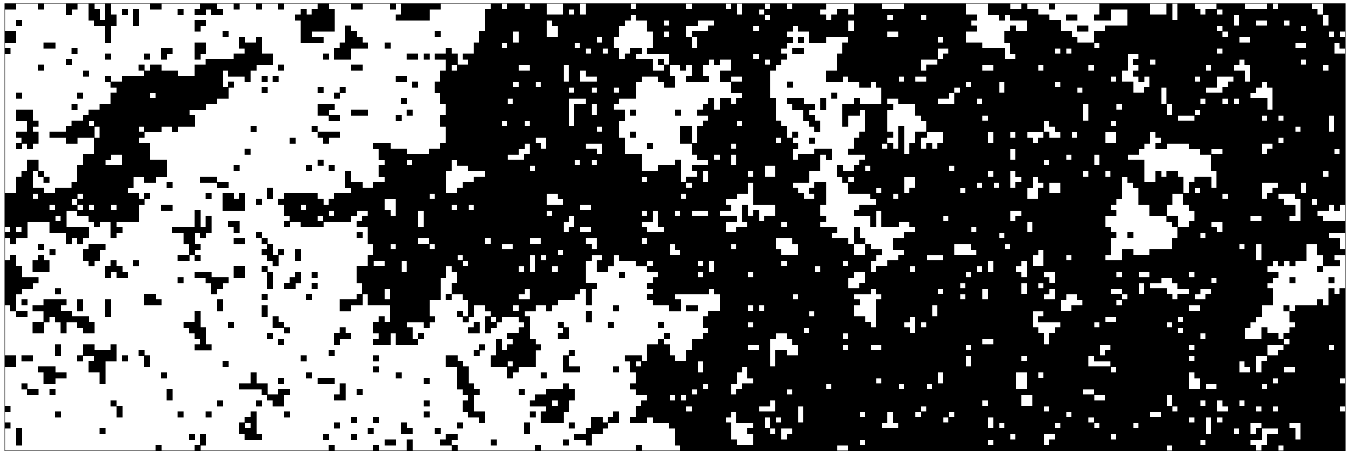
Off-critical SLEs: Massive SLE(4)

with M. Bauer, Saclay.



A sample of critical Ising configuration: SLE=interfaces

A possible approach (via field theory + probability) to the description of off-critical interfaces (SLEs) in the scaling regime near a critical point.

Many examples:

– Ising model at $T \longrightarrow T_c$;

– Percolation at $p \longrightarrow p_c$;

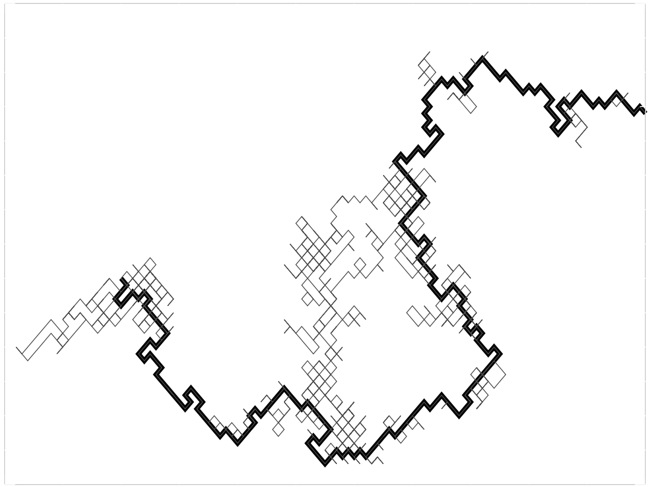
– LERW at $x \longrightarrow x_c$;

– SAW at different fugacity;

– etc...

Break conformal invariance

Loop Erased Random Walks:



Weight: $w_\gamma = \sum_{r \to \gamma} x^{|r|}$

At criticality x_c

Plan:

- Toy model: random walks
- SAW as an example
- Drift, partition function and field theory
- Massive SLE(4)
- Massive LERW

A toy model: random walks

• Let $X_N = a\sum_{j=1}^{N}\varepsilon_j$ be a (biaised) random walks of step a, with $\varepsilon_j = \pm$ with proba p, $1-p$.
Set $X_t = a(N_+ - N_-)$ and $t = a^2N = a^2(N_+ + N_-)$.
The scaling limit is for $N \to \infty$, $a \to 0$ at t fixed:
At "criticality" $p_c = 1/2$, X_t =Brownian motion: $(dX_t)^2 = dt$.

• "Near criticality" $p \to 1/2$ as $a \to 0$?
The ratio of proba of a walk at p and p_c is $(2p)^{N_+}(2(1-p))^{N_-}$, so:

$$\mathbf{E}_p[\cdots] = \mathbf{E}_{p=1/2}[M\cdots] \quad \text{with} \quad M \equiv e^S = (2p)^{N_+}(2(1-p))^{N_-}$$

To get a finite weight M, we need $p \simeq (1+a\mu)/2$ as $a \to 0$ (scaling limit).
The 'action' is then $S_t = \mu X_t - \mu^2 t/2$ and in the continuum:

$$\mathbf{E}_\mu[\cdots] = \mathbf{E}_{Br}[M_t\cdots] \quad \text{with} \quad M_t \equiv e^{S_t} = e^{\mu X_t - \mu^2 t/2} \quad \text{(a martingale)}$$

W.r.t. $\mathbf{E}_\mu$, X_t now has a drift: $dX_t = dB_t + \mu dt$.

• A martingale for some stochastic process is a time dependent random variable, say $t \longrightarrow M_t$, such that its expectation conditioned on the process up to time s is M_s, i.e. statistically conserved quantity.

• Going off-criticality amounts to weight by a martingale (which depends on the perturbation). It modifies (adds a drift to) the stochastic equation.

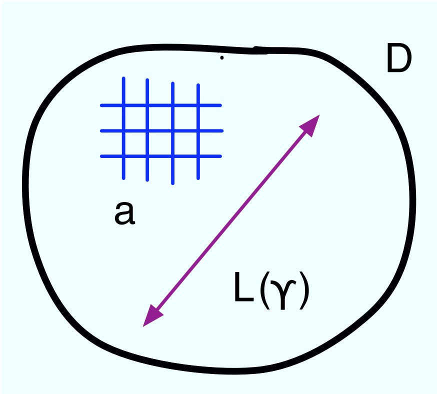
This is the approach we (try to) adapt to off-critical SLEs.......

• Grisanov theorem:
If M_t martingale for Brownian motion X_t: $M_t^{-1} dM_t = F_t dX_t$,
then w.r.t. to $\hat{\mathbf{E}}[\cdots] = \mathbf{E}_{Br}[M_t \cdots]$,
X_t satisfies $dX_t = d\hat{B}_t + F_t dt$ with $\hat{B}_t$ a $\hat{\mathbf{E}}$-Brownian motion.

In physics, this is known as the M.S.R. path integral representation.

• Other works on off-critical SLEs: Nolin-Werner, Makarov-Smirnov, ...

SAW as an example



SAW= self-avoiding walk
Weight: $w_\gamma = x^{|\gamma|}$ with $|\gamma| =$ nbr. steps.
Proba: w_γ / Z_D
with $Z_D = \sum_\gamma w_\gamma$ the partition function.

At criticality $x = x_c$, conjecturally SAW=SLE(8/3) in the continuum, $a \to 0$.

- How to take the scaling limit?

$$Z_D / Z_D^{crit.} = Z_D^{crit.-1} \sum_\gamma x^{|\gamma|} = Z_D^{crit.-1} \sum_\gamma x_c^{|\gamma|} (x/x_c)^{|\gamma|} = \mathbf{E}_{crit.}[(x/x_c)^{|\gamma|}]$$

At criticality, $|\gamma|$ is related to the macroscopic size via the fractal dimension: $|\gamma| \simeq [\ell(\gamma)/a]^{d_\kappa}$. The scaling limit is $\frac{x - x_c}{x_c} \simeq -a^{d_\kappa} \mu$ and:

$$Z_D / Z_D^{crit.} = \mathbf{E}_{crit.}[e^{-\mu L_D(\gamma)}] \quad , \quad L_D(\gamma) \simeq a^{d_\kappa} |\gamma| \equiv \text{'natural parametrisation'}$$

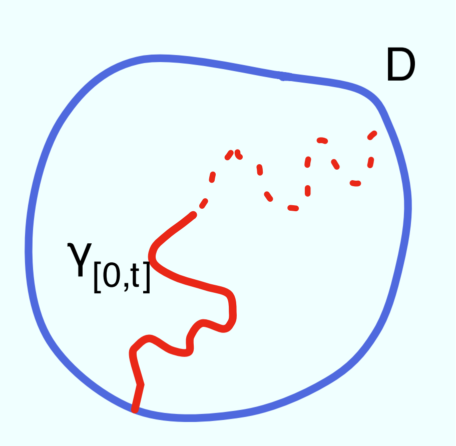
• Off-critical weight, off-critical drift and partition function.

(For SAW) the off-critical weighting is by the natural parametrization:
$\mathbf{E}_\mu[O] = Z_D^{-1}\mathbf{E}_{crit.}[e^{-\mu L_D(\gamma)}O]$ with $Z_D = \mathbf{E}_{crit.}[e^{-\mu L_D(\gamma)}]$.
If the observable O only depends on the curve up to time t ($\mathcal{F}_t$-measurable):

$$\mathbf{E}_\mu[O] = \mathbf{E}_{crit.}[M_t\, O] \quad \text{with} \quad M_t = Z_D^{-1}\,\mathbf{E}_{crit.}[e^{-\mu L_D(\gamma)}|\mathcal{F}_t]$$

Girsanov theorem tells that the off-critical drift is $M_t^{-1}dM_t$.

Since $L_D(\gamma)$ counts the number of steps, we expect an additivity property:
$L_D(\gamma_{[0,s]}) = L_D(\gamma_{[0,t]}) + L_{D\backslash\gamma_{[0,t]}}(\gamma_{[t,s]})$



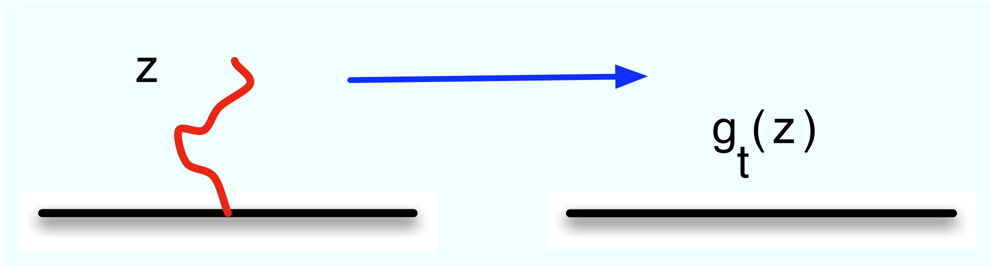
For the off-critical weighting martingale $\Longrightarrow M_t = e^{-\mu L_D(\gamma_{[0,t]})}\ \frac{Z_{D\backslash\gamma_{[0,t]}}}{Z_D}$

Ie.$\rightarrow$ ”surface energy” + ”ratio of partition function”.

Off-critical SLE and field theory

Loewner equation:

$$\frac{dg_t(z)}{dt} = \frac{2}{g_t(z) - \xi_t}$$



ξ_t: driving source.

- At criticality, ξ_t is a 1D Brownian motion, $\mathbf{E}[\xi_t^2] = \kappa t$.
- Off-criticality, ξ_t not a Brownian motion (it depends on the perturbation):

— short distance: by scaling we expect: $\lambda^{-1}\xi_{\lambda^2 t} \to \sqrt{\kappa} B_t$ as $\lambda \to 0$.

— decomposition: from above, we expect: $d\xi_t = \sqrt{\kappa} dB_t + F_t dt$

F_t off-critical drift term (perturbation dependent).

— But off-critical measure can be singular w.r.t. critical one
(cf percolation [Nolin-Werner] or infinite domain).

• From field theory, the off-critical drift reads

.
$$F_t = M_t^{-1} dM_t = \kappa \partial_{\xi_t} \log(e^{E_t(\gamma_{[0,t]})}\, \mathcal{Z}_t).$$

The off-critical weighting martingale $M_t = e^{E_t}\, \mathcal{Z}_t$ with $\mathcal{Z}_t = Z_t^{\text{off-crit.}} / Z_t^{\text{crit.}}$, the ratio of the partition functions in the domain cut along the curve.

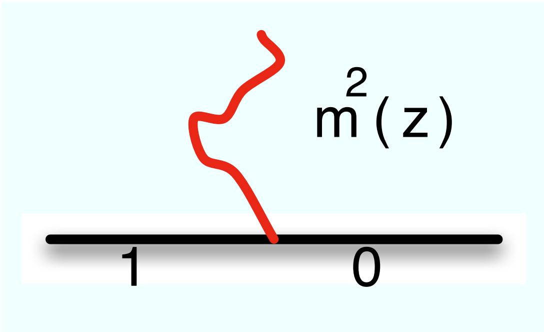
• Previous discussion follows (naively) from basic stat. mech. principles. (because the measure on curves induced by the Boltzmann weights is the ratio of partition functions).

• The measure on curves is induced (via the Loewner equation) from that on the driving source ξ_t solution of the SDE: $d\xi_t = \sqrt{\kappa} dB_t + F_t dt$.

Massive SLE(4)

To appear...

SLE(4) in a perturbed environment.



Massive bosonic free field

$$S = \int \frac{d^2x}{8\pi} [(\partial X)^2 + m^2(x) X^2]$$

with Dirichlet boundary conditions.

- SLE(4) = discontinuity curve of a massless boson (Sheffield-Schramm).
- At criticality, Gaussian free field (in upper half plane H):

$$\langle X(z) \rangle_H = \sqrt{2}\, Im(\log z) \quad , \quad \langle X(z) X(w) \rangle_H^{conn.} = G_0(z,w) = -\log \left| \frac{z-w}{z-\bar{w}} \right|^2$$

Perturbing (composite) operator: $X^2(z) = \lim_{w \to z} X(z) X(w) + \log |z-w|^2$.

- Off-critical drift $F_t = 4\partial_\xi \log Z_t^{[m]}$ with $Z_t^{[m]} = \langle e^{-\int \frac{d^2x}{8\pi} m^2(x) X^2(x)} \rangle_{H_t}$.

H_t = the cutted upper half plane.

• First order computation.

X^2 is not a scalar but $\langle X^2(z)\rangle_{H_t}$ is a (local) martingale:

$$\langle X^2(z)\rangle_{H_t} = \varphi_t(z)^2 + \log\rho_t(z), \quad \varphi_t(z) = \sqrt{2}\, Im(\log g_t(z)), \quad \rho_t(z) = \text{conf. radius}$$

To 1rst order, $Z_t^{[m]} = 1 - \int \frac{d^2x}{8\pi} m^2(x) \langle X^2(z)\rangle_{H_t} + \cdots$.

As a CFT correlation, $\varphi_t(z)$ is a martingale: $d\varphi_t(z) = \theta_t(z) dB_t$.

To 1rst order, the drift is: $F_t = -\int \frac{d^2x}{4\pi} m^2(x)\ \theta_t(x)\ \varphi_t(x) + \cdots$.

• All order computation.

Computable since the theory is Gaussian (only connected diagrams contribute), and to all orders:

$$F_t = -\int \frac{d^2x}{4\pi} m^2(x)\ \theta_t(x)\ \Phi_t^{[m]}(x) = -\int \frac{d^2x}{4\pi} m^2(x)\ \Theta_t^{[m]}(x)\ \varphi_t(x)$$

with $(-\Delta + m^2)\Phi_t^{[m]} = 0$ in H_t with Dirichlet b.c.

• Massive $\langle X(x)\rangle_{H_t}$ = (conditioned) proba for γ to be on the right of x.

It should be/It is a (massive) martingale with this drift.

This follows from Hadamard formula.

This is the way Makarov-Smirnov computed the drift.

• Perfect matching (should be true from basic stat. mech.):

$$\mathbf{E}^{[m]}[\ \langle\cdots(\text{any X correlation})\cdots\rangle_{H_t}\ ] = \langle\cdots(\text{any X correlation})\cdots\rangle_{H}$$

$\longrightarrow$ Reorganisation of the statistical sum:

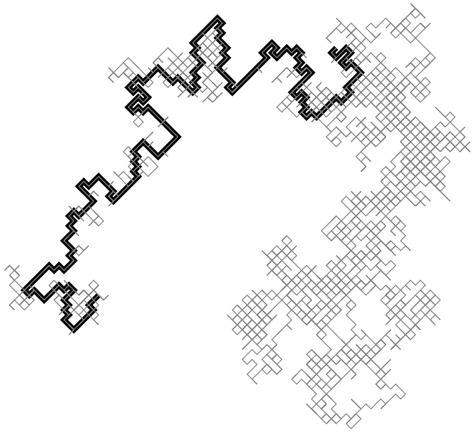
– first sum over fields at fixed interfaces

– then on possible interfaces shapes.

• The curve γ is (a.s.) the discontinuity curve.

• Massive harmonic navigator (with killing)

Massive LERW



LERW=random walk with loop erased.

Weight: $w_\gamma = \sum_{r\to\gamma} x^{|r|}$

At criticality x_c

SLE(2) = [c=-2] = symplectic fermions.

- Scaling limit $\frac{x-x_c}{x_c} = -a^2\mu$ and the partition function $Z_D = \sum_r x^{|r|}$:

$$Z_D = \mathbf{E}_{Br.}[e^{-\mu\tau_D}] = \mathbf{E}_{Br.}[\exp(-\int d^2x\ \mu(x)\ \ell_D(x))]$$

Brownian local time $\ell_D(x)$ is conformal of dimension zero.

- Fugacity perturbation is the massive perturbation:

$$S = \int \frac{d^2x}{8\pi}[(\partial\chi^+)(\partial\chi^-) + \mu(x)(\chi^+\chi^-)], \quad Z_t = \langle \psi^+\ e^{-\int \frac{d^2x}{8\pi}\mu(x)(\chi^+\chi^-)} \int \psi^- \rangle$$

$\psi^\pm = \partial\chi^\pm$ are creating/annihilating the curve.

- and a similar story.......

.... Thank You ....