2nd INSTANS Conference on

**Exact Results in Low-Dimensional Quantum Systems**

8 – 12 September 2008

Galileo Galilei Institute of Theoretical Physics, Florence

# FROM PARTICLES TO FIELDS IN TWO-DIMENSIONAL QFT

Gesualdo Delfino

SISSA-Trieste

Based on :

GD, G.Niccoli, Nucl.Phys.B '08

GD, Nucl.Phys.B in press

Many of the exact results in low-dimensional systems come from exact solutions in $1+1$–dimensional QFT

| at criticality | away from criticality |
|---|---|
| Conformal Field Theory | Integrable Field Theory |
| operator algebra | particle solution |
| Virasoro representations | S−matrix |
| no direct notion of particles | no direct notion of fields |

Recovering the space of fields from the $S$-matrix is important in principle and in practice

**Space of conformal fields** (BPZ '84)

Correspondence with the lowest weight representations of the Virasoro algebra. The space of fields splits into **families**, each one consisting of a **primary** $\Phi_0$ and infinitely many **descendants**

$$\Phi(x) = L_{-i_1} \dots L_{-i_I} \bar{L}_{-j_1} \dots \bar{L}_{-j_J} \, \Phi_0(x)$$

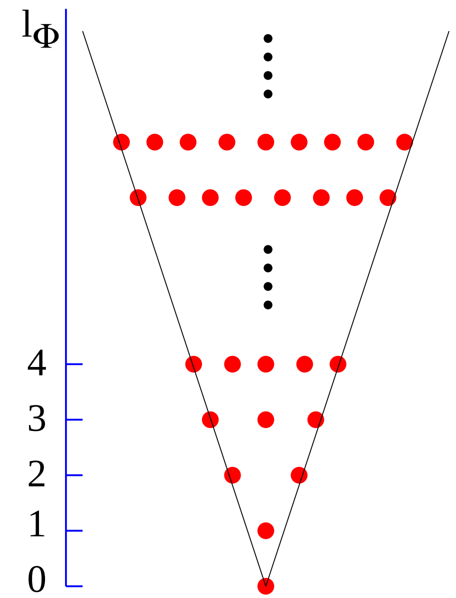
$$0 < i_1 \leq i_2 \leq \dots \leq i_I \qquad 0 < j_1 \leq j_2 \leq \dots \leq j_J$$

$$(l_\Phi, \bar{l}_\Phi) = \left( \sum_{n=1}^{I} i_n \, , \sum_{n=1}^{J} j_n \right)$$ left and right level

$$(\Delta_\Phi, \bar{\Delta}_\Phi) = (\Delta_{\Phi_0} + l_\Phi, \bar{\Delta}_{\Phi_0} + \bar{l}_\Phi)$$ conformal dimensions

$X_\Phi = \Delta_\Phi + \bar{\Delta}_\Phi$ scaling dimension

$s_\Phi = \Delta_\Phi - \bar{\Delta}_\Phi$ spin




The actual 'filling' of each level depends on the existence of null vectors

The role of internal symmetries is best understood using modular invariance (Cardy '86)

## Two-dimensional massive field theories

Since a massive theory is a perturbation of a CFT, it should have the same field content (up to internal symmetry breaking effects).

How to see this now that conformal symmetry is gone?

**Integrable theories.** The $S$-matrix is known exactly

$$|A_a(\theta_1)A_b(\theta_2)\rangle_{in} = S_{ab}^{cd}(\theta_1-\theta_2)|A_c(\theta_1)A_d(\theta_2)\rangle_{out}$$

Field content and correlation functions must be determined from particle dynamics

I'll consider a large class of theories (includes sine-Gordon) characterized by

1) abelian internal symmetry group $G$

2) asymptotic diagonality, i.e. $\lim_{\theta\to\pm\infty} S_{ab}^{cd}(\theta) = e^{\pm 2i\pi\alpha_{ab}}\delta_a^c\delta_b^d$

## **Form factor equations** (Karowski, Weisz '78; Smirnov '80s)

$$F^{\Phi}_{a_1\ldots a_n}(\theta_1,\ldots,\theta_n) = \langle 0|\Phi(0)|A_{a_1}(\theta_1)\ldots A_{a_n}(\theta_n)\rangle \qquad (= 0 \quad \text{if} \quad \textstyle\sum_{j=1}^n C_{a_j} \neq -C_\Phi)$$

$$F^{\Phi}_{a_1\ldots a_n}(\theta_1+\Lambda,\ldots,\theta_n+\Lambda) = e^{s_\Phi \Lambda} F^{\Phi}_{a_1\ldots a_n}(\theta_1,\ldots,\theta_n) \tag{1}$$

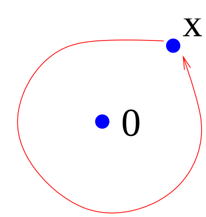
$$F^{\Phi}_{\ldots a_i a_{i+1}\ldots}(\ldots,\theta_i,\theta_{i+1},\ldots) = S^{b_i b_{i+1}}_{a_i a_{i+1}}(\theta_i-\theta_{i+1})\, F^{\Phi}_{\ldots b_{i+1} b_i\ldots}(\ldots,\theta_{i+1},\theta_i,\ldots) \tag{2}$$

$$F^{\Phi}_{a_1\ldots a_n}(\theta_1+2i\pi,\theta_2,\ldots,\theta_n) = e^{-2i\pi\gamma_{\Phi,a_1}} F^{\Phi}_{a_2\ldots a_n a_1}(\theta_2,\ldots,\theta_n,\theta_1) \tag{3}$$

$$\mathrm{Res}_{\theta_a-\theta_b = iu^c_{ab}}\, F^{\Phi}_{a b a_1\ldots a_n}(\theta_a,\theta_b,\theta_1,\ldots,\theta_n) = i\Gamma^c_{ab}\, F^{\Phi}_{c a_1\ldots a_n}(\theta_c,\theta_1,\ldots,\theta_n) \tag{4}$$

$$\mathrm{Res}_{\theta'=\theta+i\pi}\, F^{\Phi}_{\bar{a} a a_1\ldots a_n}(\theta',\theta,\theta_1,\ldots,\theta_n) = i\left[\delta^{b_1}_{a_1}\ldots\delta^{b_n}_{a_n} - e^{2i\pi\gamma_{\Phi,a}} S^{b_1\ldots b_n}_{a_1\ldots a_n}(\theta|\theta_1,\ldots,\theta_n)\right] F^{\Phi}_{b_1\ldots b_n}(\theta_1,\ldots,\theta_n) \tag{5}$$

$\gamma_{\Phi,a}$ semi-locality index




$$\langle\cdots\Phi_1(z e^{2i\pi},\bar{z}^{-2i\pi})\Phi_2(0)\cdots\rangle = e^{2i\pi\gamma_{\Phi_1,\Phi_2}}\langle\cdots\Phi_1(z,\bar{z})\Phi_2(0)\cdots\rangle$$

$\gamma_{\Phi,a} = \gamma_{\Phi,\varphi_a}$ $\qquad C_{\varphi_a} = C_a$ $\qquad s_{\varphi_a} = 0$

The form factor equations contain only the spin and charge data of the field
$\longrightarrow$ **infinitely many solutions**. Look at asymptotic behavior

## The operators $\Lambda_a$ (GD '08)

$$\lim_{\theta_n\to+\infty} F^{\Phi}_{a_1\ldots a_n}(\theta_1,\ldots,\theta_{n-1},\theta_n+2i\pi) = \prod_{j=1}^{n-1} e^{-2i\pi\alpha_{a_j a_n}} \lim_{\theta_n\to+\infty} F^{\Phi}_{a_n a_1\ldots a_{n-1}}(\theta_n+2i\pi,\theta_1,\ldots,\theta_{n-1})$$

$$= e^{-2i\pi\gamma_{\Phi,a_n}} \prod_{j=1}^{n-1} e^{-2i\pi\alpha_{a_j a_n}} \lim_{\theta_n\to+\infty} F^{\Phi}_{a_1\ldots a_n}(\theta_1,\ldots,\theta_n)$$

$$\Longrightarrow \qquad F^{\Phi}_{a_1\ldots a_n}(\theta_1,\ldots,\theta_n) = f^{\Phi,a_n}_{a_1\ldots a_{n-1}}(\theta_1,\ldots,\theta_{n-1})\, e^{y_{\Phi,a_n}\theta_n}\,, \qquad \theta_n\to+\infty\,, \qquad n>1$$

$$y_{\Phi,a_n} = -\gamma_{\Phi,a_n} - \sum_{j=1}^{n-1}\alpha_{a_j a_n} + k_{\Phi,a_n} \qquad\qquad k_{\Phi,a_n}\in\mathbf{Z}$$

$f^{\Phi,a_n}_{a_1\ldots a_{n-1}}(\theta_1,\ldots,\theta_{n-1})$ satisfy the equations for $F^{\Phi_{(a_n)}}_{a_1\ldots a_{n-1}}(\theta_1,\ldots,\theta_{n-1})$ with

$$s_{\Phi_{(a)}} = s_\Phi - y_{\Phi,a} \qquad\qquad \gamma_{\Phi_{(a)},b} = \gamma_{\Phi,b} - \alpha_{ab} \qquad\qquad C_{\Phi_{(a)}} = C_\Phi + C_a$$

$$\Lambda_a F^{\Phi}_{a_1\ldots a_m a}(\theta_1,\ldots,\theta_m,\theta) \equiv \lim_{\theta\to+\infty} e^{-y_{\Phi,a}\theta} F^{\Phi}_{a_1\ldots a_m a}(\theta_1,\ldots,\theta_m,\theta) = F^{\Phi_{(a)}}_{a_1\ldots a_m}(\theta_1,\ldots,\theta_m), \quad m\geq 0$$

$$\Lambda_a\Phi = \Phi_{(a)}$$

Introduce a "dual charge" $\tilde{C}$ (defined $mod\,\mathbf{Z}$), so that

$$\gamma_{\Phi_1,\Phi_2} = C_{\Phi_1}\tilde{C}_{\Phi_2} + \tilde{C}_{\Phi_1}C_{\Phi_2}$$

$$s_\Phi = C_\Phi \tilde{C}_\Phi + n_\Phi \qquad\qquad n_\Phi \in \mathbf{Z}$$

$$\Theta \equiv T_\mu^\mu \qquad (C_\Theta = \tilde{C}_\Theta = s_\Theta = 0) \qquad\qquad \Theta_a \equiv \Theta_{(a)} = \Lambda_a \Theta$$

$\alpha_{ab} = -\gamma_{\Theta_a,b} = -\tilde{C}_{\Theta_a}C_b$ $S$-matrix determines values of $\tilde{C}$

$s_{\Theta_a} = -\alpha_{aa} + n_{\Theta_a}$ $\Theta_a$ parafermionic if $\alpha_{aa} \notin \mathbf{Z}/2$ (cfr Smirnov '90)

$$y_{\Phi,a} = s_\Phi - s_{\Phi_{(a)}} = -[C_a\tilde{C}_\Phi + (C_\Phi + C_a)\tilde{C}_{\Theta_a}] + k_{\Phi,a} \qquad\qquad k_{\Phi,a} = n_\Phi - n_{\Phi_{(a)}}$$

$\mathcal{F} =$ space of fields

$= \text{space of form factor solutions} = \bigoplus_{C,\tilde{C},n} \mathcal{F}^n_{C,\tilde{C}}$ $\qquad \dim \mathcal{F}^n_{C,\tilde{C}} = \infty$

$$\Lambda_a \; : \; \mathcal{F}^n_{C,\tilde{C}} \longrightarrow \mathcal{F}^{n-k_a}_{C+C_a,\,\tilde{C}+\tilde{C}_{\Theta_a}\,(mod\,\mathbf{Z})}$$

$$\mathcal{F}^n_{C,\tilde{C}} = \bigoplus_{k_a \in \mathbf{Z}} \mathcal{F}^{n,k_a}_{C,\tilde{C}}$$

- **Lower bound on asymptotic behavior**

$$\langle\Phi_{(a)}\rangle = \Lambda_a F_a^\Phi(\theta) = F_a^\Phi(0) \lim_{\theta\to+\infty} e^{(-y_{\Phi,a}+s_\Phi)\theta} = F_a^\Phi(0) \lim_{\theta\to+\infty} e^{s_{\Phi_{(a)}}\theta}$$

$$\langle\Phi_{(a)}\rangle < \infty \quad \Longrightarrow \quad s_{\Phi_{(a)}} \leq 0 \quad \text{i.e.}$$

$$y_{\Phi,a} \geq s_\Phi \qquad \text{if} \qquad F_a^\Phi(0) \neq 0$$

– $y_{\Phi,a}$ takes integer-spaced values and is bounded from below, like the conformal level

– the form factor eqs (Lorentz and analytic properties) imply a semi-infinite integer gradation for the space of fields

- **Upper bound on asymptotic behavior** (GD, Mussardo '95)

From the spectral representation of correlation functions in reflection-positive theories

$$y_{\Phi,a} \leq X_\Phi/2 \qquad \text{if} \qquad s_\Phi = 0$$

$$\Longrightarrow \qquad 0 \leq y_{\Phi,a} < 1 \qquad \text{for } \Phi \text{ scalar, relevant with } F_a^\Phi \neq 0$$

– since $y_{\Phi,a}$ takes integer-spaced values, only the minimal value is selected

– explains the observed effectiveness of the upper bound

**Eigenfields of $\Lambda_a$:** $$\Lambda_a \phi_a = \lambda_{\phi_a} \bar{\phi}_a$$

$\bar{\phi}_a$ $C, \tilde{C}$-conjugate of $\phi_a$, $s_{\phi_a} = s_{\bar{\phi}_a} = 0$

$\longrightarrow$ $y_{\phi_a,a} = 0$ minimal asymptotic behavior allowed for scalar fields

$$\phi_a \in \begin{cases} \mathcal{F}^{0,0}_{-C_a,0} \bigoplus \mathcal{F}^{0,0}_{0,-\tilde{C}_{\ominus a}} & \text{if} \quad C_a \neq 0 \\ \bigoplus_C \mathcal{F}^{0,0}_{C,0} \bigoplus_{\tilde{C}} \mathcal{F}^{0,0}_{0,\tilde{C}} & \text{if} \quad C_a = 0 \end{cases} \qquad \bar{\phi}_a = \phi_{\bar{a}}$$

$$\Lambda_a F_a^{\phi_a} = \begin{cases} \lambda_{\phi_a} \langle \phi_{\bar{a}} \rangle \\ F_a^{\phi_a} \end{cases} \Longrightarrow \qquad \lambda_{\phi_a} = \frac{F_a^{\phi_a}}{\langle \phi_{\bar{a}} \rangle} \quad \text{universal number}$$

For example: $$\lim_{\theta_2 \to +\infty} F^{\phi_a}_{\bar{a}a}(\theta_1, \theta_2) = \frac{F_a^{\phi_a} F_{\bar{a}}^{\phi_{\bar{a}}}}{\langle \phi_{\bar{a}} \rangle}$$

Example of asymptotic factorization (Smirnov '90; Koubek,Mussardo '93; GD,Simonetti,Cardy '96; GD,Niccoli '05; Balog,Weisz '07)

Call $\Omega_a$ the space of solutions of the form factor equations satisfied by the eigenfield(s) $\phi_a$ (i.e. same spin=0, same $C$ and $\tilde{C}$)

| **criticality (CFT)** | **off-criticality (IFT)** |
|---|---|
| $V_\Delta = \bigoplus_{l=0}^{\infty} V_{\Delta,l}$ | $\Omega_a = \bigoplus_{k=0}^{\infty} \Omega_{a,k}$ |
| $l = \bar{l}$=level | $k \equiv k_{\Phi,a} = y_{\Phi,a}$ |
| primary $\in V_{\Delta,0}$ | $\phi_a \in \Omega_{a,0}$ |
| $(\partial\bar{\partial})^j \, V_{\Delta,l} \subseteq V_{\Delta,l+j}$ | $(\partial\bar{\partial})^j \, \Omega_{a,k} \subseteq \Omega_{a,k+j}$ |

## Natural conjectures

– Critical and off-critical field spaces are isomorphic. In particular

$$\Omega_a = \bigoplus_{\Delta \in \mathcal{D}_a} V_\Delta$$

– Simplest cases: $k = l$ $\Omega_{a,l} = \bigoplus_{\Delta \in \mathcal{D}_a} V_{\Delta,l}$

– More generally it could be $k - l$ fixed and positive for some $\Delta$ (e.g. irrelevant primaries)

– $\Omega_{a,0}$ should be spanned by primary fields. Do the basis of $\phi_a$'s and that of primaries coincide when $\dim \Omega_{a,0} > 1$ ?

**Sine-Gordon** $\mathcal{A}_{SG} = \int d^2x \, \left(\frac{1}{2}(\partial_\mu\varphi)^2 - \tau \cos\beta\varphi\right)$

Particles: soliton/antisoliton $A_\pm$

Amplitudes:

$$S_{++}^{++}(\theta) = S_0(\theta) = -\exp\left\{-i\int_0^\infty \frac{dx}{x}\frac{\sinh\frac{x}{2}\left(1-\frac{\xi}{\pi}\right)}{\sinh\frac{x\xi}{2\pi}\cosh\frac{x}{2}}\sin\frac{\theta x}{\pi}\right\}$$

$$S_{+-}^{+-}(\theta) = -\frac{\sinh\frac{\pi\theta}{\xi}}{\sinh\frac{\pi}{\xi}(\theta - i\pi)}S_0(\theta)$$

$$S_{+-}^{-+}(\theta) = -\frac{\sinh\frac{i\pi^2}{\xi}}{\sinh\frac{\pi}{\xi}(\theta - i\pi)}S_0(\theta) \qquad \xi = \frac{\pi\beta^2}{8\pi - \beta^2}$$

Asymptotic phases: $\alpha_{ab} = -\frac{1}{4}\left(1+\frac{\pi}{\xi}\right)ab = -\frac{2\pi}{\beta^2}ab \qquad a,b = \pm 1$

$$C_\pm = \pm 1 \qquad \longrightarrow \qquad \tilde{C}_{\Theta_a} = \frac{2\pi}{\beta^2}a \,(mod\,\mathbf{Z}) \qquad s_{\Theta_a} = \frac{2\pi}{\beta^2} + n_{\Theta_a}$$

Primaires: $U_{m,\nu}(x) = e^{i\left[\frac{2\pi}{\beta}m\tilde{\varphi}(x)+\nu\beta\varphi(x)\right]} \qquad C_{U_{m,\nu}} = m \qquad \tilde{C}_{U_{m,\nu}} = \nu\,(mod\,\mathbf{Z})$

Predicted eigenfields of $\Lambda_\pm$: $\phi_\pm = U_{\mp 1,0} + U_{0,\mp 2\pi/\beta^2} = e^{\mp i\frac{2\pi}{\beta}\tilde{\varphi}} + e^{\mp i\frac{2\pi}{\beta}\varphi}$

Can be checked in principle against the Lukyanov-Zamolodchikov form factors

| Model | $G$ | $a$ | $C_a$ | $\alpha_{ab}$ | $\tilde{C}_{\Theta_a}$ | $s_{\Theta_a}(mod\,\mathbf{Z})$ | $\phi_a$ |
|---|---|---|---|---|---|---|---|
| sine-Gordon | $U(1)$ | $\pm 1$ | $a$ | $-\frac{2\pi}{\beta^2}ab$ | $\frac{2\pi}{\beta^2}a$ | $\frac{2\pi}{\beta^2}$ | $e^{\mp i\frac{2\pi}{\beta}\tilde{\varphi}} + e^{\mp i\frac{2\pi}{\beta}\varphi}$ |
| minimal $\mathbf{Z}_N$ | $\mathbf{Z}_N$ | $1,\ldots,N$-1 | $a$ | $\frac{ab}{N}$ | $-\frac{a}{N}$ | $-\frac{a^2}{N}$ | $\sigma_{N-a} + \mu_{N-a}$ |
| no symmetry | $I$ | mod. dep. | 0 | 0 | 0 | 0 | all primaries |

Conjecture: the eigenfields of $\Lambda_{a_1}\ldots\Lambda_{a_m}$ are primaries with charge $-\sum_{i=1}^{m} C_{a_i}$

## Theories without internal symmetries

$C_a = 0 \quad \forall a \quad \longrightarrow \quad$ no mass degeneracy; $\quad \alpha_{ab} = 0 \quad \forall a, b$

$$S_{ab}(\theta) = \prod_{\gamma \in \mathcal{G}_{ab}} t_\gamma(\theta) \qquad\qquad t_\gamma(\theta) = \frac{\tanh \frac{1}{2}(\theta + i\pi\gamma)}{\tanh \frac{1}{2}(\theta - i\pi\gamma)}$$

$C_\Phi = \tilde{C}_\Phi = 0 \quad \forall \Phi \quad \longrightarrow \quad s_\Phi \in \mathbf{Z}, \qquad y_{\Phi,a} = k_{\Phi,a} \in \mathbf{Z} \qquad \forall \Phi$

$$\mathcal{F} = \bigoplus_{s \in \mathbf{Z},\, k \geq s} \mathcal{F}_{0,0}^{s,k} \qquad\qquad \Omega_a = \bigoplus_{k \geq 0} \mathcal{F}_{0,0}^{0,k} = \bigoplus_{\Delta} \bigoplus_{l \geq 0} V_{\Delta,l} \quad \forall a$$

**Ising model with magnetic field at $T = T_c$:**

$\mathcal{G}_{11} = \{2/3, 2/5, 1/15\} \quad \longrightarrow \quad$ 8 particles $\qquad$ (Zamolodchikov '88)

$$\lambda_\phi = \frac{F_1^\phi}{\langle \phi \rangle} = \begin{cases} -0.640902.. \\ \\ -3.70658.. \end{cases}$$

(GD, Mussardo '95; GD, Simonetti '96)

$|F_1^\sigma / \langle \sigma \rangle| = 0.6408(3), \quad |F_1^\varepsilon / \langle \varepsilon \rangle| = 3.707(7) \qquad$ (Caselle,Hasenbusch '00, lattice)

– The solutions of $\Lambda\phi = \lambda_\phi \phi$ are the primaries

## **Checking out levels: the Lee-Yang model** (GD, Niccoli '08)

$$\mathcal{A} = \mathcal{A}_{\mathcal{M}_{2,5}} + g \int d^2x\, \varphi(x)$$

**Critical theory** ($g = 0$): $c = -22/5$ simplest non-trivial fixed point

two primaries: $I$ $(\Delta = \bar{\Delta} = 0)$ $\varphi_{1,2} \equiv \varphi$ $(\Delta = \bar{\Delta} = -1/5)$

**Massive theory**: $S(\theta) = \frac{\tanh \frac{1}{2}\left(\theta + \frac{2i\pi}{3}\right)}{\tanh \frac{1}{2}\left(\theta - \frac{2i\pi}{3}\right)}$ (Cardy, Mussardo '89)

Expectation for scalar sector: $k = l$ $\mathcal{F}_{0,0}^{0,l} = \bigoplus_{\Delta = 0, -1/5} V_{\Delta, l}$

We show this and its analogue for any value of the spin by form factor counting

**Differences with previous counting works**:

– Cardy, Mussardo '90: thermal Ising (free, chiral)

– Koubek '95: Lee-Yang (no notion of levels in the massive theory)

– Smirnov '95; Jimbo, Miwa, Takeyama '03: generalization to sine-Gordon and restrictions (again no levels)

**Form factor parameterization** (Al.Zamolodchikov '91):

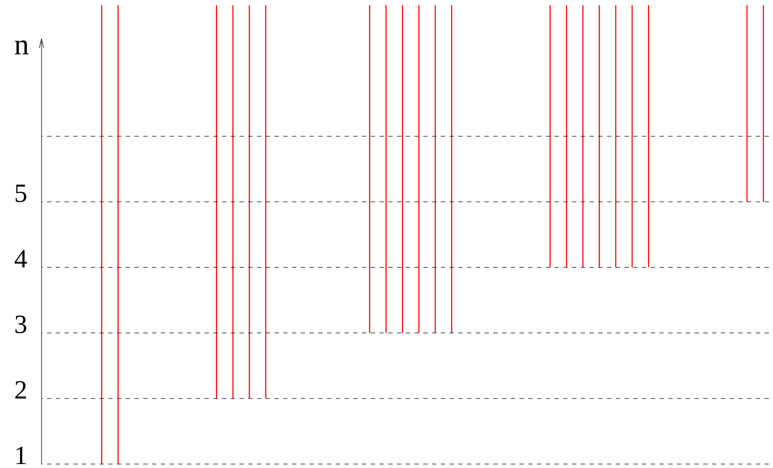
$$F_n^{\Phi}(\theta_1,\ldots,\theta_n) = \langle 0|\Phi(0)|\theta_1\ldots\theta_n\rangle = U_n^{\Phi}(\theta_1,\ldots,\theta_n)\prod_{1\le i<j\le n}\frac{F_{min}(\theta_i-\theta_j)}{\cosh\frac{\theta_i-\theta_j}{2}\left[\cosh(\theta_i-\theta_j)+\frac{1}{2}\right]}$$

$$F_{min}(\theta) = -i\sinh\frac{\theta}{2}\,\exp\left\{2\int_0^\infty \frac{dt}{t}\frac{\cosh\frac{t}{6}}{\cosh\frac{t}{2}\sinh t}\sin^2\frac{(i\pi-\theta)t}{2\pi}\right\}$$

$U_n^{\Phi}(\theta_1,\ldots,\theta_n)$ linear combinations of symmetric polynomials $\sigma_k^{(n)}(e^{\theta_1},\ldots,e^{\theta_n})$

**Counting:**

The basis of "kernel fields"




A basis for initial conditions is

$$F_n^{(a_1,..,a_{N-1}|A)}(\theta_1,...,\theta_n) = \begin{cases} 0, & n<N \\ (\sigma_N^{(N)})^A\prod_{1\le i\le N-1}(\sigma_i^{(N)})^{a_i}\prod_{1\le i<j\le N}F_{min}(\theta_i-\theta_j), & n=N \end{cases}$$

$a_1,\ldots,a_{N-1}$ non-negative integers, $A\in\mathbf{Z}$ $\qquad s=\sum_{i=1}^{N-1} i a_i + NA$

$$F_N^{(a_1,..,a_{N-1}|A)}(\theta_1+\alpha,\ldots,\theta_j+\alpha,\theta_{j+1},\ldots,\theta_n)\sim e^{y_j\alpha},\quad \alpha\to+\infty,\quad 1\le j\le N-1$$

$$y_j=\sum_{i=1}^{j-1} i a_i + j\left(\sum_{i=j}^{N-1} a_i + A + N - j\right) \qquad y\equiv \mathrm{Max}\{y_j\}_{j=\{1,\ldots,N-1\}}$$

Define the non-negative integers $\quad l = \text{Max}\{s, y, 0\} \quad \bar{l} = l - s$

Count how many solutions of type $(l, \bar{l})$ can be obtained in terms of $a_1, \ldots, a_{N-1}, A$

**Result :**

$$\begin{aligned} d(l,\bar{l}) &\equiv \#\text{ of solutions of type } (l,\bar{l}) \\ &= d_I(l) d_I(\bar{l}) + d_\varphi(l) d_\varphi(\bar{l}) \end{aligned}$$

$$d_I(l) = \sum_{N=0}^{\infty} P(N, l - N(N+1)) \qquad d_\varphi(l) = \sum_{N=0}^{\infty} P(N, l - N^2)$$

$P(N, M) = \#$ of partitions of the integer $M$ into the integers $1, 2, \ldots, N$

$d_\Phi(l) = \#$ of conformal descendants of the primary $\Phi$ at level $(l, 0)$

$\Longrightarrow \quad d(l, \bar{l})$ coincides with the total number of fields at level $(l, \bar{l})$ in $\mathcal{M}_{2,5}$

– The critical and off-critical spaces of fields are isomorphic

– The integers $l, \bar{l}$ we defined in the massive theory are the conformal levels ($l = k$ in the scalar case)

– A complete identification of the solutions exists for $l, \bar{l} \leq 7$ (GD,Niccoli '05)

## Summary

The program of deriving the field content from the $S$-matrix is in an advanced state for massive integrable theories

– A large class of these theories admits a general classification of fields in terms of charges, spin and asymptotic behavior of form factors

– The asymptotic behavior admits a lower bound and takes integer-spaced values reproducing the semi-infinite level gradation familiar from CFT

– The form factor equations allow for the introduction of operators $\Lambda_a$ mapping fields into fields

– The scalar eigenfields of the operators $\Lambda_a$ have minimal (constant) asymptotic behavior and appear to select primary fields in the massive theory

– The isomorphism of critical and off-critical field spaces has been shown for a theory emanating from a non-trivial RG fixed point

## Perspectives

– Any integrable theory

– Dropping integrability ?

– Higher dimensions ?