Entanglement Entropy at 2D quantum critical points, topological fluids and quantum Hall fluids

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Collaborators and References

- Stefanos Papanikolaou, Kumar Raman, Benjamin Hsu, Shiying Dong, Robert G. Leigh and Sean Nowling (UIUC),
- Michael Mulligan and Eun-Ah Kim (Stanford), Joel E. Moore (University of California Berkeley), Paul Fendley (Virginia)

- S. Dong, E. Fradkin, R. G. Leigh and S. Nowling, JHEP **05**, 016 (2008).
- S. Papanikolaou, K. S. Raman, and E. Fradkin, Phys. Rev. B **76**, 224421 (2007).
- E. Fradkin and J. E. Moore, Phys. Rev. Lett. **97**, 050404 (2006).
- P. Fendley and E. Fradkin, Phys. Rev. B **72**, 024412 (2005).

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- Topological Quantum Computing?

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    Evidence for $q = e/4$ vortex.
    Shot noise @ point contact in a 2DEG (Heiblum)
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    Is the plateau at $\nu = 12/5$ a parafermion state?
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- The effective action that respects the symmetries is Abelian Chern-Simons gauge theory $U(1)_m$ for the hydrodynamic gauge field $\mathcal{A}_\mu$

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- Matching the correlation functions of the RK and Lifshitz models, one finds $\kappa = \frac{1}{8\pi}$.

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- Universal $O(1)$ term in *topological phases* in 2D

$$S = \alpha\ L - \gamma + O(L^{-1}), \qquad \text{Kitaev and Preskill, Levin and Wen}$$

α is non universal and γ depends only on topological invariants

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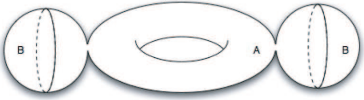

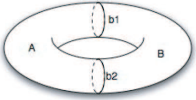

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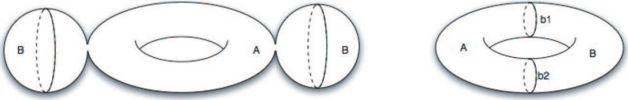

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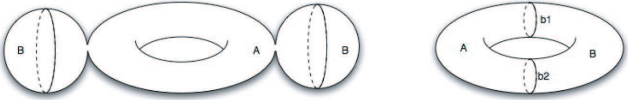

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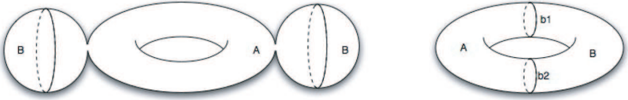

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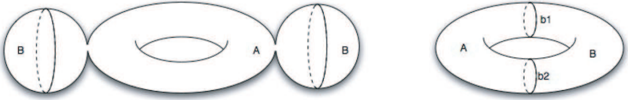

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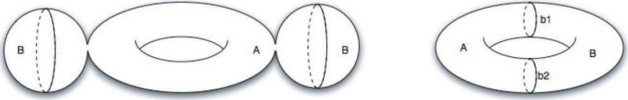

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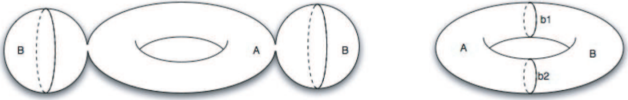

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  - Case I: There is a pair of $\hat{\alpha}$ and $\hat{\alpha}^*$ in A and in B. Each pair can fuse into the identity or into the adjoint. For $k \geq 2$, the Hilbert space on S^2 with 2 pairs of $\hat{\alpha}$ and $\hat{\alpha}^*$'s is two dimensional. The entanglement entropy depends on the quantum dimensions of the conformal block.
  - Case II: There are two $\hat{\alpha}$'s in A and two $\hat{\alpha}^*$'s in B. The entropy now depends on which channels (representation) the quasiparticles fuse and on the choice of state (conformal block).

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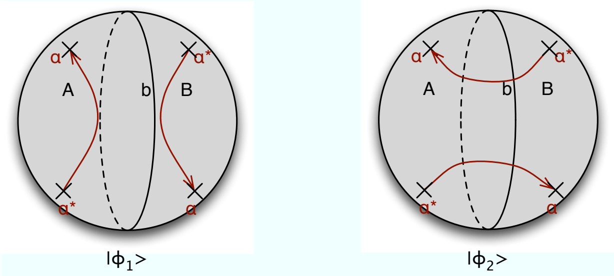

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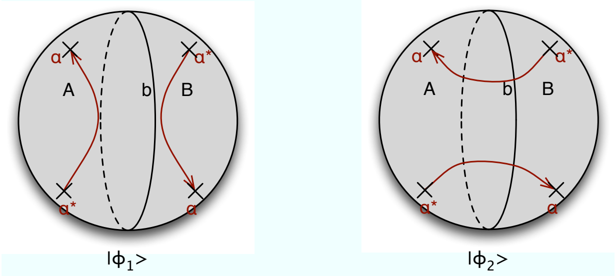

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Reduced density matrix: $\rho_A = aa^*\rho_{11} + ab^*\rho_{12} + a^*b\rho_{21} + bb^*\rho_{22}$

Entanglement Entropy: $S_A = \ln \mathcal{S}_0{}^0 - \lambda_1 \ln \lambda_1 - (d_{\hat{\alpha}}^2 - 1)\lambda_2 \ln \lambda_2$

$$\lambda_1 = \frac{|ad_{\hat{\alpha}} + b|^2}{|ad_{\hat{\alpha}} + b|^2 + (d_{\hat{\alpha}}^2 - 1)|b|^2}, \quad \lambda_2 = \frac{|b|^2}{|ad_{\hat{\alpha}} + b|^2 + (d_{\hat{\alpha}}^2 - 1)|b|^2}.$$

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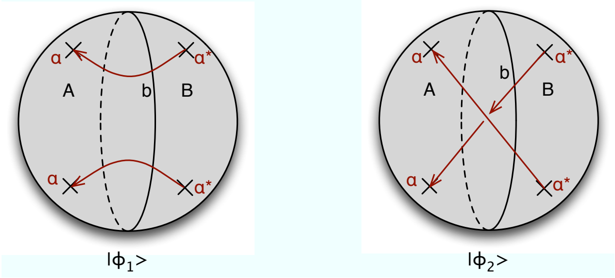

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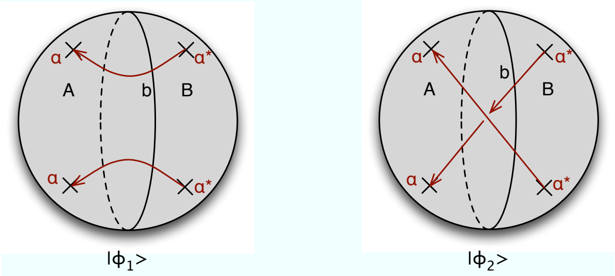

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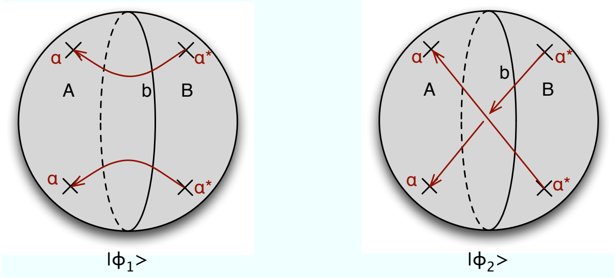

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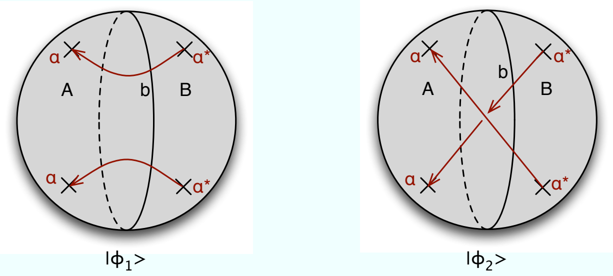

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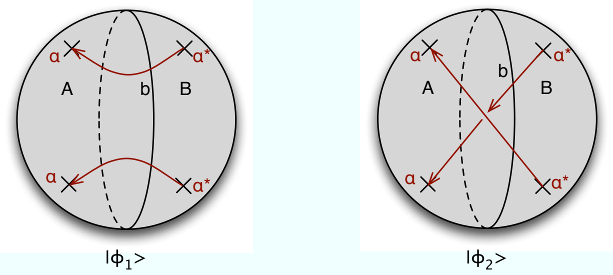

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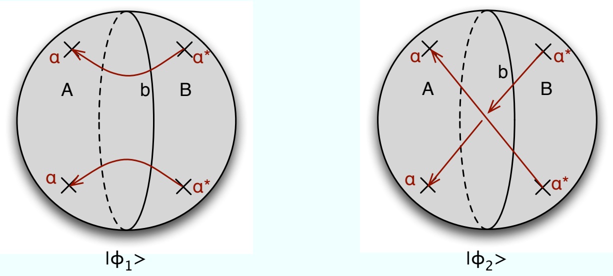

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