,L$

nearest neighbor sites: $|\ell_i - \ell_j| = 1$



Boundary Weights:

(Behrend, Pearce 2001)



$$= \sqrt{\frac{\sin(r\pm 1)\lambda}{\sin r\lambda}\,\frac{\sin(\xi\pm u)\sin(r\lambda+\xi\mp u)}{\sin^2\lambda}}$$

fusion: $\mathbf{D}^{q+1} \sim \mathbf{D}^{q}\mathbf{D}^{1} + \mathbf{D}^{q-1}$, $\tilde{\mathbf{d}}^{q} \sim \mathbf{D}^{q+1}\mathbf{D}^{q-1}$

Integrability: YBE $\Rightarrow [\mathbf{D}^{q}(u), \mathbf{D}^{q'}(v)] = 0$

basis of eigenstates independent of u

Analyticity: each entry of $\mathbf{D}^{1}(u)$ is entire function (zeros); each entry of $\mathbf{D}^{q}(u)$ ($q > 1$) is the ratio of an entire function by some known function

Periodicity: $u + \pi \equiv u$

Functional equations $q = 1, \ldots, L-2$

$$\tilde{\mathbf{d}}^{q}(u - \frac{\lambda}{2})\tilde{\mathbf{d}}^{q}(u + \frac{\lambda}{2}) = \left[\mathbf{1} + \tilde{\mathbf{d}}^{q-1}(u)\right]\left[\mathbf{1} + \tilde{\mathbf{d}}^{q+1}(u)\right]$$

$$\tilde{\mathbf{d}}^{0}(u) = \tilde{\mathbf{d}}^{L-1}(u) = 0$$

- true for the eigenvalues of $\tilde{\mathbf{d}}^{q}(u)$
- solution given by the analytic properties in $u \in \mathbb{C}$:

  ** $L-2$ analyticity strips

  ** zeros of eigenvalues $D^{q}(u)$, zeros of the numerical factors

  ** (with boundaries) additional numerical factors with zeros/poles

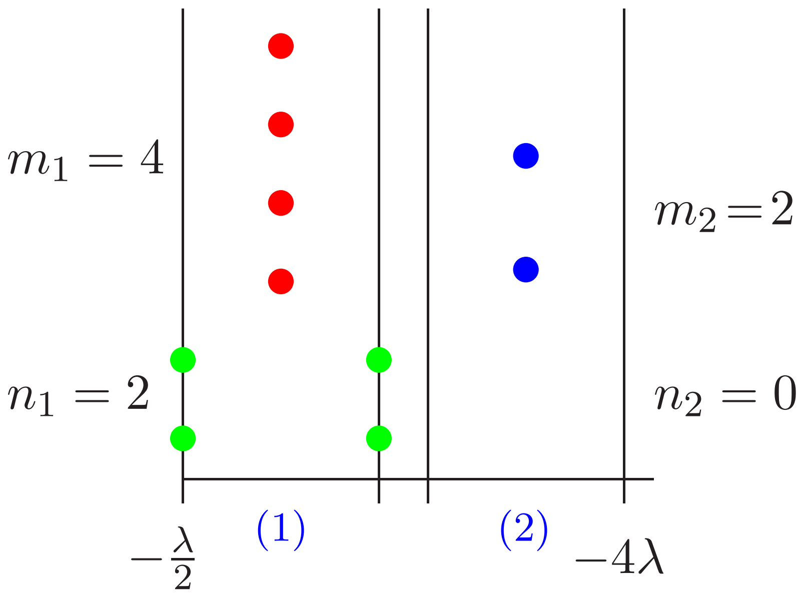
A_4: zeros of the eigenvalues of $\mathrm{D}(u)$

Two analyticity strips: periodicity $u+\pi \equiv u$; $\lambda = \pi/5$

$$(1) \quad -\frac{\lambda}{2} < \mathrm{Re}(u) < \frac{3\lambda}{2}, \qquad (2) \quad 2\lambda < \mathrm{Re}(u) < 4\lambda$$

$$\begin{aligned} m_i &= \{\text{number of } 1\text{-strings in strip } i = 1,2\} \\ n_i &= \{\text{number of } 2\text{-strings in strip } i = 1,2\} \end{aligned}$$

$$(\boldsymbol{m},\boldsymbol{n})-\textbf{system:} \begin{cases} m_1 + n_1 = \frac{N+m_2}{2} \\ m_2 + n_2 = \frac{m_1}{2} \end{cases} \Longrightarrow \quad m_1,\ m_2, \in 2\mathbb{N}$$

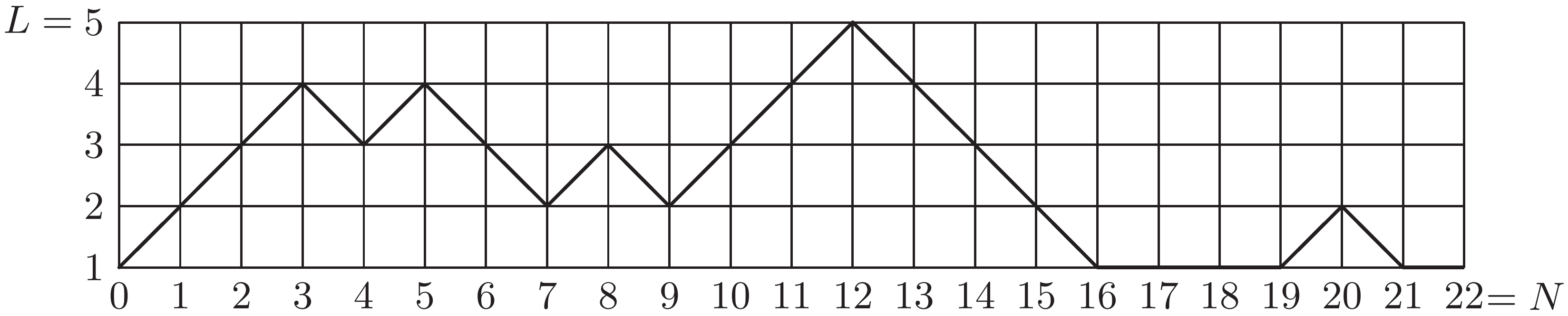


Relative order

Paths

paths on A_L: only diagonal paths are permitted; initial and final point at height=1

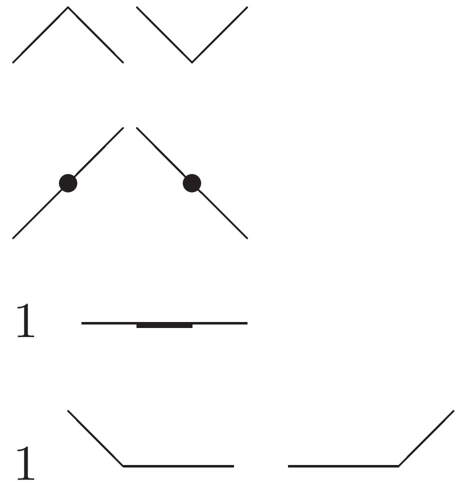
paths on T_L: diagonal paths are permitted; horizontal paths permitted at heigth=1; initial and final point at height=1; fixed shape rectangle $L = \lfloor \frac{N}{2} \rfloor$



Local energy density

(Baxter80, ABF84)

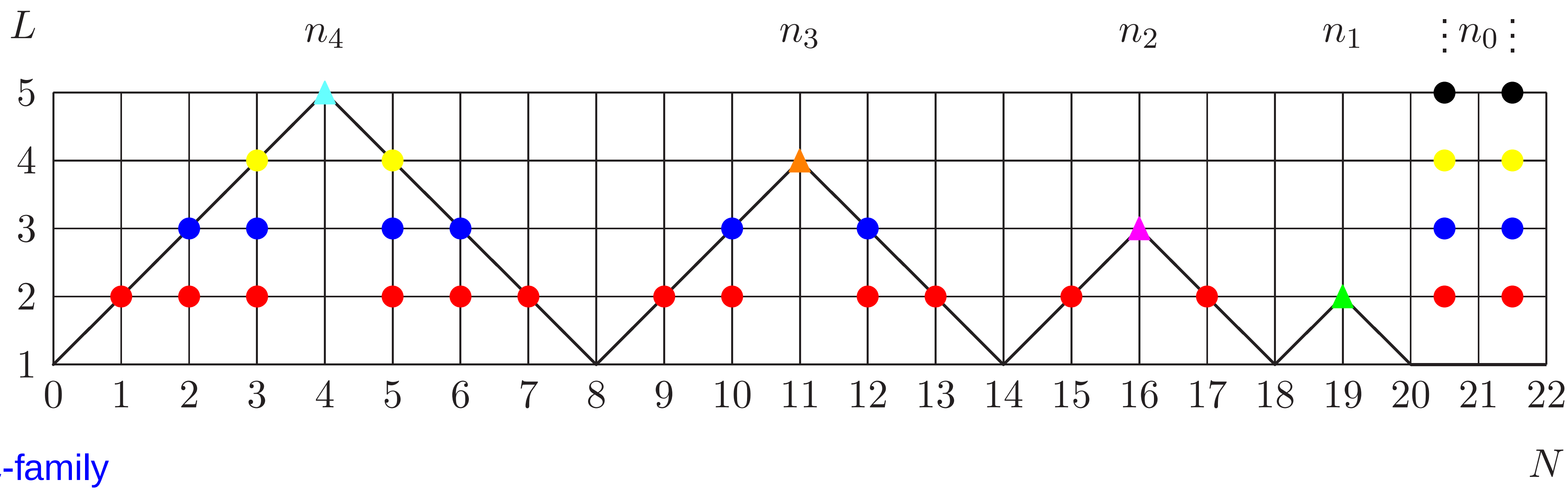
$$h(\sigma_{j-1}, \sigma_j, \sigma_{j+1}) = \begin{cases} 0 & \text{if} \quad \sigma_{j+1} = \sigma_{j-1} \\ 1 & \text{if} \quad \sigma_{j+1} - \sigma_{j-1} = \pm 2 \\ 1 & \text{if} \quad (\sigma_{j-1}, \sigma_j, \sigma_{j+1}) = (1,1,1) \\ \frac{1}{2} & \text{if} \quad (\sigma_{j-1}, \sigma_j, \sigma_{j+1}) = (2,1,1) \text{ or } (1,1,2) \end{cases}$$



1-dim configurational sums $\quad E(\sigma) = \frac{1}{2} \sum_{j=1}^{N} j \, h(\sigma_{j-1}, \sigma_j, \sigma_{j+1})$

Quasi-particles

(Warnaar 1995)



n-family

- $L-1$ types of pure particles (pyramids): ▲ ▲ ▲ ▲ $\cdots$
- tower particle (T_L case only)

$$n_a \quad = \quad \{\# \text{ of particles of type } a = 0, 1, 2, \ldots, L-1\}$$

m-family: in sites where there is a straight line segment or in the middle of horizontal lines

- $L-2$ types of dual particles (strings): ● ● ● $\cdots$

$$m_a \quad = \quad \{\# \text{ of dual particles of type } a = 0, 1, 2, \ldots, L-2\}$$

- Geometric packing constraints

$$
\begin{aligned}
N &= n_0 + 2(L-1)n_{L-1} + \ldots + 6n_3 + 4n_2 + 2n_1 \\
m_1 &= n_0 + 2(L-2)n_{L-1} + \ldots + 4n_3 + 2n_2 \\
m_2 &= n_0 + 2(L-3)n_{L-1} + \ldots + 2n_3 \\
&\vdots \\
m_{L-2} &= n_0 + 2n_{L-1} \\
m_0 &= n_0
\end{aligned}
$$

A_L case: $m_0 = n_0 = 0$

- (m,n) system

$$m_a + n_a = \tfrac{1}{2}N\delta(a,1) + \tfrac{1}{2}\sum_{b=1}^{L-2} A_{a,b}m_b \qquad a = 1,2,\ldots,L-2$$

$A_{a,b}:\ A_{L-2}$ adjacency matrix

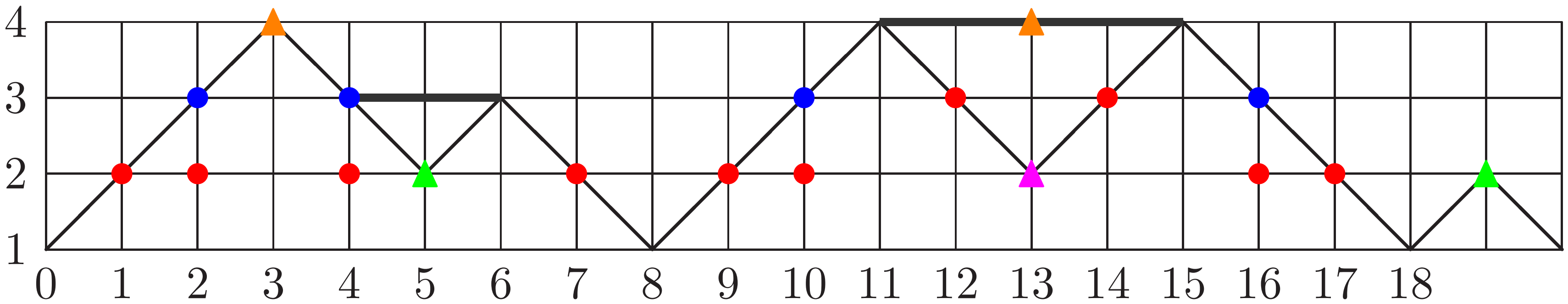
$$m_a + n_a = \tfrac{1}{2}N\delta(a,1) + \tfrac{1}{2}\sum_{b=1}^{L-1} A_{a,b}m_b \qquad a = 0,1,2,\ldots,L-2$$

$A_{a,b}:\ T'_{L-1}$ adjacency matrix

- Interaction: n-family particles can be sliced and diced and turned upside-down (geometric packing constraints are respected).

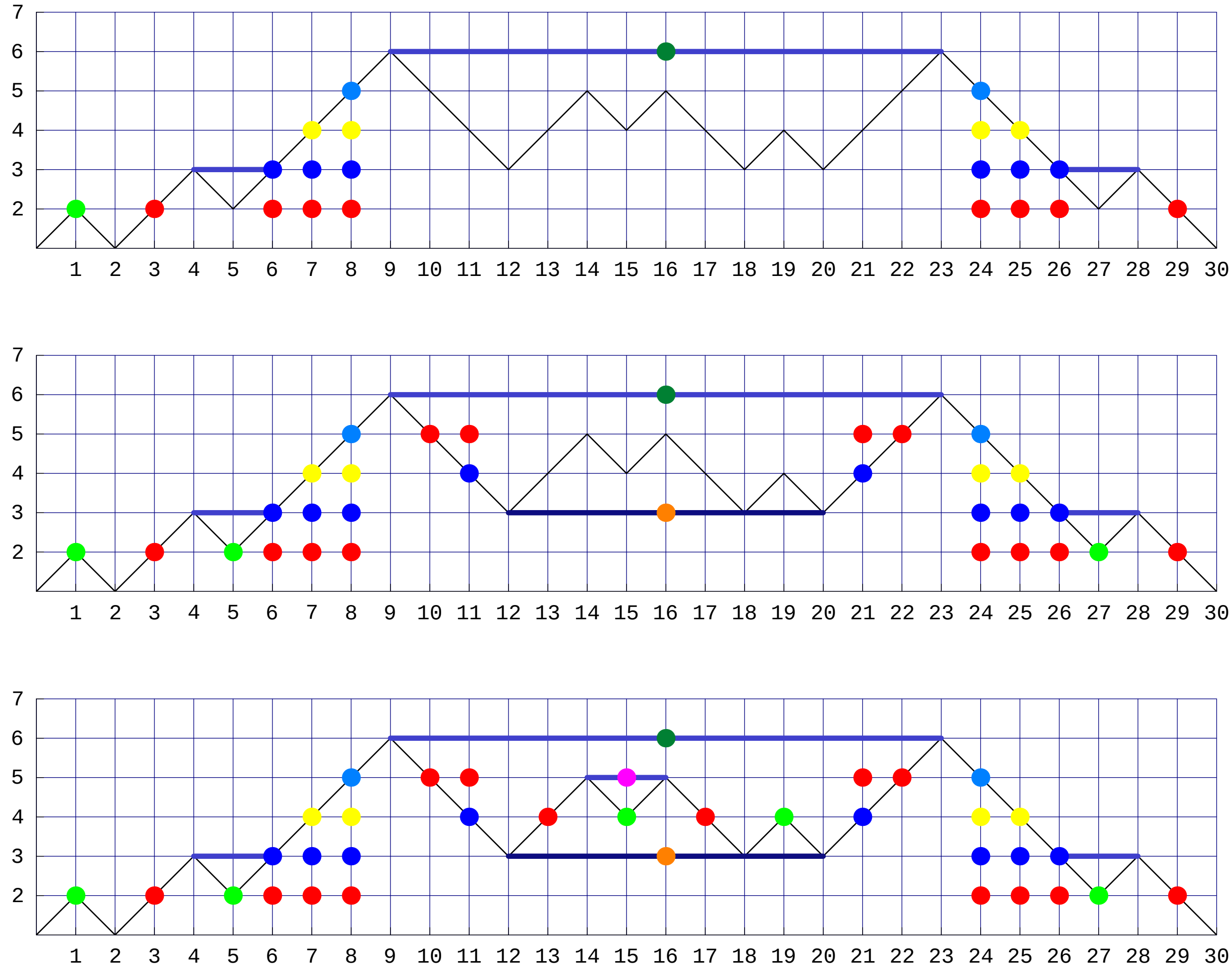
General paths: interactions

Path $=$ {non-interacting particles} $+$ {complexes of overlapping particles} (c.f. Warnaar 1995)



baseline and maximum peak

Decomposition Algorithm

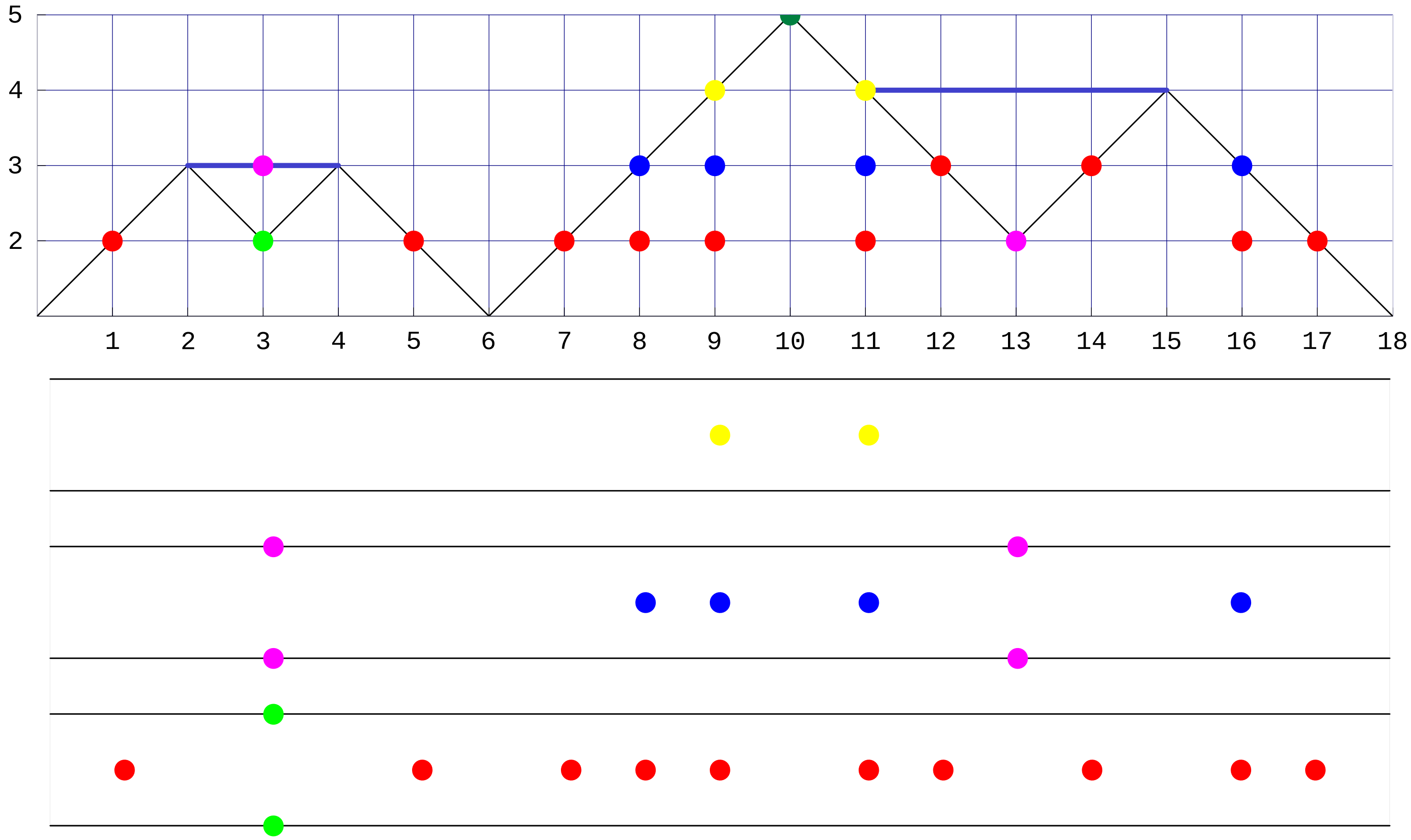


1. Identify any tower particles and decorate them with dual-particles. These tower particles automatically separate the path into non-overlapping complexes with respect to the initial baseline at height 1.
2. For each current baseline, separate the pure particles from the complexes and decorate the pure particles. A complex with respect to the current baseline is any path that is not a pyramid.
3. For each current complex, identify the left-most and right-most global maxima and connect these with a new baseline. The left-most and right-most global maxima may coincide and in this case no new baseline is drawn.
4. From each left (right) maxima, outline the profile of the complex moving continuously down and to the left (right) drawing new baselines as needed. Decorate the sliced particles correponding to these maxima.
5. Stand on your head, identify all the current baselines and go to 2.

Energy-preserving bijection: RSOS paths $\leftrightarrow$ strings

There is a *natural* energy-preserving bijection between RSOS paths and string patterns: a n-particle (m-particle) of type a at position j corresponds to a 2-string (1-string) in strip a; the relative order within a strip is preserved.

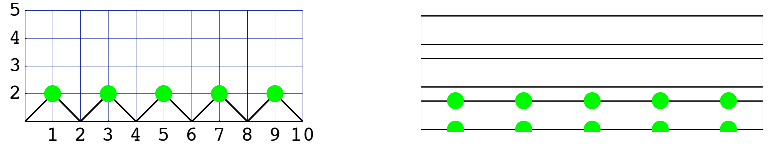
The RSOS particle decomposition matches the pattern of the zeros of the transfer matrix eigenvalues.



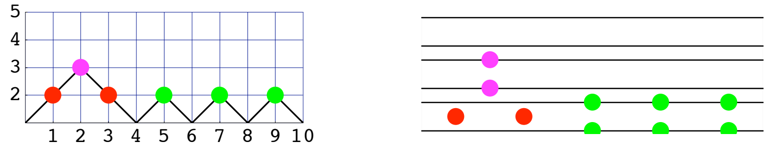
In the example, $m_1 = 10$, $m_2 = 4$, $m_3 = 2$ and $E = 50$.

more data

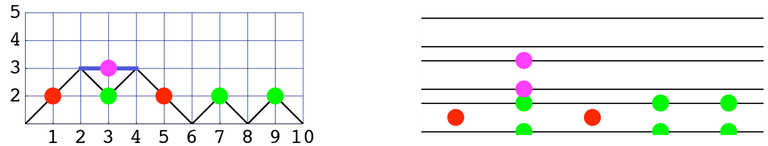
In[98]:= **chiralPaths[1, 1, 5, 8]**



1:   E=0   m={0, 0, 0}   n={5, 0, 0, 0}

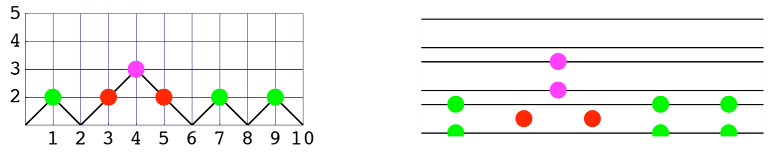


2:   E=2   m={2, 0, 0}   n={3, 1, 0, 0}

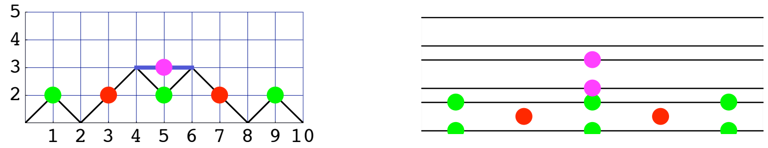


3:   E=3   m={2, 0, 0}   n={3, 1, 0, 0}

4:   E=4   m={2, 0, 0}   n={3, 1, 0, 0}

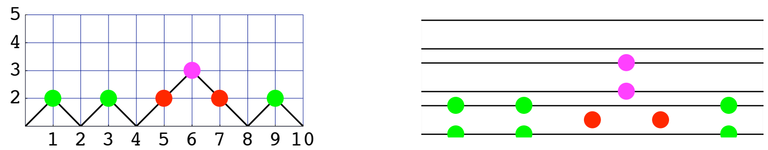


5: E=4 m={2, 0, 0} n={3, 1, 0, 0}



6: E=5 m={2, 0, 0} n={3, 1, 0, 0}

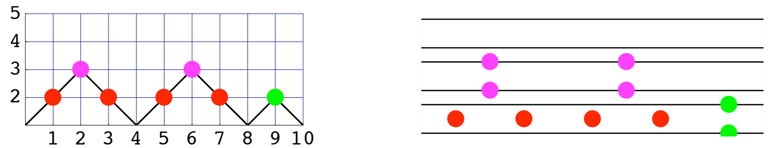
7: E=6 m={4, 2, 0} n={2, 0, 1, 0}



8: E=6 m={2, 0, 0} n={3, 1, 0, 0}

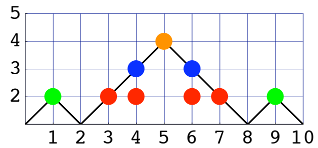
9: E=7 m={4, 2, 0} n={2, 0, 1, 0}

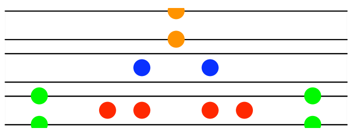
10: E=8 m={4, 2, 0} n={2, 0, 1, 0}



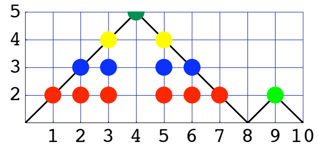
11: E=8 m={4, 0, 0} n={1, 2, 0, 0}

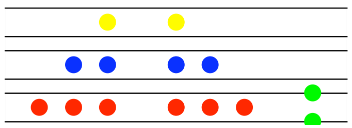
12: E=9 m={4, 2, 0} n={2, 0, 1, 0}





13: E=10 m={4, 2, 0} n={2, 0, 1, 0}

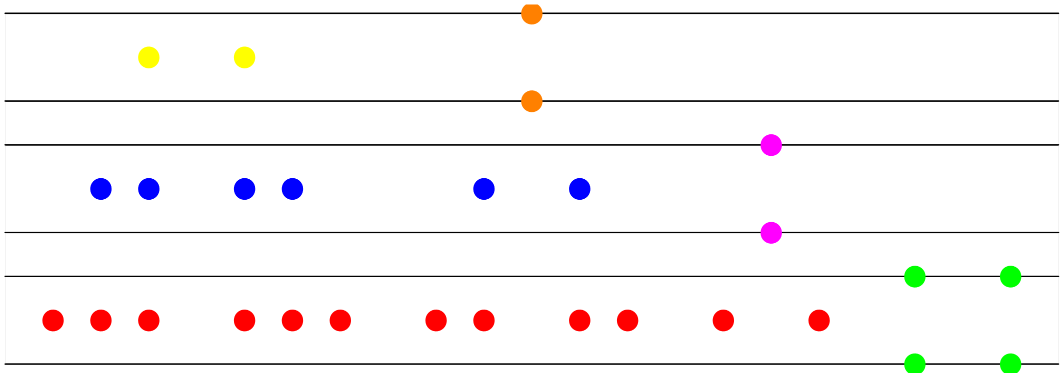




14: E=12 m={6, 4, 2} n={1, 0, 0, 1}

Minimal Energy Configurations

For given particle/string content, the minimal energy occurs when the particles are ordered from largest to smallest and the 1-strings are all to the left of the 2-strings (creation energy).



The minimal energy is

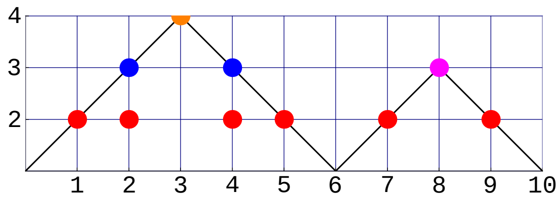
$$E = \frac{1}{4} \boldsymbol{m} \, C \, \boldsymbol{m}$$

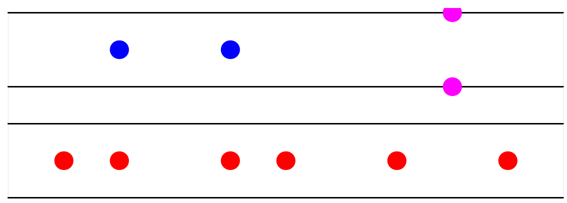
where $C = 2I - A = \{$Cartan matrix of $A_{L-2}\}$

Particle Dynamics:

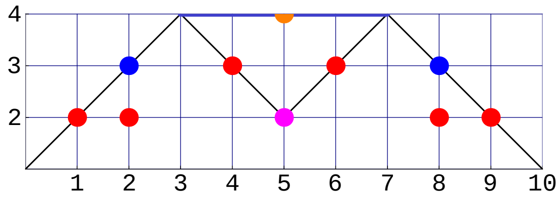
relative order within a strip does matter!

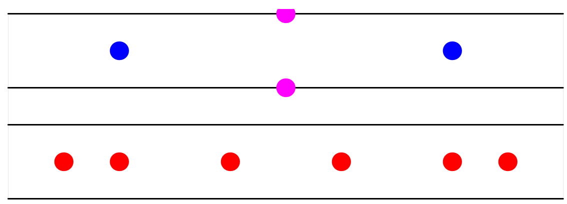
$E = 14$



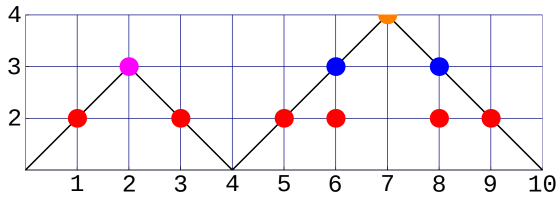


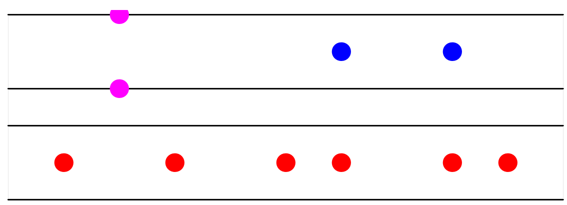
$E = 15$



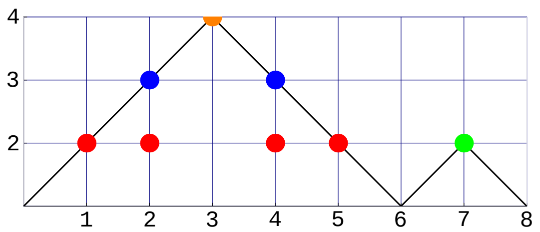


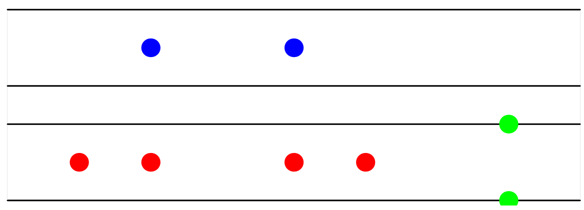
$E = 16$



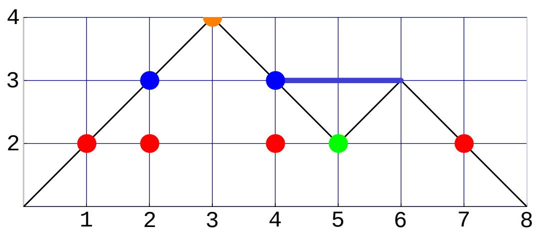


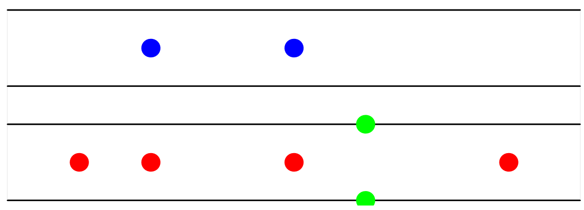
$E = 6$



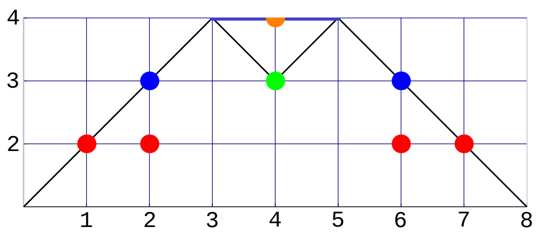


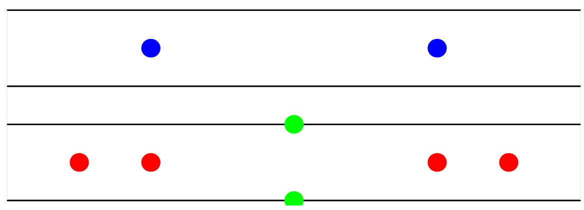
$E = 7$





$E = 8$



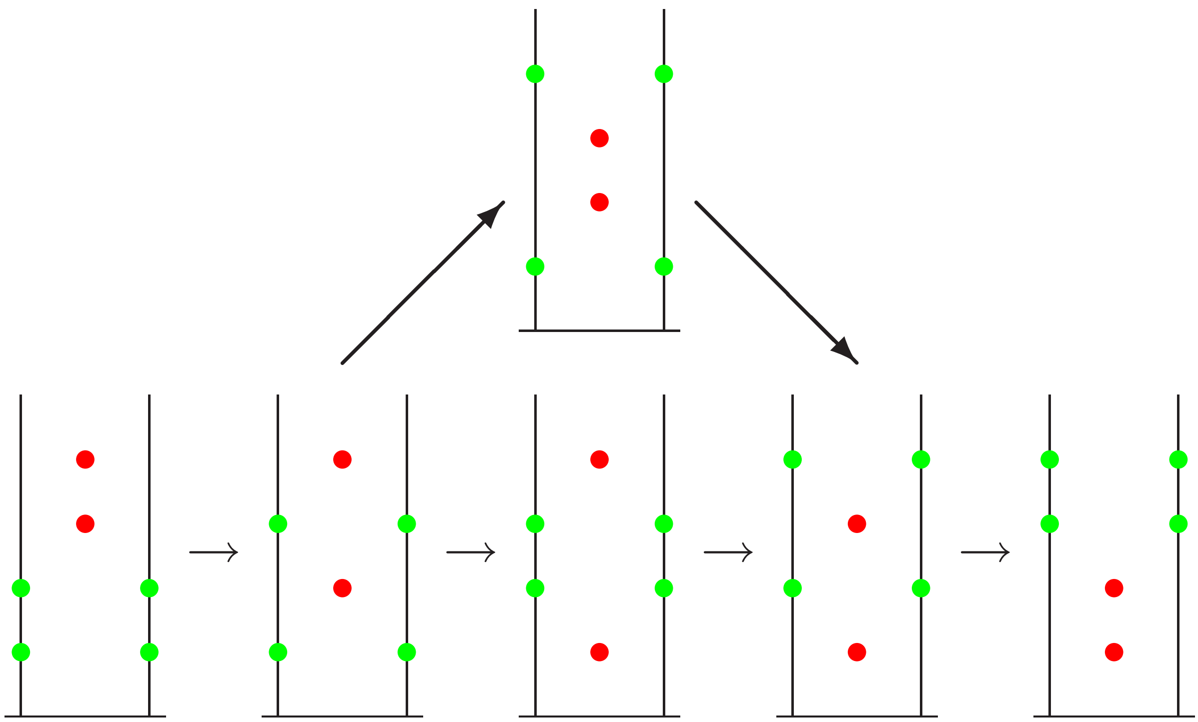


q-Binomials

$$\begin{bmatrix} m+n \\ m \end{bmatrix}_q = \sum_{I_1=0}^{n} \sum_{I_2=0}^{I_1} \cdots \sum_{I_m=0}^{I_{m-1}} q^{I_1+\ldots+I_m} = \begin{cases} \dfrac{(q)_{m+n}}{(q)_m\,(q)_n}, & m,n \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

q-factorial: $(q)_m = \prod_{j=1}^{m} (1-q^j)$

Example: $\left[\begin{smallmatrix}4\\2\end{smallmatrix}\right]_q = 1 + q + 2q^2 + q^3 + q^4$



$$1 \quad + \quad q \quad + \quad 2q^2 \quad + \quad q^3 \quad + \quad q^4$$

Integer and non-negative quantum numbers

$I_j^{(a)} = \{$number of 2-strings above the 1-string labelled j in strip $a\}$

Energy eigenstate characterised by $I^{(a)} = (I_1^{(a)}, I_2^{(a)}, \ldots, I_{m_a}^{(a)})$

Within one strip $n_a \geqslant I_1^{(a)} \geqslant I_2^{(a)} \geqslant \cdots \geqslant I_{m_a}^{(a)} \geqslant 0$

Finitized fermionic characters

minimal energy configuration + "excitation energies (q-binomials)" = spectrum of E (lattice fermionic gas)

Generating function $\sum q^E$: $$\chi_0^{(N)}(q) = q^{-c/24} \sum_{(m,n)} q^{\frac{1}{4}\,\boldsymbol{m}^T\, C\, \boldsymbol{m}} \prod_{a=1}^{L-2} \begin{bmatrix} m_a + n_a \\ m_a \end{bmatrix}_q$$

(m, n)-system $$m_a + n_a = \tfrac{1}{2} N \delta(a,1) + \tfrac{1}{2} \sum_{b=1}^{L-2} A_{a,b} m_b \qquad a = 1, 2, \ldots, L-2$$

$m_a\,, n_a$ bounded by N; in particular $$\tfrac{N}{2} \geqslant m_1 \geqslant m_2 \geqslant \ldots \geqslant m_{L-2} \geqslant 0$$

Finite versus finitized

Limit $N \to \infty$, with N even, produces the CFT vacuum fermionic character

$$\chi_0(q) = q^{-c/24} \sum_{\substack{m_a = 0 \text{ (mod 2)} \\ m_1, m_2, \ldots, m_{L-2} \geq 0}} \frac{q^{\frac{1}{4}\,\boldsymbol{m}^T\, C\, \boldsymbol{m}}}{(q)_{m_1}} \prod_{a=2}^{L-2} \begin{bmatrix} \frac{1}{2}(m_{a-1} + m_{a+1}) \\ m_a \end{bmatrix}_q$$

A_L model TBA equations

Scaling energy: $$E = -\int_{-\infty}^{\infty} \frac{dy}{\pi^2} e^{-y} \log(1+\hat{d}_1(y)) + \frac{2}{\pi} \sum_{k=1}^{m_1} e^{-y_k^{(1)}}$$

TBA equations: $q = 1, \ldots, L-2$

$$\log \hat{d}_q(x) = -4e^{-x}\delta_{q,1} + \sum_{j=1}^{L-2} A_{q,j} \sum_{k=1}^{m_j} \log\tanh[\tfrac{1}{2}(x - y_k^{(j)})] + \sum_{j=1}^{L-2} A_{q,j} \int_{-\infty}^{\infty} \frac{dy}{2\pi} \frac{\log(1+\hat{d}_j(y))}{\cosh(x-y)}$$

Counting function

$$\begin{aligned} \Psi_q(x) &= i\log\hat{d}_q(x - i\frac{1}{2}\pi) = 4e^{-x}\delta_{q,1} + i\sum_{j=1}^{L-2} A_{q,j} \sum_{k=1}^{m_j} \log\tanh(\frac{x - y_k^{(j)}}{2} - i\frac{\pi}{4}) \\ &\quad - \fint_{-\infty}^{\infty} \frac{dy}{2\pi} \frac{\log(1+\hat{d}_j(y))}{\sinh(x-y)} \end{aligned}$$

Quantization conditions (auxiliary equations) $\Psi_q(y_k^{(q)}) = \Big[1 + 2(I_k^{(q)} + m_q - k)\Big]\pi$

Solution of TBA

Energy eigenvalues can be worked out exactly

$$E = -\frac{1}{24}\left(1 - \frac{6}{L(L+1)}\right) + \frac{1}{4}\boldsymbol{m}^T C\, \boldsymbol{m} + \sum_{a=1}^{L-2}\sum_{k=1}^{m_a} I_k^{(a)}$$

$$n_a \geqslant I_1^{(a)} \geqslant I_2^{(a)} \geqslant \ldots \geqslant I_{m_q}^{(a)} \geqslant 0 \qquad (\text{for } a = 1\,,\ n_1 = \infty)$$

$\sum q^E$ gives the vacuum sector fermionic character

$$\chi_0(q) = q^{-c/24} \sum_{\substack{m_a = 0 \text{ (mod 2)} \\ m_1, m_2, \ldots, m_{L-2} \geq 0}} \frac{q^{\frac{1}{4}\boldsymbol{m}^T C \boldsymbol{m}}}{(q)_{m_1}} \prod_{a=2}^{L-2} \begin{bmatrix} \frac{1}{2}(m_{a-1} + m_{a+1}) \\ m_a \end{bmatrix}_q$$

TBA, integrals of motion and particle interpretation

... ricapitoliamo ...

- transfer matrix of integrable lattice model
- paths, quasi-particles and (m, n)-system
- energy-preserving bijection: RSOS paths $\longleftrightarrow$ zeros of transfer matrix eigenvalues; paths encode the classification of states
- finitized fermionic characters, lattice gas interpretation
- continuum scaling limit, thermodynamic Bethe Ansatz equations
- conformal partition function from TBA
- equivalence between Virasoro characters and 1-d configurational sums demonstrated in a lattice approach working entirely at criticality, with open boundary conditions

- extension to other bound. cond. including periodic case, massive and massless cases
- extension to other models (non-unitary, parafermions ...)