Topological order from quantum loops and nets

Paul Fendley

It has proved to be quite tricky to T-invariant spin models whose quasiparticles are non-abelian anyons.

Here I'll describe the simplest (so far!) such models with non-abelian topological order in the ground state.

They

1. require only interactions around a face (e.g. four-spin interactions on the square lattice)

2. are naturally expressed in terms of loops and nets simultaneously

3. possess "quantum self-duality"

Outline:

1. Quantum loops

2. Crashing the $d = \sqrt{2}$ barrier

3. Quantum nets

4. Quantum self-duality

Paper: arXiv:0804.0625 (Annals of Physics)

Essential ingredients:

Coupled Potts models: with J. Jacobsen

The Temperley-Lieb algebra and the chromatic polynomial: with V. Krushkal

Quantum Potts nets: with E. Fradkin

The Potts model and the BMW algebra: with N. Read

Why quantum loops?

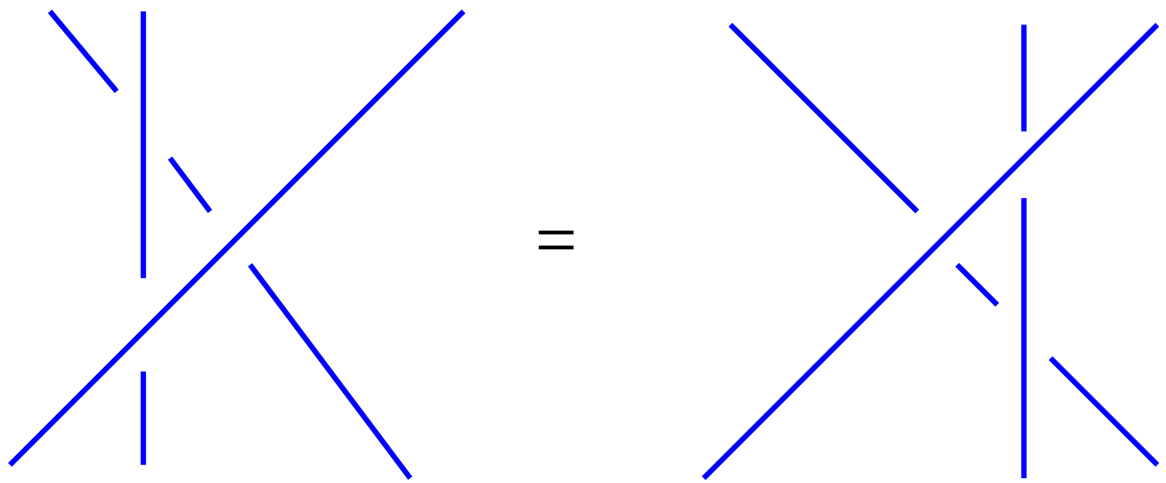
The statistics of non-abelian anyons follows from their behavior of the wavefunction under braiding of their worldlines.

Brading is a purely topological property, and so if realizable, might prove the basis for a fault-tolerant quantum computer.

Each braid describes how the wavefunction behaves under exchange of particles. With non-abelian anyons, there is a degenerate Hilbert space, so braiding can change the particles' state!

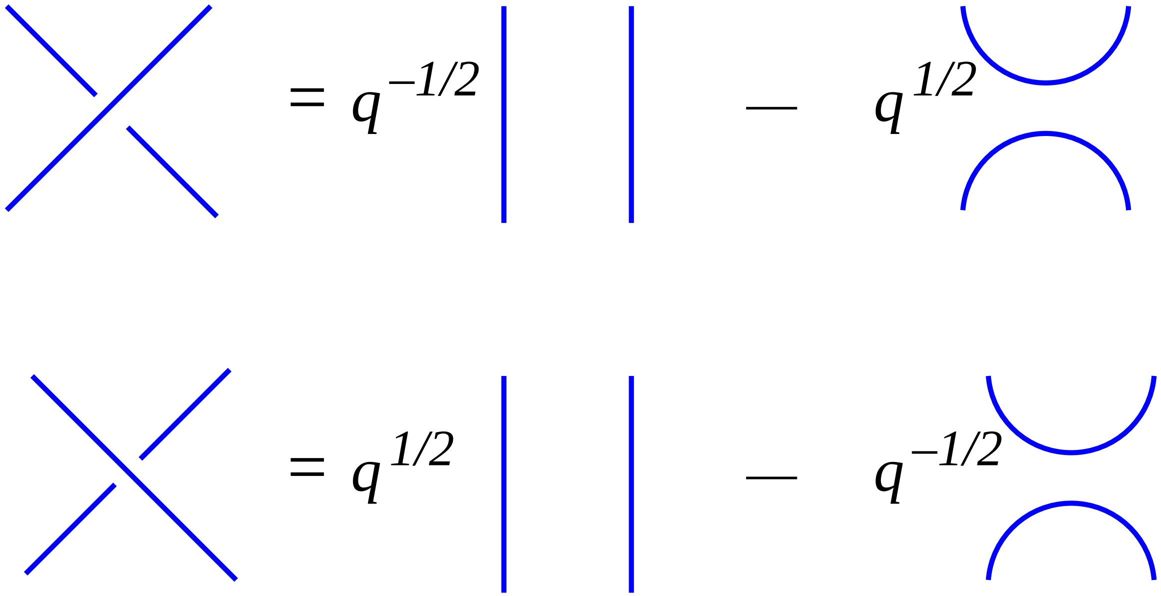
It is convenient to project the world lines of the particles onto the plane. Think of one direction as time, so that the braids become overcrossings and undercrossings

The braids must satisfy the consistency condition



which in closely related contexts is called the Yang-Baxter equation.

A simple way of satisfying the consistency conditions leads to the Jones polynomial in knot theory. Replace the braid with the linear combination



so that the lines no longer cross. q is a parameter which is a root of unity in the cases of interest: Fibonacci anyons corresponds to $q = e^{i\pi/5}$.

This gives a representation of the braid group if the resulting loops satisfy d-isotopy.

- isotopy: Configurations related by deforming without making any lines cross receive the same weight.

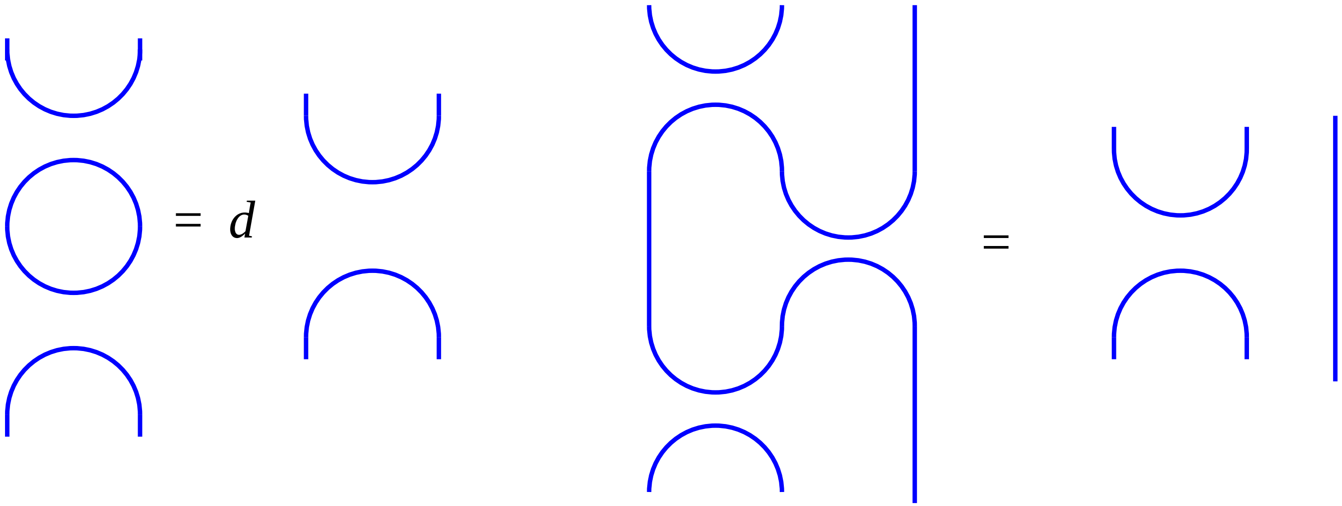
- d: A configuration with a closed loop receives weight

$$d = q + q^{-1}$$

relative to the configuration without the loop.

d is the quantum dimension of the anyon. The dimension of the $\mathcal{N}$-anyon Hilbert space grows as $d^{\mathcal{N}}$; think of it as the number of anyons created and annihilated in the loop.

If you like algebras, the proper framework to analyze this is the Temperley-Lieb algebra, which graphically is



The task is now to find a lattice model whose quasiparticles have such braiding.

The clever idea of the the quantum loop model is to use these pictures to build the model:

1. find a 2d classical loop model where loops are critical but local degrees of freedom are not (e.g. percolation)

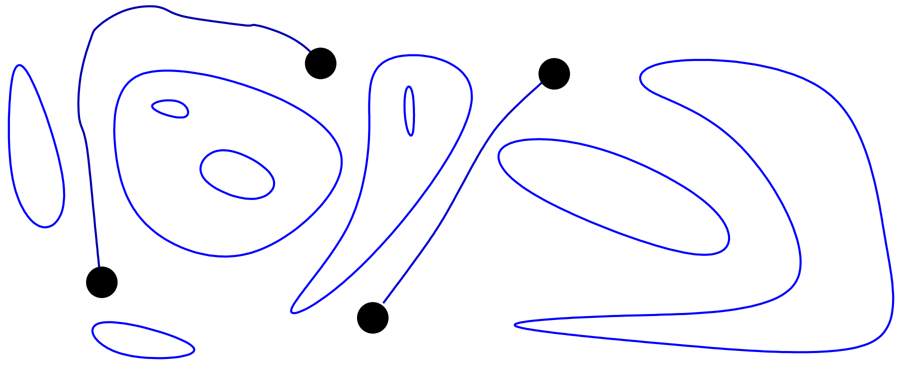
2. use each loop configuration as a basis element of the quantum Hilbert space

3. find a Hamiltonian whose ground state a sum over loop configurations with the appropriate weighting, so that

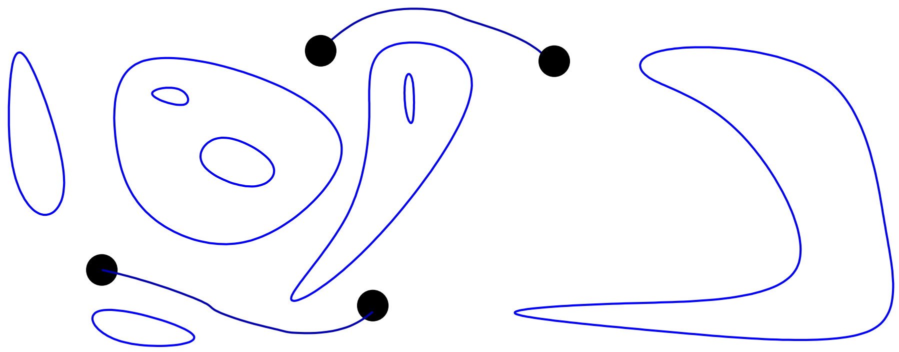
4. if you “cut” a loop, you end up with two deconfined anyonic excitations

Kitaev; Moessner and Sondhi; Freedman

The excitations with non-abelian braiding are defects in the sea of loops.



After braiding, the four quasiparticles can be attached in the other way!



The strategy here is backwards.

1. First we choose a basis for the space of states:
   completely packed loops

2. Then choose a ground-state wavefunction where each loop receives a weight d:

$$|g.s.\rangle = \sum_{\mathcal{L}} d^{n_{\mathcal{L}}} |\mathcal{L}\rangle$$

   where $n_{\mathcal{L}}$ is the number of loops in configuration $\mathcal{L}$.

3. Then we find an inner product where the loops are critical in the corresponding classical model, but local degrees of freedom are not:
   loops are not an orthonormal basis.

4. Then we find a hermitian T-invariant Hamiltonian which annihilates this ground state.

To have non-abelian braiding, the quantum loop models need to be gapped and have topological order.

Loops of all sizes must appear in the ground state. This behavior is necessary to get topological order – otherwise a length scale appears.

This length scale physically is the confinement length.

When defining the quantum loop model, it is not enough to state that loops form a basis of the Hilbert space: one must also define the inner product as well.

The naive inner product makes each loop configuration orthonormal.

This doesn't work. For $d > \sqrt{2}$, it turns out that "short loops" are favored, so there is a confinement length. For $1 < d \leq \sqrt{2}$, the model is gapless. For $d = 1$, the model is abelian.

There are two ways of crashing through the $d = \sqrt{2}$ barrier to find quantum loop models whose deconfined excitations are Fibonacci anyons:

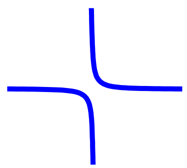
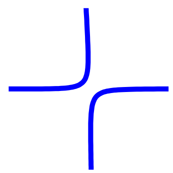
- Allow the loops to branch, so that they are not really loops, but rather nets.

- Change the inner product in the quantum-mechanical model.

  It turns out that the two are essentially the same!

In the completely packed loop model, every link of the lattice is covered by a loop.

The only degrees of freedom are therefore the two choices of how the loops avoid each other at each vertex:

$|1\rangle =$  $|\widehat{1}\rangle =$ 

There is thus a quantum two-state system at every vertex.

If we set $\langle 1|\widehat{1}\rangle = 0$, then we have the $d = \sqrt{2}$ barrier.

So instead, don't make them orthogonal!

$$\begin{pmatrix} \langle 1|1\rangle & \langle 1|\widehat{1}\rangle \\ \langle\widehat{1}|1\rangle & \langle\widehat{1}|\widehat{1}\rangle \end{pmatrix} = \begin{pmatrix} 1 & \lambda \\ \lambda^* & 1 \end{pmatrix}$$

For this to be positive definite, $|\lambda| < 1$.

Keep the ground state

$$|\Psi\rangle = \sum_{\mathcal{L}} d^{n_{\mathcal{L}}} |\mathcal{L}\rangle$$

so that now

$$\langle\Psi|\Psi\rangle = \sum_{\mathcal{L}} \sum_{\mathcal{M}} d^{n_{\mathcal{L}}+n_{\mathcal{M}}} \lambda^{n_X}$$

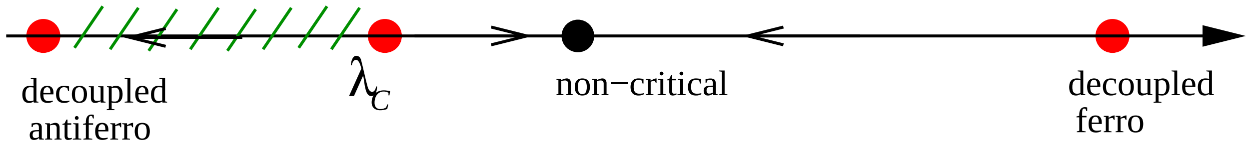
is a sum over two flavors of loops $\mathcal{L}$ and $\mathcal{M}$, which are different at n_X vertices.

Correlators in the quantum ground state are the same as those in the classical two-flavor loop model with partition function $Z = \langle\Psi|\Psi\rangle$.

Good news #1:

The corresponding classical loop model with $d = 2\cos(\pi/(k+2))$ is critical when $\lambda < \lambda_c$, where

$$\lambda_c = -\sqrt{2}\sin\left(\frac{\pi(k-2)}{4(k+2)}\right)$$



Fendley and Jacobsen

Moreover, correlators of local operators are exponentially decaying for $\lambda < \lambda_c$.

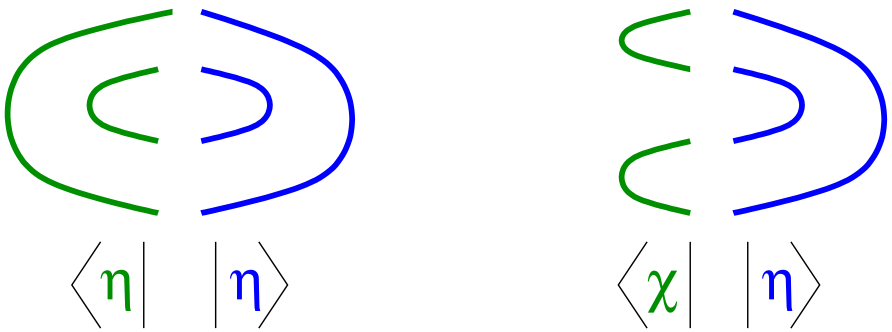
The ground state of the quantum model therefore is a sum over loops of all length scales.

The excitations should be deconfined!

Good news #2:

This inner product has nice topological properties.

Consider two four-anyon states with inner products:



$|\eta\rangle$ and $|\chi\rangle$ are topologically equivalent to $|1\rangle$ and $|\widehat{1}\rangle$, and $\langle\chi|\eta\rangle$ is topologically equivalent to a single loop. Thus we indeed want $\langle\widehat{1}|1\rangle \neq 0$.

In fact, maybe

$$\begin{aligned}\lambda &= \frac{\langle\widehat{1}|1\rangle}{\sqrt{\langle 1|1\rangle\,\langle\widehat{1}|\widehat{1}\rangle}} = \frac{\langle\chi|\eta\rangle}{\sqrt{\langle\chi|\chi\rangle\,\langle\eta|\eta\rangle}} \\ &= \pm\frac{1}{d}\end{aligned}$$

???

Good news #1 (and a careful study of amplitudes in the $SU(2)_2$ TQFT) means we should choose λ negative.

Setting $\lambda = -1/d$ leads to...

Good news #3:

Loops are nets!

Two natural orthonormal bases:

- $(|0\rangle, |1\rangle)$, where

$$|0\rangle = \frac{1}{\sqrt{d^2-1}}\left(d|\widehat{1}\rangle + |1\rangle\right)$$

- $(|\widehat{0}\rangle, |\widehat{1}\rangle)$, where

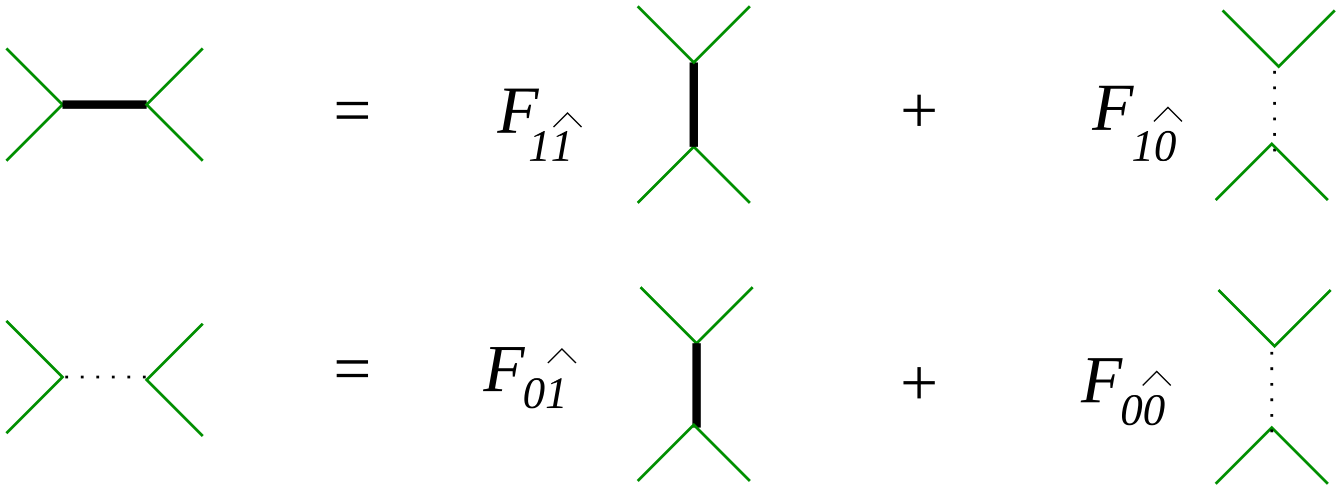
$$|\widehat{0}\rangle = \frac{1}{\sqrt{d^2-1}}\left(d|1\rangle + |\widehat{1}\rangle\right)$$

This indeed yields $\langle 0|1\rangle = \langle\widehat{0}|\widehat{1}\rangle = 0$ and $\langle 1|1\rangle = \langle\widehat{1}|\widehat{1}\rangle = 1$.

The unitary transformation relating the two bases is

$$F = \begin{pmatrix} \langle\widehat{0}|0\rangle & \langle\widehat{0}|1\rangle \\ \langle\widehat{1}|0\rangle & \langle\widehat{1}|1\rangle \end{pmatrix} = \frac{1}{d}\begin{pmatrix} 1 & \sqrt{d^2-1} \\ \sqrt{d^2-1} & -1 \end{pmatrix}$$

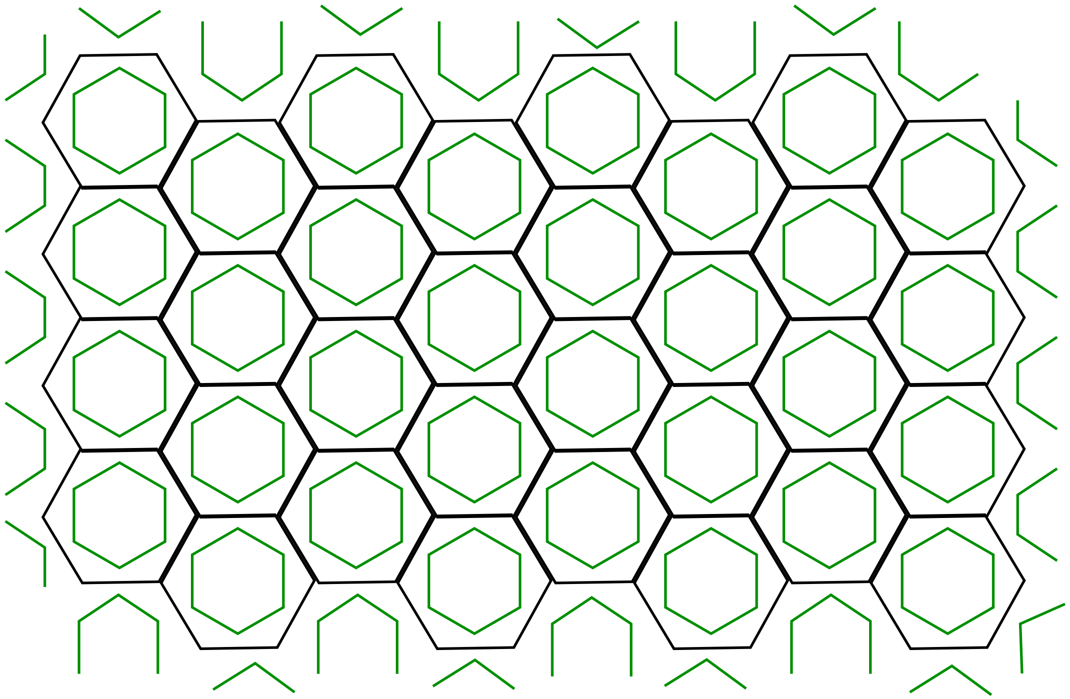
This F is the fusion matrix for anyons from quantum loops!



When lines meet at a vertex, they fuse to one of two states:

$$\frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1$$

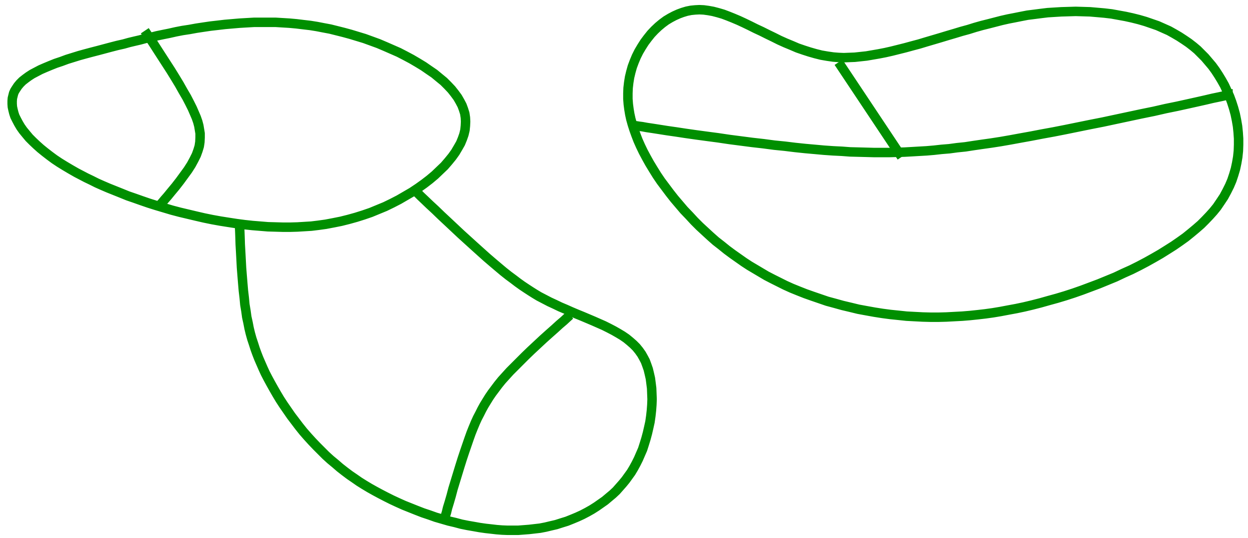
This suggests that we represent the state $|1\rangle$ as a filled link on the net lattice,
e.g. if all vertices are in state $|1\rangle$:



Vertices of the loop lattice correspond to edges of the net lattice, so loops on Kagome correspond to nets on the honeycomb.

I call these nets because when the ground state $|\Psi\rangle$ is written in this orthonormal basis, there cannot be a single state $|1\rangle$ touching a vertex!

States which do contribute to $|\Psi\rangle$ look like



The weight of each loop configuration in the ground state is still $d^{n_{\mathcal{L}}}$.

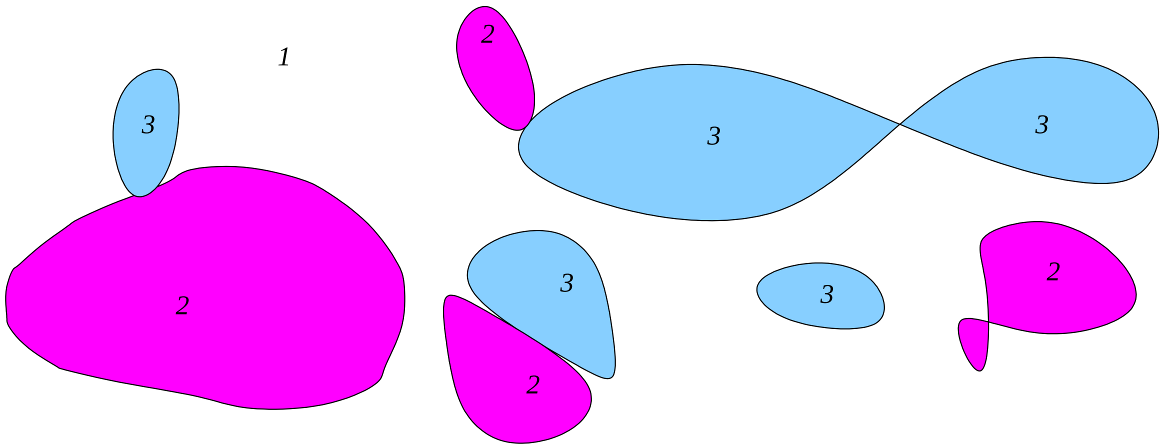
Going to the orthonormal basis gives the weight of each net $|N\rangle$ to be

$$\langle N|\Psi\rangle = \left(\frac{1}{\sqrt{d^2-1}}\right)^{L_N} \chi_{\widehat{N}}(d^2)$$

where $\chi_{\widehat{N}}(d^2)$ is the chromatic polynomial, and L_N is the length of the net (the number of links covered).

In the Fibonacci case, this is almost the same as the ground state of Levin and Wen's exactly solvable string-net model.

The chromatic polynomial only depends on the topology of N. When Q is an integer, $\chi(Q)$ is the number of ways of coloring each region with Q colors such that adjacent regions have different colors.



Clasically, think of these loops as domain walls in the low-temperature expansion of the Q-state Potts model.

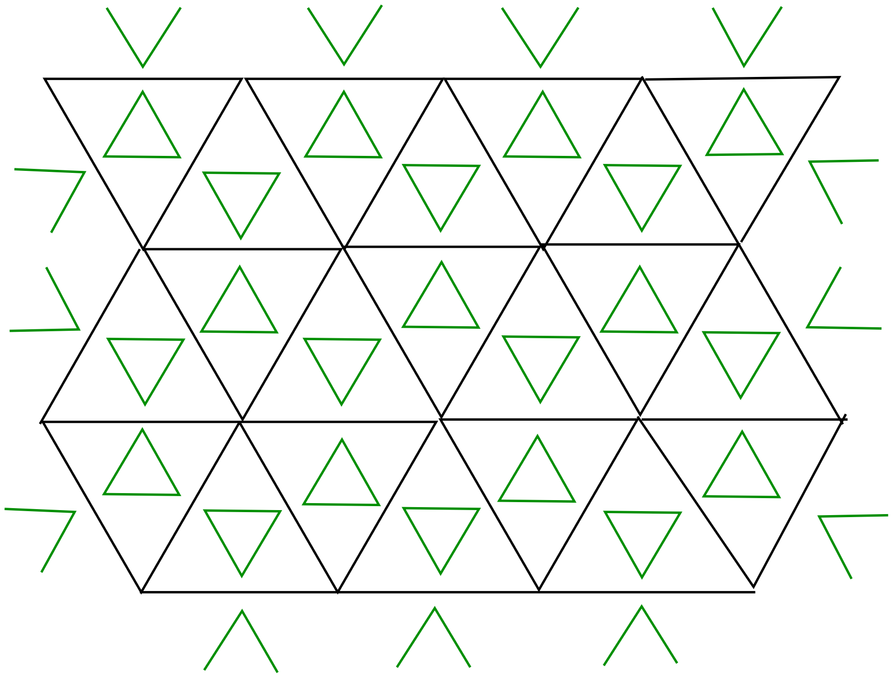
Good news #4:

Quantum self-duality means that on the square lattice, only four-spin interactions are required in the Hamiltonian!

In Levin and Wen's exactly solvable "string-net" models, 12-spin interactions are required.

Instead of writing the ground state $|\Psi\rangle$ in terms of nets, can also write them in terms of dual nets $|D\rangle$, in the $(\widehat{0}, \widehat{1})$ basis.

The dual nets live on the links of the dual of the net lattice, e.g. for loops on Kagomé when all vertices are in state $|\widehat{1}\rangle$:



The weight of each dual net $|D\rangle$ in the ground state is

$$\langle D|\Psi\rangle = \left(\frac{1}{\sqrt{d^2-1}}\right)^{L_D} \chi_{\widehat{D}}(d^2)$$

This is the same ground state $|\Psi\rangle$ in a new basis!

This quantum self-duality is highly non-obvious, and extremely useful.

A Hamiltonian H with Ψ a ground state can be found simply by demanding that H annihilate all states which are not nets and annihilate all states which are not dual nets.

For the square lattice:

$$H = \sum_{+} [P_1 P_0 P_0 P_0 + \text{rotations}] + \sum_{\square} [P_{\hat{1}} P_{\hat{0}} P_{\hat{0}} P_{\hat{0}} + \text{rotations}]$$

where P_i projects onto the states $|i\rangle$, and $P_{\hat{i}} = F P_i F$.

This is very much a non-abelian version of Kitaev's toric code.

Conclusions

- With the right inner product, we can crash the $d = \sqrt{2}$ barrier and find T-invariant lattice models with e.g. Fibonacci anyons.
- With the right inner product, loops and nets are equivalent.
- With the right inner product, the models exhibit quantum self-duality. The Hamiltonian needs involve only four-spin interactions.
- However, because $d > 1$, here the ground state should support non-abelian anyons!
- Pound your head on the wall enough, and sometimes the wall cracks before your head...