The entanglement entropy and its universal behaviour in one dimension

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Entanglement in quantum mechanics

- Entanglement occurs when a measurement in a quantum state of an observable somewhere immediately affects future measurements of observables elsewhere. Example: pair of opposite-spin photons created during some annihilation process:

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right) , \quad \langle\hat{A}\rangle = \langle\psi|\hat{A}|\psi\rangle$$

- What is special: Bell's inequality says that this cannot be described by **local variables**.

- This is particular to **pure states**. Mixed states are described by density matrices

$$\rho = \sum_\alpha p_\alpha |\psi_\alpha\rangle\langle\psi_\alpha| , \quad \langle\hat{A}\rangle = \mathrm{Tr}(\rho\hat{A})$$

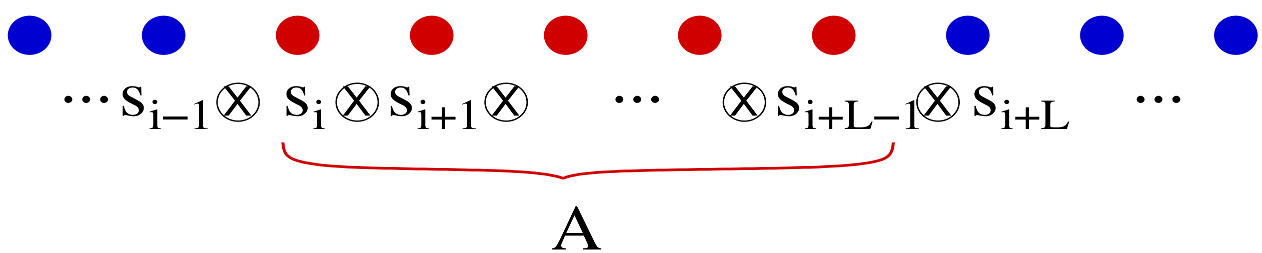
(for pure states, $\rho = |\psi\rangle\langle\psi|$; for finite temperature, $\rho = e^{-H/kT}$). A situation that looks similar to $|\psi\rangle$ but without entanglement is:

$$\rho = \frac{1}{2}\left(|\uparrow\downarrow\rangle\langle\uparrow\downarrow| + |\downarrow\uparrow\rangle\langle\downarrow\uparrow|\right)$$

How to measure (or quantify) quantum entanglement?

- Quantum entanglement is useful: at the basis of better performances of the (still theoretical) quantum computers. It is also a fundamental property of quantum mechanics.
- In **pure states**, there are various propositions for measures of quantum entanglement. Consider the **entanglement entropy**:
  - With the Hilbert space a tensor product $\mathcal{H} = s_1 \otimes s_2 \otimes \cdots \otimes s_N = A \otimes \bar{A}$, and a given state $|\mathrm{gs}\rangle \in \mathcal{H}$, calculate the **reduced density matrix**:

$$\rho_A = \mathrm{Tr}_{\bar{A}}(|\mathrm{gs}\rangle\langle\mathrm{gs}|)$$



  - The entanglement entropy is the resulting **von Neumann entropy**:

$$S_A = -\mathrm{Tr}_A(\rho_A \log(\rho_A)) = -\sum_{\substack{\text{eigenvalues of } \rho_A \\ \lambda \neq 0}} \lambda \log(\lambda)$$

The entanglement entropy

- It is the entropy that is measured in a subsystem A, once the rest of the system $\bar{A}$ – the environment – is forgotten.

  If we think A is all there is, we will think the system is in a mixed state, with density matrix given by ρ_A. The entropy of ρ_A measures how mixed ρ_A is. This mising is due to the connections, or entanglement, with the environment.
- It was proposed as a way to understand black hole entropy [Bombelli, Koul, Lee, Sorkin 1986].
- Then it was proposed as a measure of entanglement [Bennet, Bernstein, Popescu, Schumacher 1996].
- Examples:
  - Tensor product state:
    $|\mathrm{gs}\rangle = |A\rangle \otimes |\bar{A}\rangle \Rightarrow \rho_A = |A\rangle\langle A| \Rightarrow S_A = -1\log(1) = 0$.
  - The state $|\mathrm{gs}\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$:
    $$\rho_{1^{\mathrm{st}}\ \mathrm{spin}} = \frac{1}{2}(|\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow|) \Rightarrow S_{1^{\mathrm{st}}\ \mathrm{spin}} = -2\times\left(\frac{1}{2}\log\left(\frac{1}{2}\right)\right) = \log(2)$$

There are no known good measures of quantum entanglement in mixed states.

One basic property of entanglement entropy

Entanglement entropy is not "directional": $S_A = S_{\bar{A}}$. Proof:

- Consider anti-linear map $f : A \longrightarrow \bar{A}$ with $f|A\rangle = \langle A|\mathrm{gs}\rangle$. Similarly $\bar{f} : \bar{A} \longrightarrow A$ with $\bar{f}|\bar{A}\rangle = \langle \bar{A}|\mathrm{gs}\rangle$.
- Then $\rho_A : A \longrightarrow A$ is $\rho_A = \bar{f} f$ and $\rho_{\bar{A}} : \bar{A} \longrightarrow \bar{A}$ is $\rho_{\bar{A}} = f \bar{f}$.
- Hence if $\rho_A|A\rangle = \lambda|A\rangle$ then $\bar{f} f|A\rangle = \lambda|A\rangle \Rightarrow (f\bar{f})f|A\rangle = \lambda f|A\rangle$ so that $\rho_{\bar{A}} f|A\rangle = \lambda f|A\rangle$.
- Hence ρ_A and $\rho_{\bar{A}}$ have the same set of non-zero eigenvalues (with the same degeneracies).

Scaling limit

- Say $|\mathrm{gs}\rangle$ is a ground state of some local spin-chain Hamiltonian, and that the chain is infinitely long.
- An important property of $|\mathrm{gs}\rangle$ is the **correlation length** ξ:

$$\langle\mathrm{gs}|\hat{\sigma}_i\hat{\sigma}_j|\mathrm{gs}\rangle \sim e^{-|i-j|/\xi} \text{ as } |i-j| \to \infty$$

- Suppose there are parameters in the Hamiltonian such that for certain values, $\xi \to \infty$. This is a **quantum critical point**.
- We may adjust these parameters in such a way that the length L of A stays in proportion to ξ: $L/\xi = mr$.
- The resulting entanglement entropy has a **universal** part: a part that does not depend very much on the details of the Hamiltonian.
- This is the **scaling limit**, and what we obtain is a **quantum field theory**. Here: with a mass m – or with many masses m_α associated to many correlation lengths – and where r is the dimensionful length of A in the scaling limit.

Short- and large-distance entanglement entropy

Consider $\varepsilon = 1/(m_1 \xi)$, a non-universal QFT cutoff with dimenions of length. Then:

- Short distance: $0 \ll L \ll \xi$, logarithmic behavior [Holzhey, Larsen, Wilczek 1994; Calabrese, Cardy 2004]

$$S_A \sim \frac{c}{3} \log\left(\frac{r}{\varepsilon}\right)$$

- Large distance: $0 \ll \xi \ll L$, saturation

$$S_A \sim -\frac{c}{3} \log(m_1 \varepsilon) + U$$

where c is the central charge of the corresponding critical point.

The next correction term

We found [Cardy, Castro Alvaredo, D. 2007], [Castro Alvaredo, D. 2008], [D. 2008]

$$S_A \sim -\frac{c}{3}\log(m_1\varepsilon) + U - \frac{1}{8}\sum_{\alpha=1}^{\ell} K_0(2rm_\alpha) + O\left(e^{-3rm_1}\right)$$

where ℓ is the number of particles in the spectrum of the QFT, and m_α are the masses of the particles, with $m_1 \leq m_\alpha\ \forall\alpha$.

- This next correction term depends only on the particle spectrum, but not on their interaction (i.e. not on the way they scatter off each other).

- In generic QFT, the largest mass is less than twice the smallest mass. Hence, the entanglement entropy provides “clean” information about “half” of the spectrum.

Partition functions on multi-sheeted Riemann surfaces

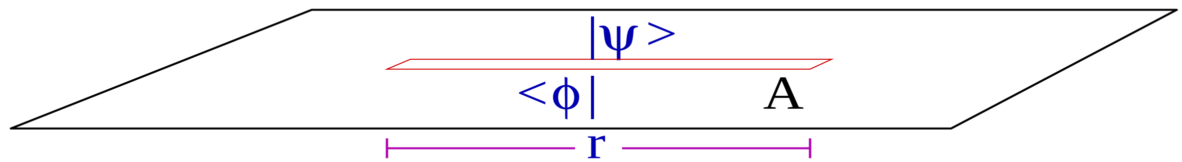
[Callan, Wilczek 1994; Holzhey, Larsen, Wilczek 1994]

- We can use the “replica trick” for evaluating the entanglement entropy:

$$S_A = -\mathrm{Tr}_A(\rho_A \log(\rho_A)) = -\lim_{n\to 1}\frac{d}{dn}\mathrm{Tr}_A(\rho_A^n)$$

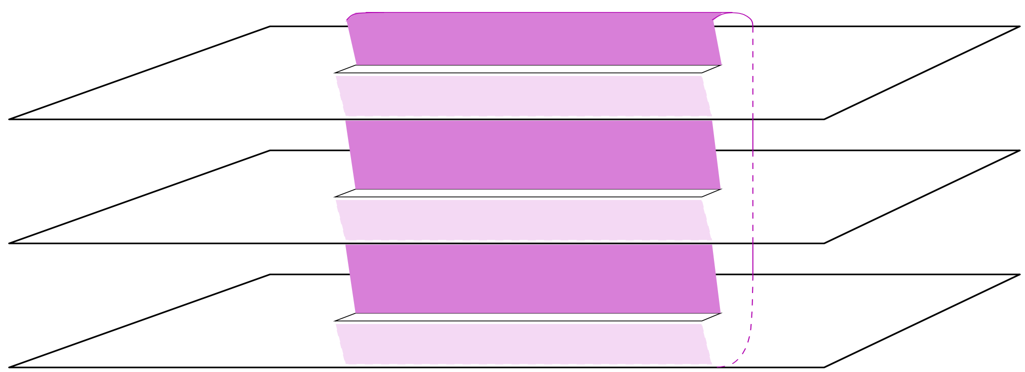
- For integer numbers n of replicas, in the scaling limit, this is a partition function on a Riemann surface:

$${}_A\langle\phi|\rho_A|\psi\rangle_A \quad \sim$$



$$\mathrm{Tr}_A(\rho_A^n) \quad \sim \quad Z_n = \int [d\varphi]_{\mathcal{M}_n} \exp\left[-\int_{\mathcal{M}_n} d^2x\, \mathcal{L}[\varphi](x)\right]$$

$$\mathcal{M}_n \quad :$$



Branch-point twist fields

[Cardy, Castro Alvaredo, D. 2007]

- Consider many copies of the QFT model on the usual $\mathbb{R}^2$:

$$\mathcal{L}^{(n)}[\varphi_1,\ldots,\varphi_n](x) = \mathcal{L}[\varphi_1](x) + \ldots + \mathcal{L}[\varphi_n](x)$$

- There is an obvious symmetry under cyclic exchange of the copies:

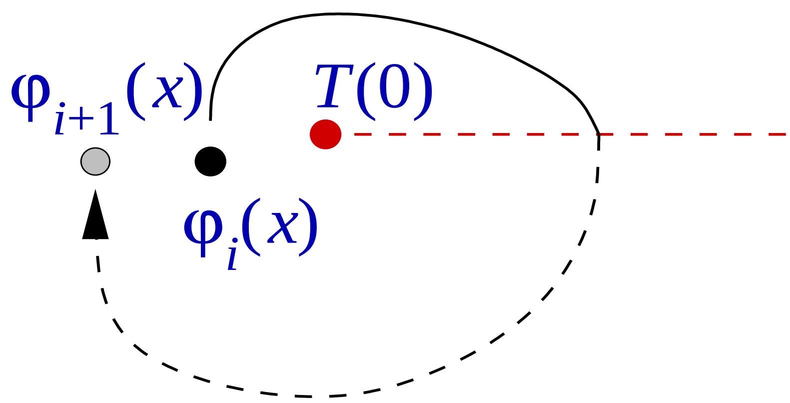
$$\mathcal{L}^{(n)}[\sigma\varphi_1,\ldots,\sigma\varphi_n] = \mathcal{L}^{(n)}[\varphi_1,\ldots,\varphi_n]\,, \quad \text{with} \quad \sigma\varphi_i = \varphi_{i+1 \bmod n}$$

- The associated **twist fields** $\mathcal{T}$, when inside correlation functions, gives

$$\langle \mathcal{T}(0) \cdots \rangle_{\mathcal{L}^{(n)}} \propto \int_{C_0} [d\varphi_1 \cdots d\varphi_n]_{\mathbb{R}^2} \exp\left[- \int_{\mathbb{R}^2} d^2x \, \mathcal{L}^{(n)}[\varphi_1, \ldots, \varphi_n](x) \right] \cdots$$

with branching conditions on the line $\mathrm{x} \in (0, \infty)$ given by

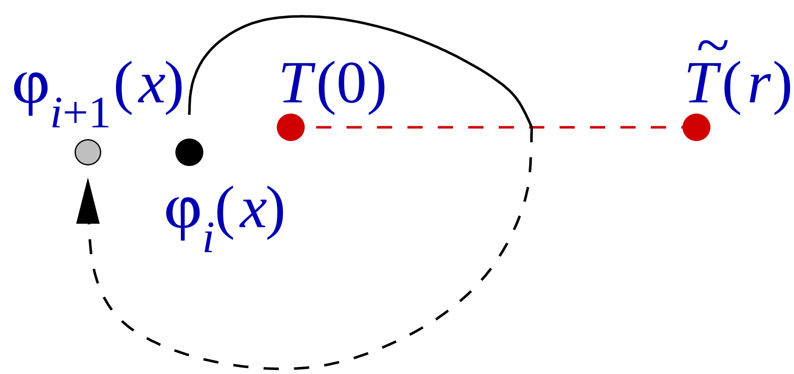
$$C_0 \;:\; \varphi_i(\mathrm{x}, 0^+) = \varphi_{i+1}(\mathrm{x}, 0^-) \quad (\mathrm{x} > 0)$$



- Another twist field $\tilde{\mathcal{T}}$ is associated to the inverse symmetry σ^{-1}, and we have

$$\begin{aligned} \langle \mathcal{T}(0)\tilde{\mathcal{T}}(r)\rangle_{\mathcal{L}^{(n)}} &\propto \int_{C_{0,r}} [d\varphi_1 \cdots d\varphi_n]_{\mathbb{R}^2} \exp\left[-\int_{\mathbb{R}^2} d^2x\, \mathcal{L}^{(n)}[\varphi_1,\ldots,\varphi_n](x)\right] \\ &= Z_n \end{aligned}$$

$C_{0,r}$:

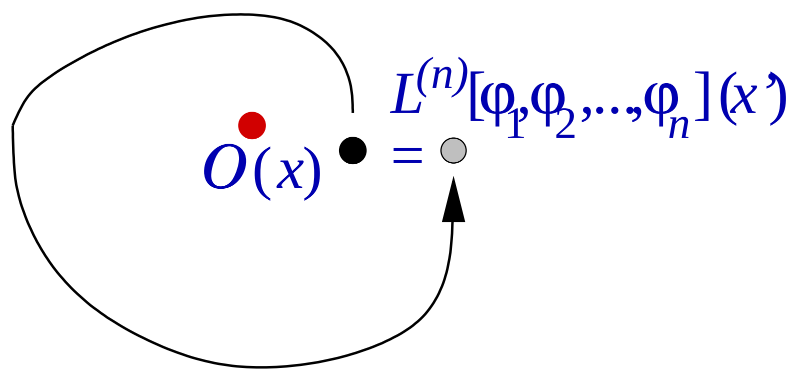


Locality in QFT

- A field $\mathcal{O}(x)$ is **local** in QFT if measurements associated to this field are quantum mechanically independent from measurements of the **energy density** (or **Lagrangian density**) at space-like distances. That is, equal-time commutation relations vanish:

$$[\mathcal{O}(\mathrm{x}, t=0), \mathcal{L}^{(n)}(\mathrm{x}', t=0)] = 0 \quad (\mathrm{x} \neq \mathrm{x}') \ .$$

- This means that:



- **Branch-point twist fields are local fields in the n-copy theory.**

Short- and large-distance entanglement entropy revisited

Hence we have

$$Z_n = D_n \varepsilon^{2d_n} \langle \mathcal{T}(0)\tilde{\mathcal{T}}(r)\rangle_{\mathcal{L}^{(n)}} \ , \quad S_A = -\lim_{n\to 1} \frac{d}{dn} Z_n$$

where D_n is a normalisation constant, and d_n is the scaling dimension of $\mathcal{T}$ [Calabrese, Cardy 2004]:

$$d_n = \frac{c}{12}\left(n - \frac{1}{n}\right)$$

- Short distance: $0 \ll L \ll \xi$, logarithmic behavior

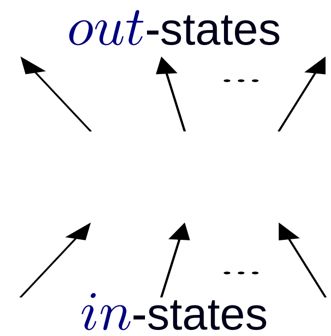
$$\langle \mathcal{T}(0)\tilde{\mathcal{T}}(r)\rangle_{\mathcal{L}^{(n)}} \sim r^{-2d_n} \ \Rightarrow \ S_A \sim \frac{c}{3}\log\left(\frac{r}{\varepsilon}\right)$$

- Large distance: $0 \ll \xi \ll L$, saturation

$$\langle \mathcal{T}(0)\tilde{\mathcal{T}}(r)\rangle_{\mathcal{L}^{(n)}} \sim \langle \mathcal{T}\rangle^2_{\mathcal{L}^{(n)}} \ \Rightarrow \ S_A \sim -\frac{c}{3}\log(m_1\varepsilon) + U$$

Asymptotic states

- In QFT, the Hilbert space is described by particles coming from the far past (in-states) or going to the far future (out-states). The overlap between in- and out-states is the **scattering matrix**.



- With particle trajectories chosen to meet all at a point in space-time, the set of all possible configurations of incoming particles (particle types and rapidities) form a basis for the Hilbert space. Idem for outgoing particles.

- These in-states or out-states are denoted $|\theta_1, \theta_2, \ldots, \theta_k\rangle^{in,out}_{\alpha_1,\alpha_2,\ldots,\alpha_k}$ with $\theta_1 > \ldots > \theta_k$ for in-states and the opposite for out-states, where θ_i's are rapidities and α_i's are particle types. Here we assume all particles of the model have non-zero mass.

- Energy and momentum of these states are the sums of those of individual particles: $E = \sum_{i=0}^{k} m_{\alpha_i} \cosh\theta_i$ and $P = \sum_{i=0}^{k} m_{\alpha_i} \sinh\theta_i$.

Form factors and two-point function

- In the n-replica model $\mathcal{L}^{(n)}$, there are n times as many particle types, which we will denote by $\mu = (\alpha, j)$ with $j = 1, \ldots, n$ the replica label.

- The two-point function of branch-point twist fields can be decomposed into the in-basis, giving a **large-distance expansion**:

$$\langle \mathcal{T}(0)\tilde{\mathcal{T}}(r)\rangle_{\mathcal{L}^{(n)}} = \langle \text{vac}|\mathcal{T}(0)\tilde{\mathcal{T}}(r)|\text{vac}\rangle =$$
$$\sum_{k=0}^{\infty} \sum_{\substack{\alpha_1,\ldots,\alpha_k \\ j_1,\ldots,j_k}} \int \frac{d\theta_1 \cdots d\theta_k}{(2\pi)^k} |F_{\mu_1,\ldots,\mu_k}(\theta_1,\ldots,\theta_k)|^2 e^{-r\sum_{i=1}^k m_{\alpha_i}\cosh\theta_i}$$

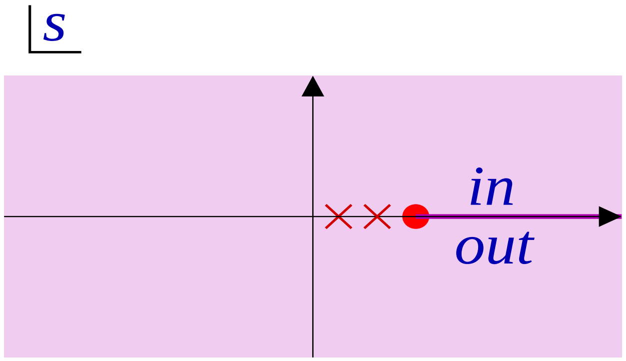
where the matrix elements involved are called **form factors** (where we choose, say, the in-states):

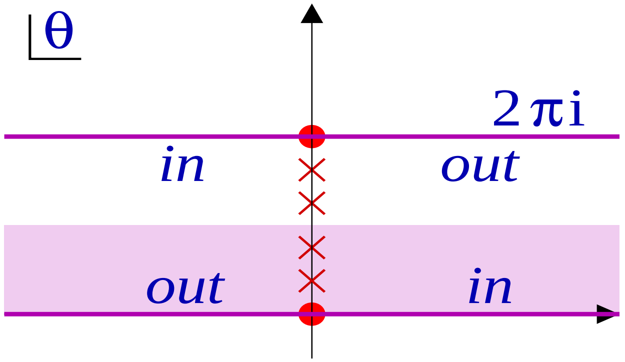
$$F_{\mu_1,\ldots,\mu_k}(\theta_1,\ldots,\theta_k) = \langle \text{vac}|\mathcal{T}(0)|\theta_1,\ldots,\theta_k\rangle^{in}_{\mu_1,\ldots,\mu_k}$$

Analytic properties of two-particle form factors

Consider $F_{\mu_1,\mu_2}(\theta_1,\theta_2) = F_{\mu_1,\mu_2}(\theta_1-\theta_2)$ (by relativistic invariance) as an analytic function of $\theta \equiv \theta_1 - \theta_2$.

- Such form factors for **usual (non-twist) fields** have a well-known analytic structure: using Mandelstam's s-variable $s = m_{\alpha_1}^2 + m_{\alpha_2}^2 + 2m_{\alpha_1}m_{\alpha_2}\cosh(\theta)$, there is a branch cut from $s=(m_{\alpha_1}+m_{\alpha_2})^2$ to ∞, just above which we are describing the physical form factor with an in-state, and just below which it is the form factor with an out-state instead. Between 0 and $(m_{\alpha_1}+m_{\alpha_2})^2$, there may be poles due to bound states, and there are no other singularities on the physical sheet.



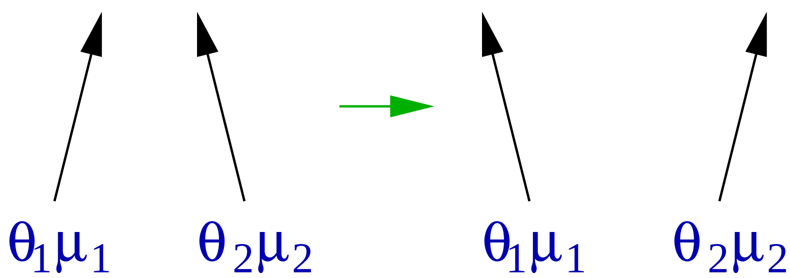


- Form factors for **branch-point twist-fields** have **modified analytic properties**.

Change of sign of θ (as usual)

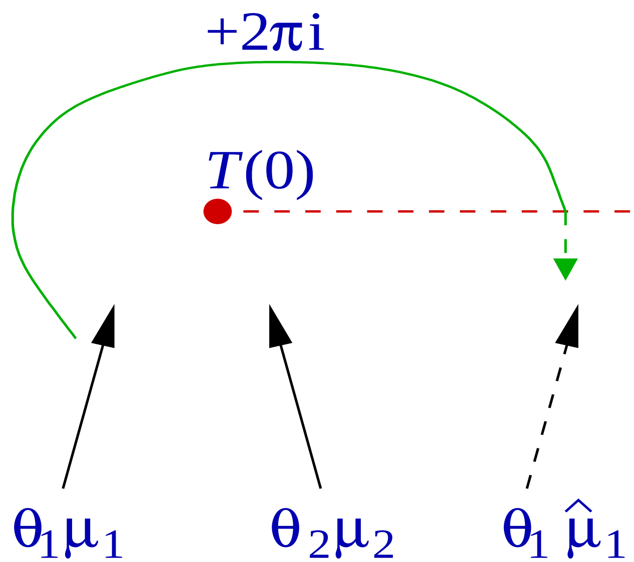
For $\theta_1 < \theta_2$:

$$
\begin{aligned}
F_{\mu_1,\mu_2}(\theta_1-\theta_2) &= \langle \text{vac}|\mathcal{T}(0)|\theta_1,\theta_2\rangle^{out}_{\mu_1,\mu_2} \\
&\overset{j_1\neq j_2}{=} \langle \text{vac}|\mathcal{T}(0)|\theta_2,\theta_1\rangle^{in}_{\mu_2,\mu_1} = F_{\mu_2,\mu_1}(\theta_2-\theta_1)
\end{aligned}
$$



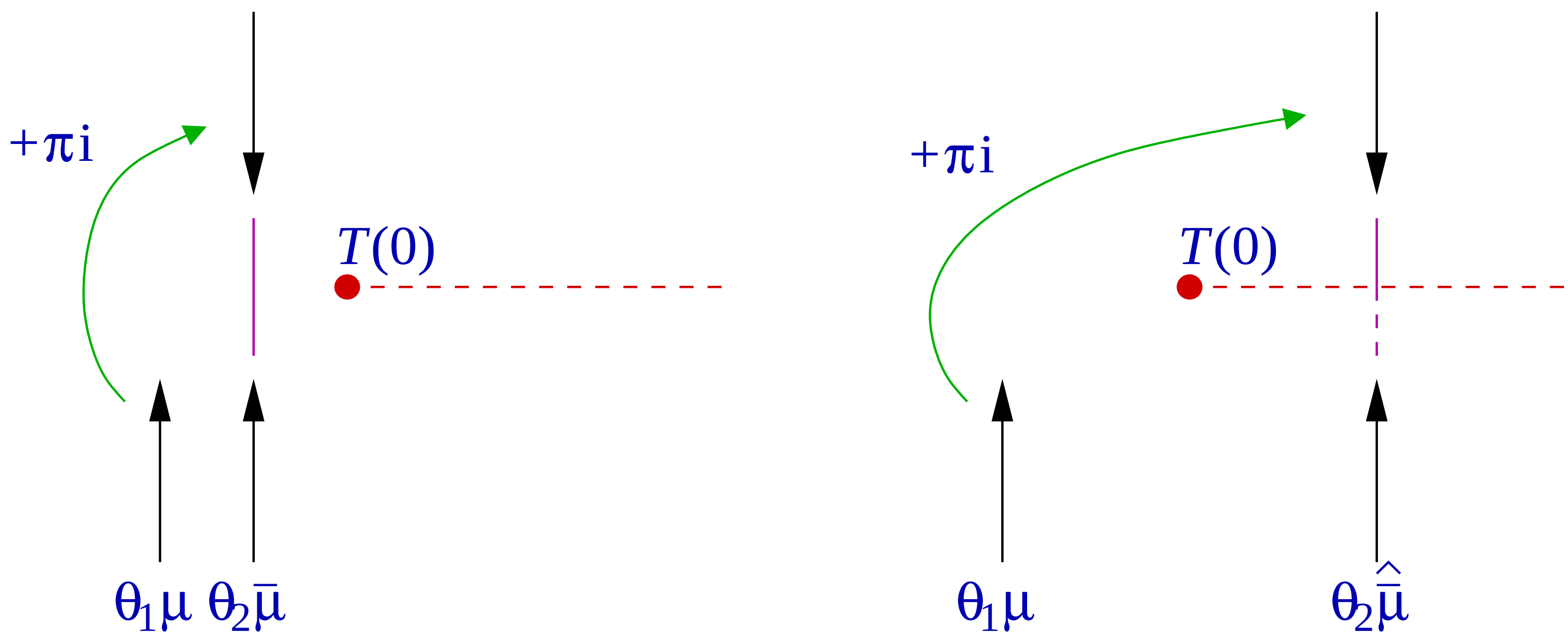
Quasi-periodicity relation (different)

$$F_{\mu_1,\mu_2}(\theta + 2\pi i) = F_{\mu_2,\hat{\mu}_1}(-\theta)\,, \quad \hat{\mu} = (\alpha, j + 1 \bmod n)$$



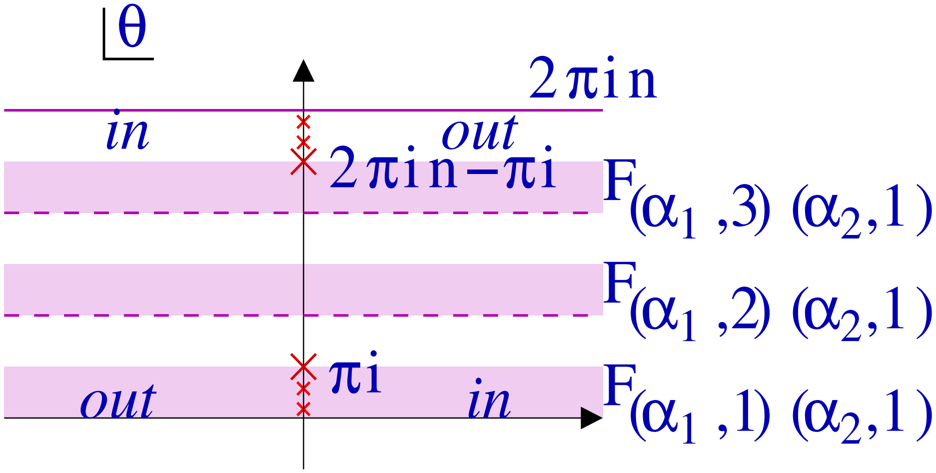
The kinematic residue equation (new)

$$-iF_{\mu_1,\mu_2}(\theta+\pi i) \sim \langle \mathcal{T} \rangle \frac{\delta_{\alpha_1,\bar{\alpha}_2}(\delta_{j_1,j_2}-\delta_{j_1+1,j_2})}{\theta}\ , \quad \bar{\alpha}_2 = \text{anti-particle of } \alpha_2$$



The structure of the two-particle form factors

Putting all that together, only $F_{(\alpha_1,1),(\alpha_2,1)}(\theta)$ matters, thanks to the relation $F_{(\alpha_1,j_1),(\alpha_2,j_2)}(\theta) = F_{(\alpha_1,1),(\alpha_2,1)}(\theta + 2\pi i(j_1 - j_2))$ for $0 \leq j_1 - j_2 \leq n-1$. It has the following analytic structure:



Correction term to the entanglement entropy

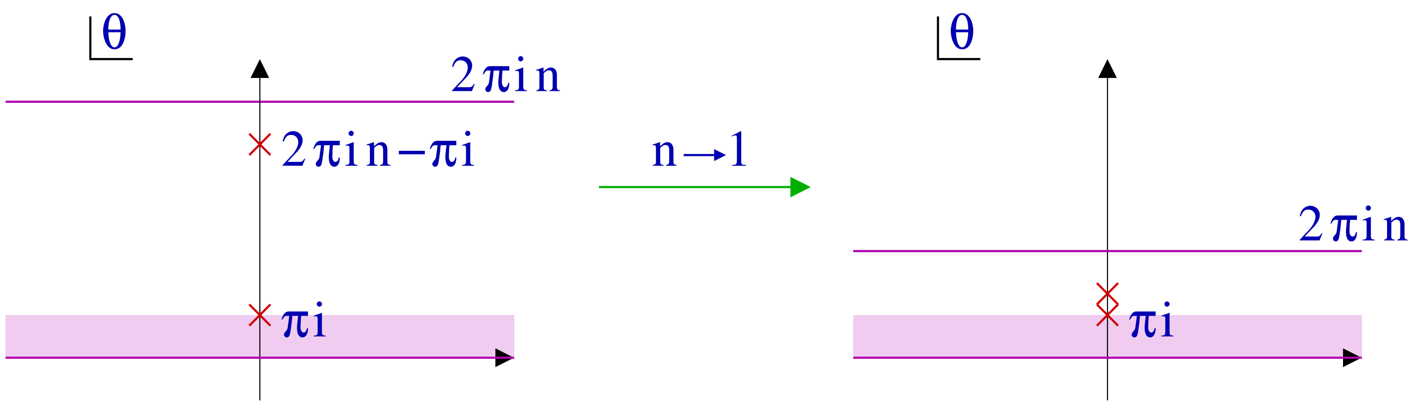
- The two-particle contribution to the entanglement entropy is

$$\frac{d}{dn}\left(\langle\mathcal{T}\rangle\frac{n}{8\pi^2}\sum_{\alpha,\beta=1}^{\ell}\int_{-\infty}^{\infty}d\theta_1 d\theta_2 f_{\alpha,\beta}(\theta_1-\theta_2,n)e^{-r(m_\alpha\cosh\theta_1+m_\beta\cosh\theta_2)}\right)_{n=1}$$

$$\langle\mathcal{T}\rangle f_{\alpha,\beta}(\theta,n)=\sum_{j=0}^{n-1}|F_{(\alpha,1),(\beta,1)}(\theta+2\pi ij)|^2$$

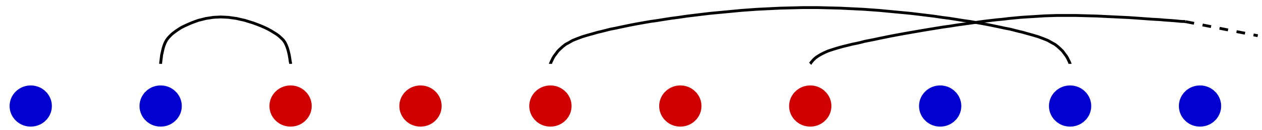
- The form factors themselves vanish like $n-1$ as $n\to 1$, because the branch-point twist field becomes the **identity field**.

- The only contribution to the entanglement entropy comes from the collision of kinematic poles at $\theta=0$, giving $\left(\frac{d}{dn}f_{\alpha,\beta}(\theta,n)\right)_{n=1}=\frac{\pi^2}{2}\delta(\theta)\delta_{\alpha,\bar{\beta}}$:



Heuristic: entanglement density and pair creations

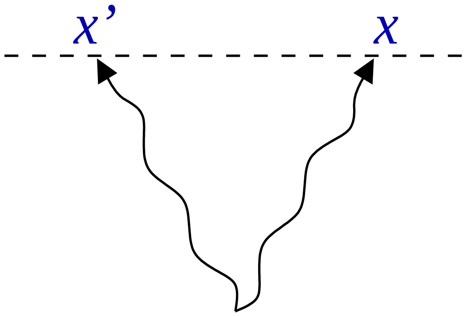
- Entanglement entropy should "count" the connections between A and $\bar{A}$:



- This should be valid for A of large extent:

$$S_A \sim \int_A dx \int_{\bar{A}} dx' s(x - x') \Rightarrow s(x) \sim -\frac{1}{2} \frac{d^2 S_{[0,x]}}{dx^2} \qquad (1)$$

- The entanglement density $s(x - x')$ should receive contributions whenever the quantum fluctuation at x is somehow correlated with that at x'.
- At large distances $x - x' \gg m^{-1}$, the main contributions should be due to particles coming from a common virtual pair created far in the past.



- The particles have to survive a time t, and the probability for this is rule by quantum uncertainty principles, $\propto e^{-Et}$ where E is the total energy of the pair, independently from the interaction between the particles.
- With velocity v, time $t = (x - x')/(2v)$ and energy $E = 2m/\sqrt{1-v^2}$, we have

$$s(x - x') \sim \sum_\alpha m_\alpha^2 \int_0^1 f(v) e^{-\frac{m_\alpha (x-x')}{v\sqrt{1-v^2}}}$$

for some $f(v)$. This has the correct behaviour $\propto e^{-2m(x-x')}$. With $f(v) = 1/(32v^3(1-v^2)^2)$, it reproduces exactly the main result.

Conclusions

- We have derived the first correction to saturation of the entanglement entropy in any two-dimensional QFT, and observed that it is very universal: it does not depend on the scattering matrix!
- This is one of the few examples where the analytic structure of matrix elements in QFT gives exact results, outside of integrable models.
- We have provided a heuristic “explanation” in terms of counting virtual pair creations far in the past – the collision of the kinematic poles seem to say that we are only considering contributions from particles coming from a common virtual pair...
- How to make this picture more precise? Can we verify the formula using other methods (there are many subtleties in the calculation)? How much of this can be generalised to higher dimensions?