Logarithmic Minimal Models, $\mathcal{W}$-Extended Fusion and Verlinde Formulas

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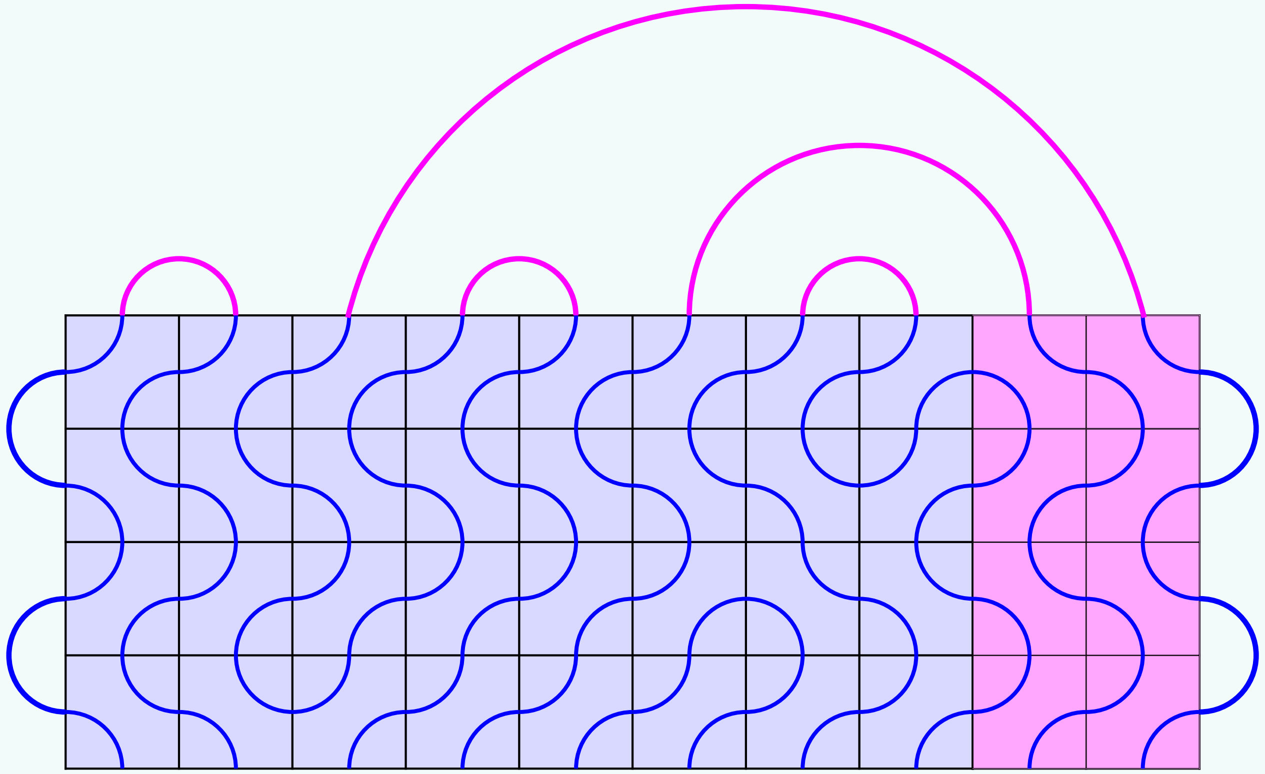

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- J.Rasmussen, PAP, *Fusion algebras of logarithmic minimal models*, J.Phys. A40 13711–33 (2007)
- PAP, J.Rasmussen, P.Ruelle, *Integrable boundary conditions and $\mathcal{W}$-extended fusion of the logarithmic minimal models $\mathcal{LM}(1,p)$*, arXiv:0803.0785, J. Phys. A (2008)
- J.Rasmussen, PAP, *$\mathcal{W}$-extended fusion of critical percolation*, arXiv:0804.4335, J. Phys. A (2008)
- J.Rasmussen, *$\mathcal{W}$-extended logarithmic minimal models*, arXiv:0805.299, Nucl. Phys. B (2008)
- PAP, J.Rasmussen, *Verlinde formula and the projective Grothendieck ring of logarithmic minimal models*, in preparation (2008)

Some Background

1957– Broadbent & Hammersley: Percolation

1972– de Gennes, des Cloizeaux: Polymers

1986– Saleur, Duplantier: Conformal theory of polymers, percolation

1993– Gurarie: Logarithmic operators in CFT

1995– Kausch: Symplectic fermions

1992– Rozansky, Read, Saleur, Schomerus, . . . Supergroup Approach to Log CFT

1996– Gaberdiel, Kausch, Flohr, Runkel, Feigin et al, Mathieu, Ridout,. . . Algebraic Log CFT

2006– Pearce, Rasmussen, Ruelle, Zuber: Lattice Approach to Log CFT

Lattice Approach:

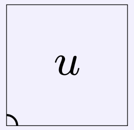
- Statistical systems with local degrees of freedom yield rational CFTs.

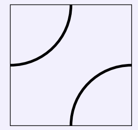
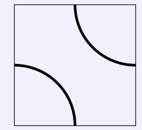
- Polymers, percolation and related lattice models do not have local degrees of freedom only nonlocal degrees of freedom (polymers, connectivities, SLE paths) and are associated with Logarithmic CFTs . . .

nonlocal lattice degrees of freedom	$\Rightarrow$	logarithmic CFT

Logarithmic Minimal Models $\mathcal{LM}(p,p')$

- **Face operators defined in planar Temperley-Lieb algebra (Jones 1999)**

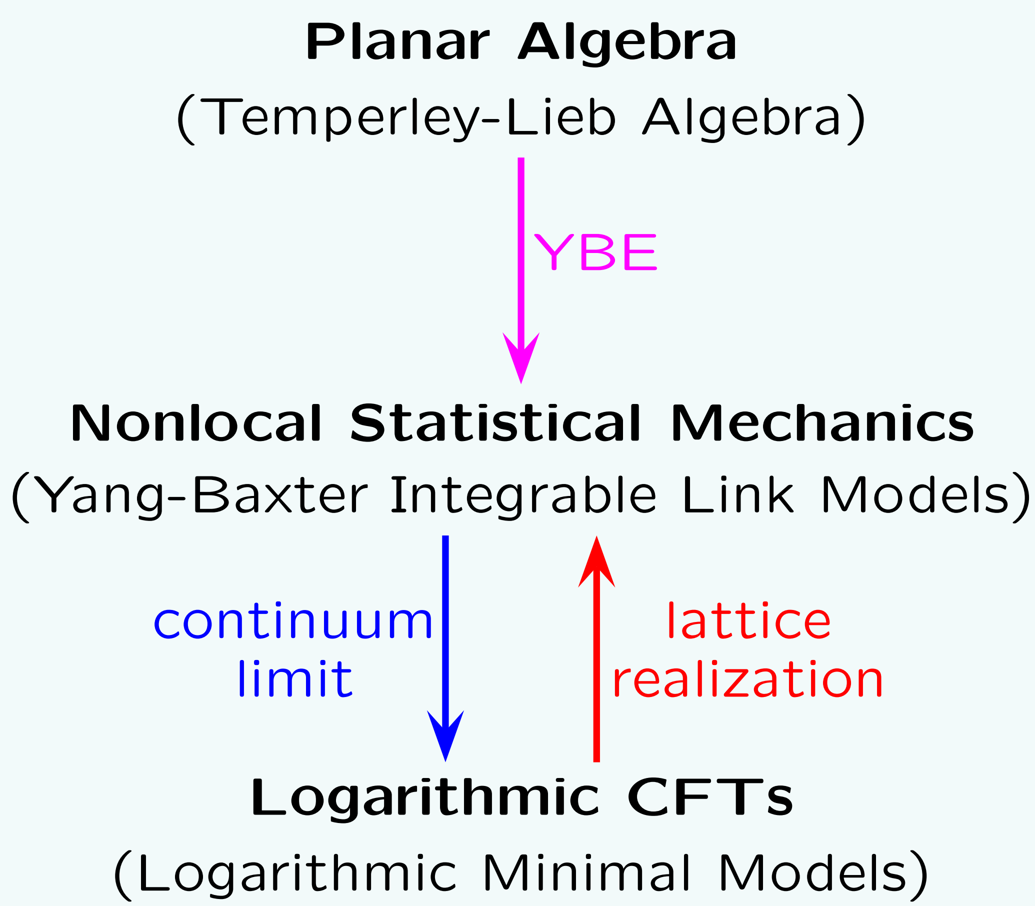
$X(u) =$ 

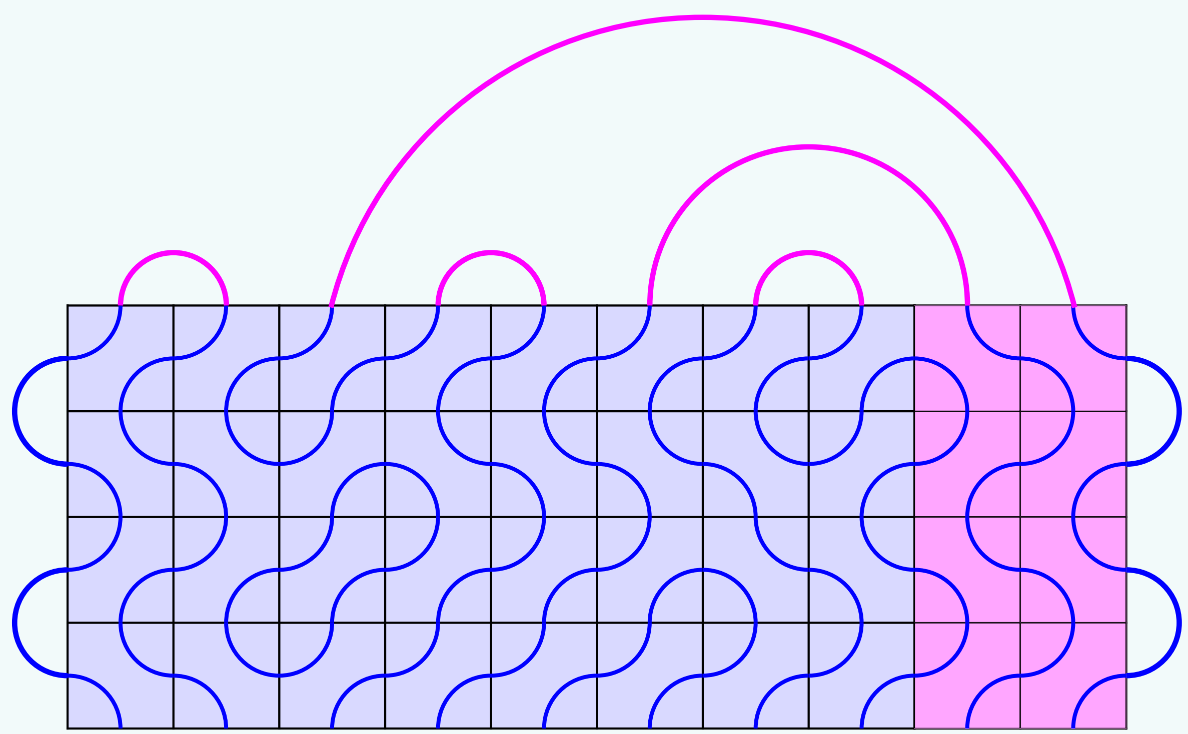
$= \dfrac{\sin(\lambda-u)}{\sin\lambda}$  $+ \dfrac{\sin u}{\sin\lambda}$  ;

$$X_j(u) = \frac{\sin(\lambda-u)}{\sin\lambda}\, I + \frac{\sin u}{\sin\lambda}\, e_j$$

$1 \le p < p'$ coprime integers, $\lambda = \dfrac{(p'-p)\pi}{p'}$ = crossing parameter

u = spectral parameter, $\beta = 2\cos\lambda$ = fugacity of loops

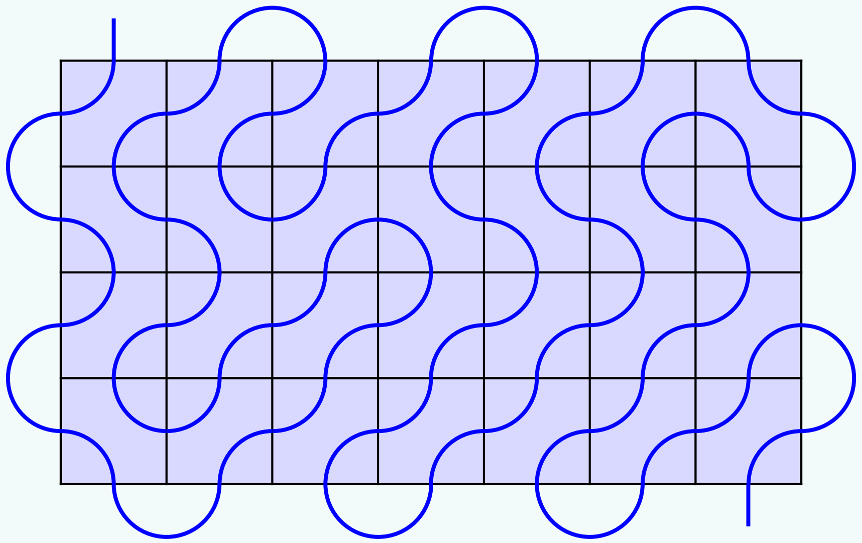




Nonlocal Degrees of Freedom

Polymers and Percolation on the Lattice

- **Critical Dense Polymers:** $(p,p')=(1,2), \qquad \lambda=\frac{\pi}{2}$

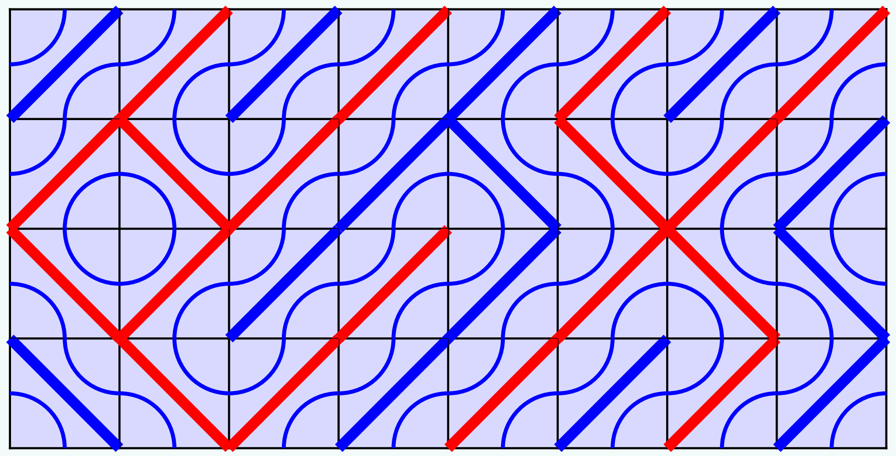


$$d_{path}^{SLE}=2-2\Delta_{p,p'-1}=2, \qquad \kappa=\frac{4p'}{p}=8$$

$\Delta_{1,1}=0$ lies outside rational $\mathcal{M}(1,2)$ Kac table

$\beta=0 \;\Rightarrow\;$ no loops $\;\Rightarrow\;$ space filling dense polymer

- **Critical Percolation:** $(p,p')=(2,3), \qquad \lambda=\frac{\pi}{3}, \qquad u=\frac{\lambda}{2}=\frac{\pi}{6}$ (isotropic)



$$d_{path}^{SLE}=2-2\Delta_{p,p'-1}=\frac{7}{4}, \qquad \kappa=\frac{4p'}{p}=6$$

$\Delta_{2,2}=\frac{1}{8}$ lies outside rational $\mathcal{M}(2,3)$ Kac table

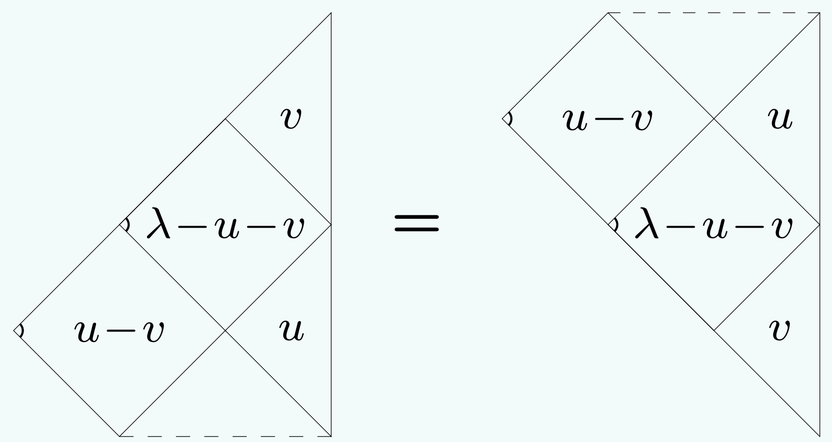
Bond percolation on the blue square lattice:

Critical probability $=p_c=\sin(\lambda-u)=\sin u=\frac{1}{2}$

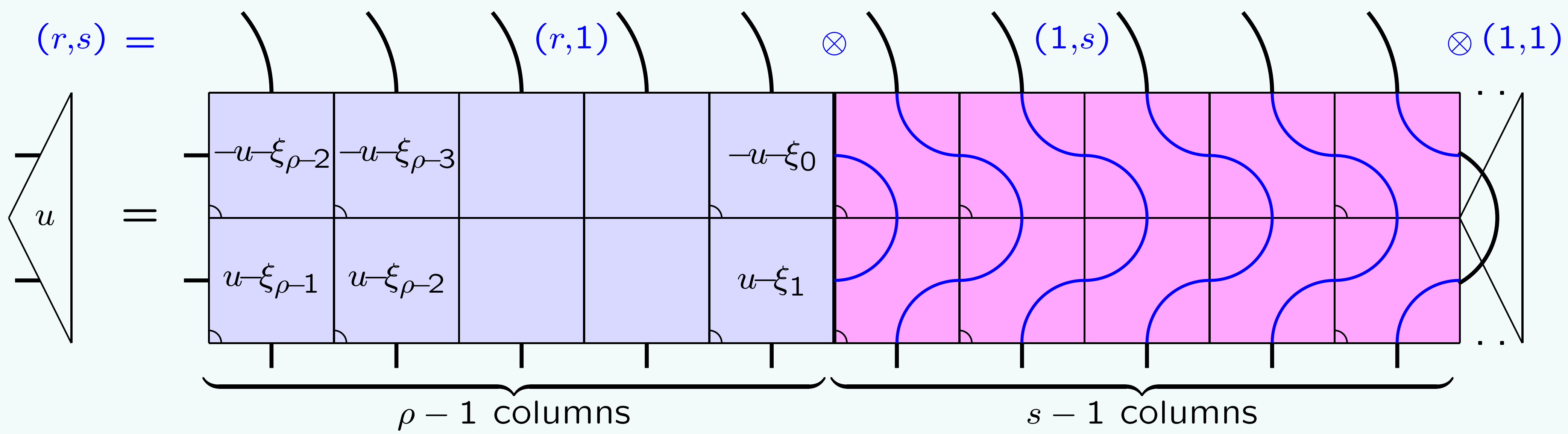
$\beta=1 \quad\Rightarrow\quad$ local stochastic process

Boundary Yang-Baxter Equation

- The Boundary Yang-Baxter Equation (BYBE) is the equality of boundary 2-tangles



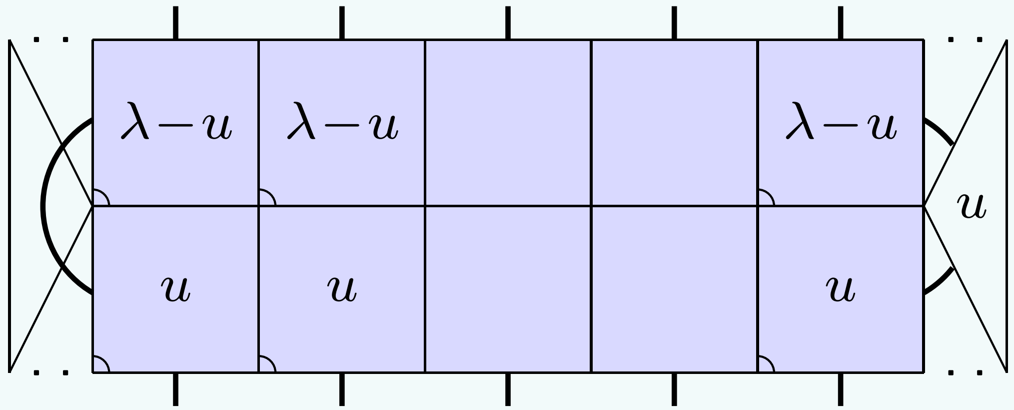
- For $r, s = 1, 2, 3, \ldots$, the $(r, s) = (r, 1) \otimes (1, s)$ BYBE solution is built as the fusion product of $(r, 1)$ and $(1, s)$ integrable seams acting on the vacuum $(1, 1)$ triangle:



- The column inhomogeneities are: $\xi_k = (k + k_0 + \frac{1}{2})\lambda$
- There is at least one choice of the integers ρ and k_0 for each r.

Double-Row Transfer Matrices

- For a strip with N columns, the double-row transfer "matrix" is the N-tangle

$\boldsymbol{D}(u) =$ 

- Using the Yang-Baxter (YBE) and Boundary Yang-Baxter Equations (BYBE) in the planar Temperley-Lieb (TL) algebra, it can be shown that, for any (r, s), these commute and are crossing symmetric

$$\boldsymbol{D}(u)\boldsymbol{D}(v) = \boldsymbol{D}(v)\boldsymbol{D}(u), \qquad \boldsymbol{D}(u) = \boldsymbol{D}(\lambda - u)$$

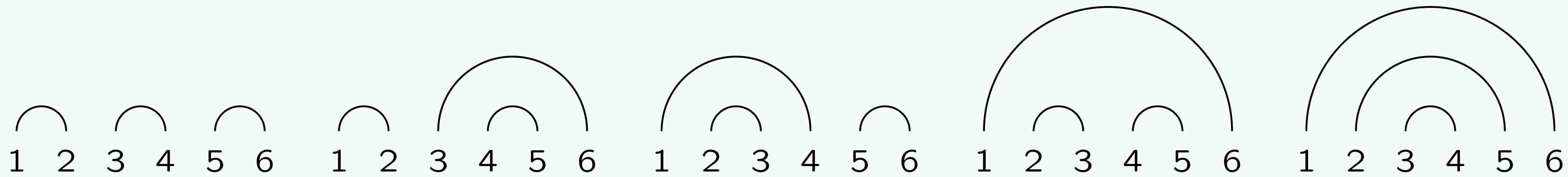
- Multiplication is vertical concatenation of diagrams, equality is the equality of N-tangles.

- In the case of one non-trivial boundary condition, the transfer matrices are found to be diagonalizable. For fusion, we take non-trivial boundary conditions on the left and right $(r', s') \otimes (r, s)$. In this case, the transfer matrices can exhibit Jordan cells and are not in general diagonalizable.

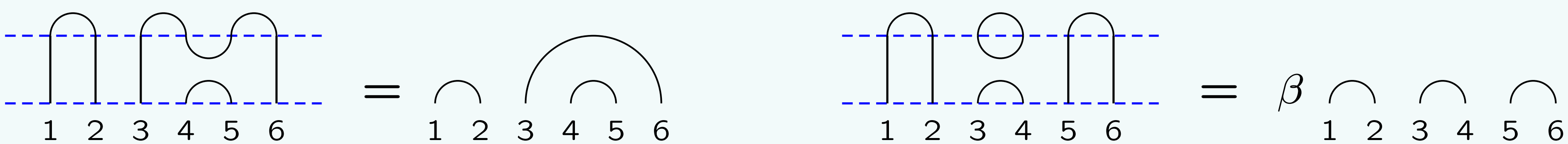
- It is necessary to act on a vector space of states to obtain *matrix representatives* and *spectra*.

Planar Link Diagrams

- The planar N-tangles act on a vector space $\mathcal{V}_N$ of *planar link diagrams*. The dimension of $\mathcal{V}_N$ is given by Catalan numbers. For $N = 6$, there is a basis of 5 link diagrams:



- The first link diagram is the reference state. Other states are generated by the action of the TL generators by concatenation from below



etc.

- The action of the TL generators on the states is nonlocal. It leads to matrices with entries $0, 1, \beta$ that represent the TL generators. For $N = 6$, the action of e_1 and e_2 on $\mathcal{V}_6$ is

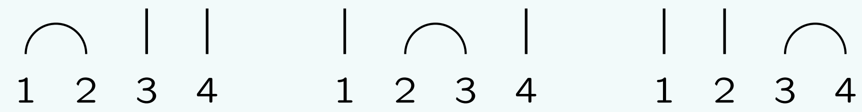
$$e_1 = \begin{pmatrix} \beta & 0 & 1 & 0 & 1 \\ 0 & \beta & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad e_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & \beta & 0 & 0 \\ 0 & 1 & 0 & \beta & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \text{etc.}$$

- The transfer matrices are built from the TL generators.

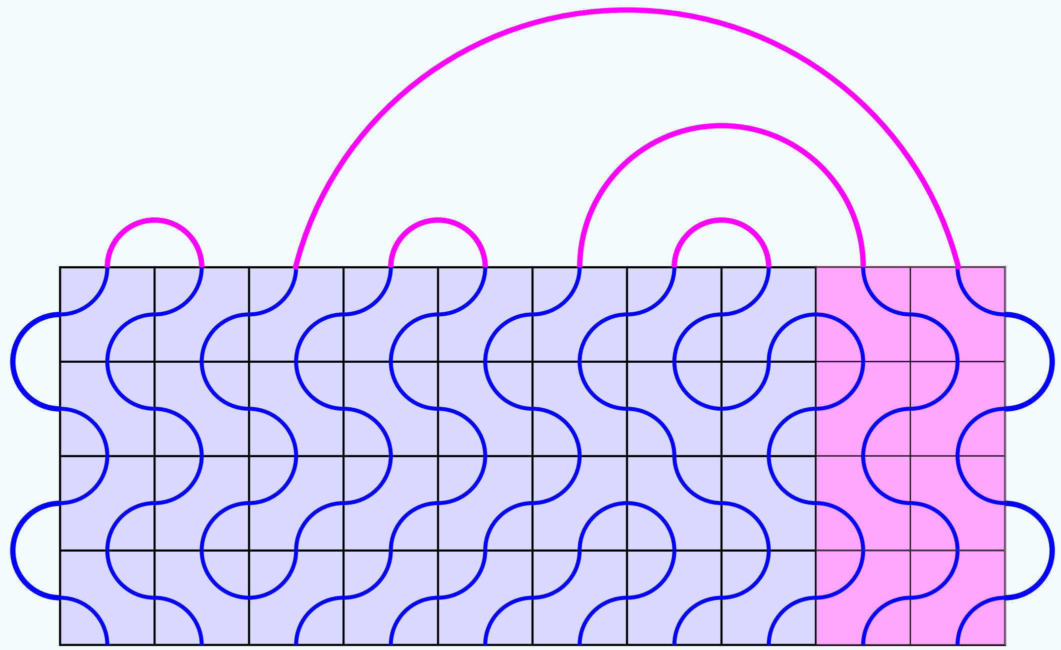
Defects

- More generally, the vector space of states $\mathcal{V}_N^{(\ell)}$ can contain ℓ *defects*:

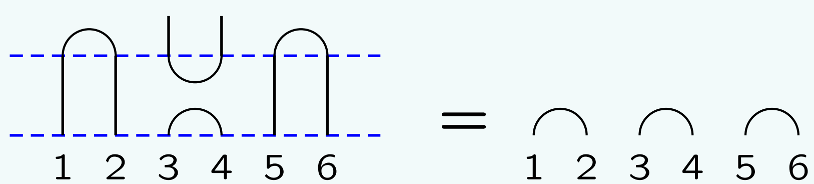
$N = 4,\ \ell = 2:$



- The ℓ defects can be closed on the right or the left. In this way, the number of defects propagating in the bulk is controlled by the boundary conditions. In particular, for $(1, s)$ boundary conditions, the $\ell = s - 1$ defects simply propagate along a boundary.



- Defects in the bulk can be annihilated in pairs but not created under the action of TL



etc.

- The transfer matrices are thus block-triangular with respect to the number of defects.

Dense Polymer Kac Table

- **Central charge:** $(p,p')=(1,2)$

$$c \;=\; 1-\frac{6(p-p')^2}{pp'} \;=\; -2$$

- **Infinitely extended Kac table of conformal weights:**

$$\begin{aligned}\Delta_{r,s} &= \frac{(p'r-ps)^2-(p-p')^2}{4pp'}\\ &= \frac{(2r-s)^2-1}{8}, \qquad r,s=1,2,3,\ldots\end{aligned}$$

- **Kac representation characters:**

$$\chi_{r,s}(q)=q^{-c/24}\,\frac{q^{\Delta_{r,s}}(1-q^{rs})}{\prod_{n=1}^{\infty}(1-q^n)}$$

- **Irreducible Representations:**

There is an irreducible representation for each distinct conformal weight. The Kac representations which happen to be irreducible are marked with a red quadrant.

s							
s	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
10	$\frac{63}{8}$	$\frac{35}{8}$	$\frac{15}{8}$	$\frac{3}{8}$	$-\frac{1}{8}$	$\frac{3}{8}$	$\cdots$
9	6	3	1	0	0	1	$\cdots$
8	$\frac{35}{8}$	$\frac{15}{8}$	$\frac{3}{8}$	$-\frac{1}{8}$	$\frac{3}{8}$	$\frac{15}{8}$	$\cdots$
7	3	1	0	0	1	3	$\cdots$
6	$\frac{15}{8}$	$\frac{3}{8}$	$-\frac{1}{8}$	$\frac{3}{8}$	$\frac{15}{8}$	$\frac{35}{8}$	$\cdots$
5	1	0	0	1	3	6	$\cdots$
4	$\frac{3}{8}$	$-\frac{1}{8}$	$\frac{3}{8}$	$\frac{15}{8}$	$\frac{35}{8}$	$\frac{63}{8}$	$\cdots$
3	0	0	1	3	6	10	$\cdots$
2	$-\frac{1}{8}$	$\frac{3}{8}$	$\frac{15}{8}$	$\frac{35}{8}$	$\frac{63}{8}$	$\frac{99}{8}$	$\cdots$
1	0	1	3	6	10	15	$\cdots$
	1	2	3	4	5	6	r

Critical Percolation Kac Table

- **Central charge:** $(p,p') = (2,3)$

$$c \;=\; 1 - \frac{6(p-p')^2}{pp'} \;=\; 0$$

- **Infinitely extended Kac table of conformal weights:**

$$\begin{aligned}\Delta_{r,s} &= \frac{(p'r-ps)^2-(p-p')^2}{4pp'}\\ &= \frac{(3r-2s)^2-1}{24}, \qquad r,s=1,2,3,\ldots\end{aligned}$$

- **Kac representation characters:**

$$\chi_{r,s}(q) = q^{-c/24}\,\frac{q^{\Delta_{r,s}}(1-q^{rs})}{\prod_{n=1}^{\infty}(1-q^n)}$$

- **Irreducible Representations:**

There is an irreducible representation for each distinct conformal weight. The Kac representations which happen to be irreducible are marked with a red quadrant.

s	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
10	12	$\frac{65}{8}$	5	$\frac{21}{8}$	1	$\frac{1}{8}$	$\cdots$
9	$\frac{28}{3}$	$\frac{143}{24}$	$\frac{10}{3}$	$\frac{35}{24}$	$\frac{1}{3}$	$-\frac{1}{24}$	$\cdots$
8	7	$\frac{33}{8}$	2	$\frac{5}{8}$	0	$\frac{1}{8}$	$\cdots$
7	5	$\frac{21}{8}$	1	$\frac{1}{8}$	0	$\frac{5}{8}$	$\cdots$
6	$\frac{10}{3}$	$\frac{35}{24}$	$\frac{1}{3}$	$-\frac{1}{24}$	$\frac{1}{3}$	$\frac{35}{24}$	$\cdots$
5	2	$\frac{5}{8}$	0	$\frac{1}{8}$	1	$\frac{21}{8}$	$\cdots$
4	1	$\frac{1}{8}$	0	$\frac{5}{8}$	2	$\frac{33}{8}$	$\cdots$
3	$\frac{1}{3}$	$-\frac{1}{24}$	$\frac{1}{3}$	$\frac{35}{24}$	$\frac{10}{3}$	$\frac{143}{24}$	$\cdots$
2	0	$\frac{1}{8}$	1	$\frac{21}{8}$	5	$\frac{65}{8}$	$\cdots$
1	0	$\frac{5}{8}$	2	$\frac{33}{8}$	7	$\frac{85}{8}$	$\cdots$
	1	2	3	4	5	6	r

Lattice Fusion and Indecomposable Representations

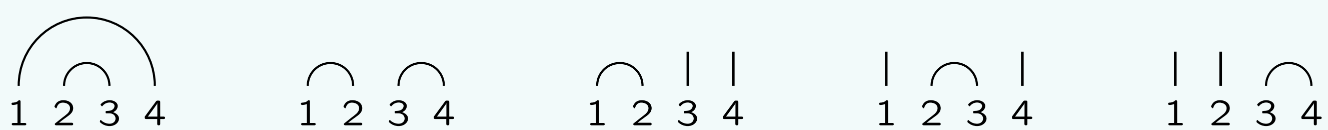
- For *Critical Dense Polymers*, the $(1,2)\otimes(1,2) = \left(-\frac{1}{8}\right)\otimes\left(-\frac{1}{8}\right) = 0\ +\ 0 = (1,1)+(1,3)$ fusion yields a reducible yet indecomposable representation. For $N=4$, the finitized partition function is ($q=$ modular parameter)

$$Z^{(N)}_{(1,2)|(1,2)}(q) = \underbrace{\chi^{(N)}_{(1,1)}(q)}_{0\text{ defects}} + \underbrace{\chi^{(N)}_{(1,3)}(q)}_{2\text{ defects}} = q^{-c/24}[(1+q^2)+(1+q+q^2)] = q^{-c/24}(2+q+2q^2)$$

- The Hamiltonian

$$\boldsymbol{D}(u)\sim e^{-u\mathcal{H}} \qquad -\mathcal{H} = \left(\begin{array}{cc|ccc} 0 & 1 & 0 & 0 & 0\\ 2 & 0 & 1 & 0 & 1\\ \hline 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 1 & 0 & 1\\ 0 & 0 & 0 & 1 & 0 \end{array}\right) + \sqrt{2}\, I \qquad -\mathcal{H}\mapsto L_0 - \frac{c}{24}$$

acts on the five states with $\ell=0$ or $\ell=2$ defects



- The Jordan canonical form for $\mathcal{H}$ has rank 2 Jordan cells

$$-\mathcal{H}\sim \left(\begin{array}{cc|ccc} 0 & 0 & 1 & 0 & 0\\ 0 & \sqrt{8} & 0 & 0 & 1\\ \hline 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & \sqrt{2} & 0\\ 0 & 0 & 0 & 0 & \sqrt{8} \end{array}\right) \sim \begin{pmatrix} 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & \sqrt{2} & 0 & 0\\ 0 & 0 & 0 & \sqrt{8} & 1\\ 0 & 0 & 0 & 0 & \sqrt{8} \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 2 & 1\\ 0 & 0 & 0 & 0 & 2 \end{pmatrix} = L_0^{(4)}$$

- The eigenvalues of $-\mathcal{H}$ approach the integer energies indicated in $L_0^{(4)}$ as $N\to\infty$.

Dense Polymer Virasoro Fusion Algebra

- The fundamental Virasoro fusion algebra of critical dense polymers $\mathcal{LM}(1,2)$ is

$$\Big\langle (2,1),(1,2)\Big\rangle \;=\; \Big\langle (r,1),(1,2k),\mathcal{R}_k;\;\; r,k\in\mathbb{N}\Big\rangle$$

- With the identifications $(k,2k')\equiv(k',2k)$, the fusion rules obtained empirically from the lattice are commutative, associative and agree with Gaberdiel and Kausch (1996)

$$(r,1)\otimes(r',1) \;=\; \bigoplus_{j=|r-r'|+1,\text{ by }2}^{r+r'-1} (j,1)$$

$$(1,2k)\otimes(1,2k') \;=\; \bigoplus_{j=|k-k'|+1,\text{ by }2}^{k+k'-1} \mathcal{R}_j$$

$$(1,2k)\otimes\mathcal{R}_{k'} \;=\; \bigoplus_{j=|k-k'|}^{k+k'} \delta^{(2)}_{j,\{k,k'\}}(1,2j)$$

$$\mathcal{R}_k\otimes\mathcal{R}_{k'} \;=\; \bigoplus_{j=|k-k'|}^{k+k'} \delta^{(2)}_{j,\{k,k'\}}\mathcal{R}_j$$

$$(r,1)\otimes(1,2k) \;=\; \bigoplus_{j=|r-k|+1,\text{ by }2}^{r+k-1} (1,2j)=(r,2k)$$

$$(r,1)\otimes\mathcal{R}_k \;=\; \bigoplus_{j=|r-k|+1,\text{ by }2}^{r+k-1} \mathcal{R}_j$$

s							
	⋮	⋮	⋮	⋮	⋮	⋮	⋰
10	$\frac{63}{8}$	$\frac{35}{8}$	$\frac{15}{8}$	$\frac{3}{8}$	$-\frac{1}{8}$	$\frac{3}{8}$	⋯
9	6	3	1	0	0	1	⋯
8	$\frac{35}{8}$	$\frac{15}{8}$	$\frac{3}{8}$	$-\frac{1}{8}$	$\frac{3}{8}$	$\frac{15}{8}$	⋯
7	3	1	0	0	1	3	⋯
6	$\frac{15}{8}$	$\frac{3}{8}$	$-\frac{1}{8}$	$\frac{3}{8}$	$\frac{15}{8}$	$\frac{35}{8}$	⋯
5	1	0	0	1	3	6	⋯
4	$\frac{3}{8}$	$-\frac{1}{8}$	$\frac{3}{8}$	$\frac{15}{8}$	$\frac{35}{8}$	$\frac{63}{8}$	⋯
3	0	0	1	3	6	10	⋯
2	$-\frac{1}{8}$	$\frac{3}{8}$	$\frac{15}{8}$	$\frac{35}{8}$	$\frac{63}{8}$	$\frac{99}{8}$	⋯
1	0	1	3	6	10	15	⋯
	1	2	3	4	5	6	r

$\mathcal{R}_k$ = indecomposable = $(1,2k-1)\oplus_i(1,2k+1)$,

$$\delta^{(2)}_{j,\{k,k'\}} = 2-\delta_{j,|k-k'|}-\delta_{j,k+k'}$$

$\mathcal{W}$-Extended Vacuum of Symplectic Fermions

- Critical dense polymers in the $\mathcal{W}$-extended picture is identified with *symplectic fermions*.
- The extended vacuum character of symplectic fermions is known to be

$$\hat{\chi}_{1,1}(q) = \sum_{n=1}^{\infty} (2n-1)\,\chi_{2n-1,1}(q)$$

This suggests the corresponding integrable boundary condition is the direct sum

$$(1,1)_{\mathcal{W}} = \bigoplus_{n=1}^{\infty} (2n-1)\,(2n-1,1) \;=\; \mathcal{W}\text{-irreducible representation}$$

- However, the BYBE is *not* linear and sums of solutions do *not* usually give new solutions. Rather, the *BYBE is closed under fusions*. If we can construct this direct sum from fusions, then automatically it will be a solution of the BYBE.

- Consider the triple fusion

$$(2n-1,1)\otimes(2n-1,1)\otimes(2n-1,1) = (1,1)\oplus 3(3,1)\oplus 5(5,1)\oplus\cdots\oplus(2n-1)(2n-1,1)\oplus\cdots$$

For large n, the coefficients stabilize and reproduce the extended vacuum $(1,1)_{\mathcal{W}}$. So the integrable boundary condition associated to the extended vacuum boundary condition is constructed by fusing three r-type integrable seams to the boundary

$$(1,1)_{\mathcal{W}} := \lim_{n\to\infty}(2n-1,1)\otimes(2n-1,1)\otimes(2n-1,1) = \bigoplus_{n=1}^{\infty}(2n-1)\,(2n-1,1)$$

$\mathcal{W}$-Extended Boundary Conditions

- The extended vacuum $(1,1)_{\mathcal{W}}$ must act as the identity. In particular

$$(1,1)_{\mathcal{W}} \,\hat{\otimes}\, (1,1)_{\mathcal{W}} = (1,1)_{\mathcal{W}}$$

where $\hat{\otimes}$ denotes the fusion multiplication in the extended picture.

- The extended vacuum has the stability property

$$(2m-1,1) \otimes (1,1)_{\mathcal{W}} = (2m-1)\left(\bigoplus_{n=1}^{\infty} (2n-1)\,(2n-1,1)\right) = (2m-1)\,(1,1)_{\mathcal{W}}$$

- The extended fusion $\hat{\otimes}$ is therefore defined by

$$(1,1)_{\mathcal{W}} \,\hat{\otimes}\, (1,1)_{\mathcal{W}} := \lim_{n\to\infty}\left(\frac{1}{(2n-1)^3}(2n-1,1) \otimes (2n-1,1) \otimes (2n-1,1) \otimes (1,1)_{\mathcal{W}}\right) = (1,1)_{\mathcal{W}}$$

- The representation content is 4 $\mathcal{W}$-irreducible and 2 $\mathcal{W}$-reducible yet $\mathcal{W}$-indecomposable representations. Additional stability properties enable us to define

$$\begin{aligned}
(1,s)_{\mathcal{W}} &:= (1,s) \otimes (1,1)_{\mathcal{W}} = \bigoplus_{n=1}^{\infty} (2n-1)\,(2n-1,s), \qquad s=1,2 \\
(2,s)_{\mathcal{W}} &:= \tfrac{1}{2}(2,s) \otimes (1,1)_{\mathcal{W}} = \bigoplus_{n=1}^{\infty} 2n\,(2n,s), \qquad s=1,2 \\
\hat{\mathcal{R}}_1 \equiv (\mathcal{R}_1)_{\mathcal{W}} &:= \mathcal{R}_1 \otimes (1,1)_{\mathcal{W}} = \bigoplus_{n=1}^{\infty} (2n-1)\,\mathcal{R}_{2n-1} \\
\hat{\mathcal{R}}_0 \equiv (\mathcal{R}_2)_{\mathcal{W}} &:= \tfrac{1}{2}\mathcal{R}_2 \otimes (1,1)_{\mathcal{W}} = \bigoplus_{n=1}^{\infty} 2n\,\mathcal{R}_{2n}
\end{aligned}$$

Symplectic Fermion Fusion Rules

- The $\mathcal{W}$-extended fusion rules follow from the Virasoro fusion rules combined with stability. The extended fusion rules and characters agree with Gaberdiel and Runkel (2008):

$\hat{\otimes}$	0	1	$-\frac{1}{8}$	$\frac{3}{8}$	$\hat{\mathcal{R}}_0$	$\hat{\mathcal{R}}_1$
0	0	1	$-\frac{1}{8}$	$\frac{3}{8}$	$\hat{\mathcal{R}}_0$	$\hat{\mathcal{R}}_1$
1	1	0	$\frac{3}{8}$	$-\frac{1}{8}$	$\hat{\mathcal{R}}_1$	$\hat{\mathcal{R}}_0$
$-\frac{1}{8}$	$-\frac{1}{8}$	$\frac{3}{8}$	$\hat{\mathcal{R}}_0$	$\hat{\mathcal{R}}_1$	$2(-\frac{1}{8})+2(\frac{3}{8})$	$2(-\frac{1}{8})+2(\frac{3}{8})$
$\frac{3}{8}$	$\frac{3}{8}$	$-\frac{1}{8}$	$\hat{\mathcal{R}}_1$	$\hat{\mathcal{R}}_0$	$2(-\frac{1}{8})+2(\frac{3}{8})$	$2(-\frac{1}{8})+2(\frac{3}{8})$
$\hat{\mathcal{R}}_0$	$\hat{\mathcal{R}}_0$	$\hat{\mathcal{R}}_1$	$2(-\frac{1}{8})+2(\frac{3}{8})$	$2(-\frac{1}{8})+2(\frac{3}{8})$	$2\hat{\mathcal{R}}_0+2\hat{\mathcal{R}}_1$	$2\hat{\mathcal{R}}_0+2\hat{\mathcal{R}}_1$
$\hat{\mathcal{R}}_1$	$\hat{\mathcal{R}}_1$	$\hat{\mathcal{R}}_0$	$2(-\frac{1}{8})+2(\frac{3}{8})$	$2(-\frac{1}{8})+2(\frac{3}{8})$	$2\hat{\mathcal{R}}_0+2\hat{\mathcal{R}}_1$	$2\hat{\mathcal{R}}_0+2\hat{\mathcal{R}}_1$

Example: Consider the extended fusion rule $1 \,\hat{\otimes}\, 1 = 0$:

$$
\begin{aligned}
(2,1)_{\mathcal{W}} \,\hat{\otimes}\, (2,1)_{\mathcal{W}} &:= \Big(\tfrac{1}{2}(2,1)\otimes(1,1)_{\mathcal{W}}\Big)\hat{\otimes}\Big(\tfrac{1}{2}(2,1)\otimes(1,1)_{\mathcal{W}}\Big) \\
&= \tfrac{1}{4}\Big((2,1)\otimes(2,1)\Big)\otimes\Big((1,1)_{\mathcal{W}}\,\hat{\otimes}\,(1,1)_{\mathcal{W}}\Big) \\
&= \tfrac{1}{4}\Big((1,1)\oplus(3,1)\Big)\otimes(1,1)_{\mathcal{W}} = \tfrac{1}{4}(1+3)(1,1)_{\mathcal{W}} = (1,1)_{\mathcal{W}}
\end{aligned}
$$

Representation Content of $\mathcal{WLM}(p,p')$

	Number	Symplectic Fermions	Critical Percolation
$\mathcal{W}$-reps	$6pp' - 2p - 2p'$	6	26
Rank 1	$2p + 2p' - 2$	4	8
Rank 2	$4pp' - 2p - 2p'$	2	14
Rank 3	$2(p-1)(p'-1)$	0	4
$\mathcal{W}$-irred chars	$2pp' + \frac{1}{2}(p-1)(p'-1)$	4	13

- Kac tables of 4 and 13 $\mathcal{W}$-irreducible characters for symplectic fermions and critical percolation:

s \ r	1	2
2	$-\frac{1}{8}$	$\frac{3}{8}$
1	0	1

s \ r	1	2
3	$\frac{1}{3}, \frac{10}{3}$	$-\frac{1}{24}, \frac{35}{24}$
2	$1, 5$	$\frac{1}{8}, \frac{21}{8}$
1	$(0)\ 2, 7$	$\frac{5}{8}, \frac{33}{8}$

$\mathcal{W}$-Ireducible Characters of Critical Percolation

- $\mathcal{W}$-irreducible representations:

$$\hat{\chi}_{\frac{1}{3}}(q) = \frac{1}{\eta(q)}\sum_{k\in\mathbb{Z}}(2k-1)q^{3(4k-3)^2/8} \qquad \hat{\chi}_{\frac{21}{8}}(q) = \frac{1}{\eta(q)}\sum_{k\in\mathbb{Z}}2k\,q^{(6k-2)^2/6}$$

$$\hat{\chi}_{\frac{10}{3}}(q) = \frac{1}{\eta(q)}\sum_{k\in\mathbb{Z}}2k\,q^{3(4k-1)^2/8} \qquad \hat{\chi}_{\frac{33}{8}}(q) = \frac{1}{\eta(q)}\sum_{k\in\mathbb{Z}}2k\,q^{(6k-1)^2/6}$$

$$\hat{\chi}_{\frac{1}{8}}(q) = \frac{1}{\eta(q)}\sum_{k\in\mathbb{Z}}(2k-1)\,q^{(6k-5)^2/6} \qquad \hat{\chi}_{-\frac{1}{24}}(q) = \frac{1}{\eta(q)}\sum_{k\in\mathbb{Z}}(2k-1)\,q^{(6k-6)^2/6}$$

$$\hat{\chi}_{\frac{5}{8}}(q) = \frac{1}{\eta(q)}\sum_{k\in\mathbb{Z}}(2k-1)\,q^{(6k-4)^2/6} \qquad \hat{\chi}_{\frac{35}{24}}(q) = \frac{1}{\eta(q)}\sum_{k\in\mathbb{Z}}2k\,q^{(6k-3)^2/6}$$

- From subfactors of $\mathcal{W}$-reducible yet $\mathcal{W}$-indecomposable representations:

$$\hat{\chi}_0(q) = 1$$

$$\hat{\chi}_1(q) = \frac{1}{\eta(q)}\sum_{k\in\mathbb{Z}}k^2\Big[q^{(12k-7)^2/24}-q^{(12k+1)^2/24}\Big]$$

$$\hat{\chi}_2(q) = \frac{1}{\eta(q)}\sum_{k\in\mathbb{Z}}k^2\Big[q^{(12k-5)^2/24}-q^{(12k-1)^2/24}\Big] \qquad \eta(q) = q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^n)$$

$$\hat{\chi}_5(q) = \frac{1}{\eta(q)}\sum_{k\in\mathbb{Z}}k(k+1)\Big[q^{(12k-1)^2/24}-q^{(12k+7)^2/24}\Big]$$

$$\hat{\chi}_7(q) = \frac{1}{\eta(q)}\sum_{k\in\mathbb{Z}}k(k+1)\Big[q^{(12k+1)^2/24}-q^{(12k+5)^2/24}\Big]$$

- These agree with Feigin, Gainutdinov, Semikhatov and Tipunin (2005).

$\mathcal{W}$-Projective Representations

- A $\mathcal{W}$-projective representation is a "*maximal*" $\mathcal{W}$-indecomposable representation in the sense that it does not appear as a subfactor of any other $\mathcal{W}$-indecomposable representation.
- Symplectic fermions has 4 projective representations $-1/8$, $3/8$, $\hat{\mathcal{R}}_0$ and $\hat{\mathcal{R}}_1$ with 3 distinct characters $\hat{\chi}_{-1/8}(q)$, $\hat{\chi}_{3/8}(q)$ and $\chi[\hat{\mathcal{R}}_0](q) = \chi[\hat{\mathcal{R}}_1](q)$.
- The $\mathcal{W}$-projective representations form a closed sub-fusion algebra *Proj* (p, p') of the $\mathcal{WLM}(p, p')$ fusion algebra.
- The $\mathcal{W}$-projective representation content is:

	Reps	Number	Symplectic Fermions	Critical Percolation
$\mathcal{W}$-proj reps	$\hat{\mathcal{R}}^{r,s}_{\kappa p,p'}$	$2pp'$	4	12
Rank 1	$\hat{\mathcal{R}}^{0,0}_{\kappa p,p'} \equiv (\kappa p, p')_{\mathcal{W}}$	2	2	2
Rank 2	$\hat{\mathcal{R}}^{a,0}_{\kappa p,p'}$, $\hat{\mathcal{R}}^{0,b}_{p,\kappa p'}$	$2(p+p'-2)$	2	6
Rank 3	$\hat{\mathcal{R}}^{a,b}_{\kappa p,p'}$	$2(p-1)(p'-1)$	0	4
$\mathcal{W}$-proj chars	$\varkappa_k$	$\frac{1}{2}(p+1)(p'+1)$	3	6

$$(\kappa p, p')_{\mathcal{W}} = (p, \kappa p')_{\mathcal{W}}, \qquad \hat{\mathcal{R}}^{a,b}_{\kappa p,p'} = \hat{\mathcal{R}}^{a,b}_{p,\kappa p'}$$

$$\kappa = 1, 2; \quad a = 1, 2, \ldots, p-1; \quad b = 1, 2, \ldots, p'-1; \quad k = 1, 2, \ldots, \tfrac{1}{2}(p+1)(p'+1)$$
$$r = 0, 1, \ldots, p; \qquad s = 0, 1, \ldots, p'$$

$\mathcal{W}$-Projective Characters and Grothendieck Ring

- The $2pp'$ $\mathcal{W}$-projective characters agree with Feigin et al (2006)

$$\varkappa^{0,0}_{\kappa p,p'}(q) \equiv \varkappa\big[\widehat{\mathcal{R}}^{0,0}_{\kappa p,p'}\big](q) = \frac{1}{\eta(q)}\sum_{k\in\mathbb{Z}}(2k-2+\kappa)q^{((2k-2+\kappa)-1)^2pp'/4}$$

$$\varkappa^{a,0}_{\kappa p,p'}(q) \equiv \varkappa\big[\widehat{\mathcal{R}}^{a,0}_{\kappa p,p'}\big](q) = \frac{2}{\eta(q)}\sum_{k\in\mathbb{Z}} q^{(a+(2k-1+\kappa)p)^2p'/4p}$$

$$\varkappa^{0,b}_{p,\kappa p'}(q) \equiv \varkappa\big[\widehat{\mathcal{R}}^{0,b}_{p,\kappa p'}\big](q) = \frac{2}{\eta(q)}\sum_{k\in\mathbb{Z}} q^{(b+(2k-1+\kappa)p')^2p/4p'}$$

$$\varkappa^{a,b}_{\kappa p,p'}(q) \equiv \varkappa[\widehat{\mathcal{R}}^{a,b}_{\kappa p,p'}](q) = \frac{2}{\eta(q)}\sum_{k\in\mathbb{Z}}\Big(q^{(ap'-bp+(2k+1-\kappa)pp')^2/4pp'}+q^{(ap'+bp+(2k+1-\kappa)pp')^2/4pp'}\Big)$$

- Only $\frac{1}{2}(p+1)(p'+1)$ of these are linearly independent because of the character identities

$$\varkappa^{a,0}_{p,p'}(q) = \varkappa^{p-a,0}_{2p,p'}(q), \qquad \varkappa^{0,b}_{p,p'}(q) = \varkappa^{0,p'-b}_{p,2p'}(q), \qquad \varkappa^{a,b}_{(3-\kappa)p,p'}(q) = \varkappa^{p-a,b}_{\kappa p,p'}(q) = \varkappa^{a,p'-b}_{\kappa p,p'}(q)$$

- The $\mathcal{W}$-projective fusion algebra *Proj* (p,p') possesses a Grothendieck ring $\mathcal{PG}(p,p')$ corresponding to the $\frac{1}{2}(p+1)(p'+1)$ independent $\mathcal{W}$-projective characters:

$$\mathcal{PG}(p,p') = \Big\langle \varkappa_k(q)\Big|_{k=1}^{\frac{1}{2}(p+1)(p'+1)}\Big\rangle$$

$$= \Big\langle \varkappa^{0,0}_{p,p'}(q),\ \varkappa^{0,0}_{2p,p'}(q),\ \varkappa^{a,0}_{p,p'}(q)\Big|_{a=1}^{p-1},\ \varkappa^{0,b}_{p,p'}(q)\Big|_{b=1}^{p'-1},\ \varkappa^{a,b}_{p,p'}(q)\Big|_{ap'+bp\le pp'}\Big\rangle$$

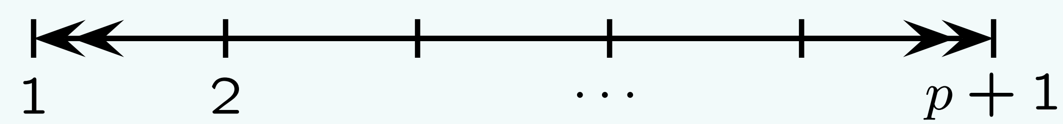
Verlinde Formula and Graph Fusion Algebra

- The modular S matrix of these characters [Feigin et al (2006)] satisfies $S^2 = I$, $S^T \neq S$.
- This modular matrix diagonalizes our fusion rules! Specifically, the conformal partition functions and Verlinde formula for the projective Grothendieck ring $\mathcal{PG}(p,p')$ are given by

$$Z_{i|j}(q) = \sum_{k=1}^{\frac{1}{2}(p+1)(p'+1)} N_{ij}{}^{k} (F\varkappa)_k(q), \qquad N_{ij}{}^{k} = \sum_{m=1}^{\frac{1}{2}(p+1)(p'+1)} \frac{S_{im}S_{jm}S_{mk}}{S_{1m}}$$

- The fundamental fusion matrix of $\mathcal{PG}(1,p)$ is

$$N_2 = \begin{pmatrix} 0 & 1 & & & & \\ 2 & 0 & 1 & & & \\ & 1 & \cdot & & & \\ & & & \cdot & & \\ & & & \cdot & 1 & \\ & & & 1 & 0 & 2 \\ & & & & 1 & 0 \end{pmatrix} :$$

$A_{(1,p)} =$ 

- The quantum dimensions of $\mathcal{PG}(1,p)$ are

$$\frac{S_{im}}{S_{1m}} = (2-\delta_{i,1}-\delta_{i,p+1}) \cos\frac{(i-1)(m-1)\pi}{p}, \qquad i,m = 1,2,\ldots,p+1$$

- For symplectic fermions $\mathcal{PG}(1,2)$, the graph fusion matrices (cf. Ising) are

$$N_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad N_2 = \begin{pmatrix} 0 & 1 & 0 \\ 2 & 0 & 2 \\ 0 & 1 & 0 \end{pmatrix} = F, \qquad N_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

- For $\mathcal{PG}(p,p')$, the fundamental fusion graph is given by a coset-type graph.

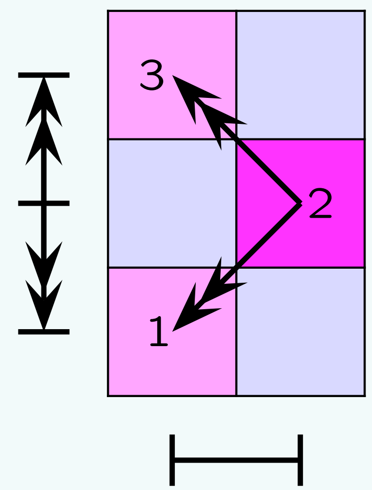
Projective Grothendieck Kac Tables

- The conformal weights of the projective Grothendieck characters of $\mathcal{PG}(p,p')$ are

$$\Delta_{r,s} = \frac{(p'r-ps)^2-(p-p')^2}{4pp'}, \qquad r=0,1,\ldots,p; \qquad s=0,1,\ldots,p'$$

Dense Polymers/ Symp Fermions:

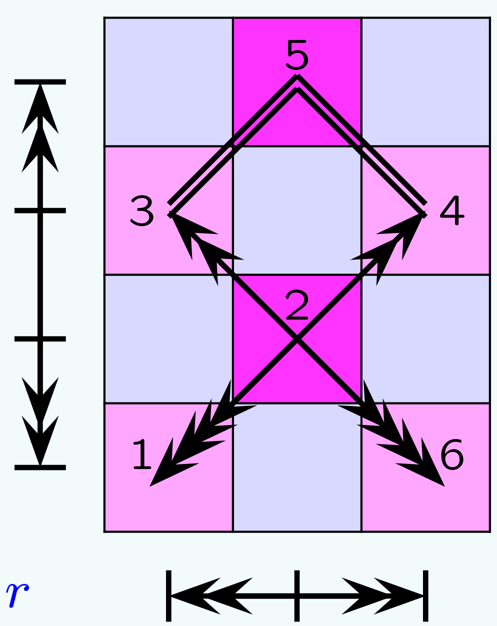
s		
2	$\frac{3}{8}$	$-\frac{1}{8}$
1	0	0
0	$-\frac{1}{8}$	$\frac{3}{8}$
	0	1 r



$$N_2 = \begin{pmatrix} 0 & 1 & 0 \\ 2 & 0 & 2 \\ 0 & 1 & 0 \end{pmatrix} = F$$

Percolation:

s			
3	$\frac{35}{24}$	$\frac{1}{3}$	$-\frac{1}{24}$
2	$\frac{5}{8}$	0	$\frac{1}{8}$
1	$\frac{1}{8}$	0	$\frac{5}{8}$
0	$-\frac{1}{24}$	$\frac{1}{3}$	$\frac{35}{24}$
	0	1	2 r

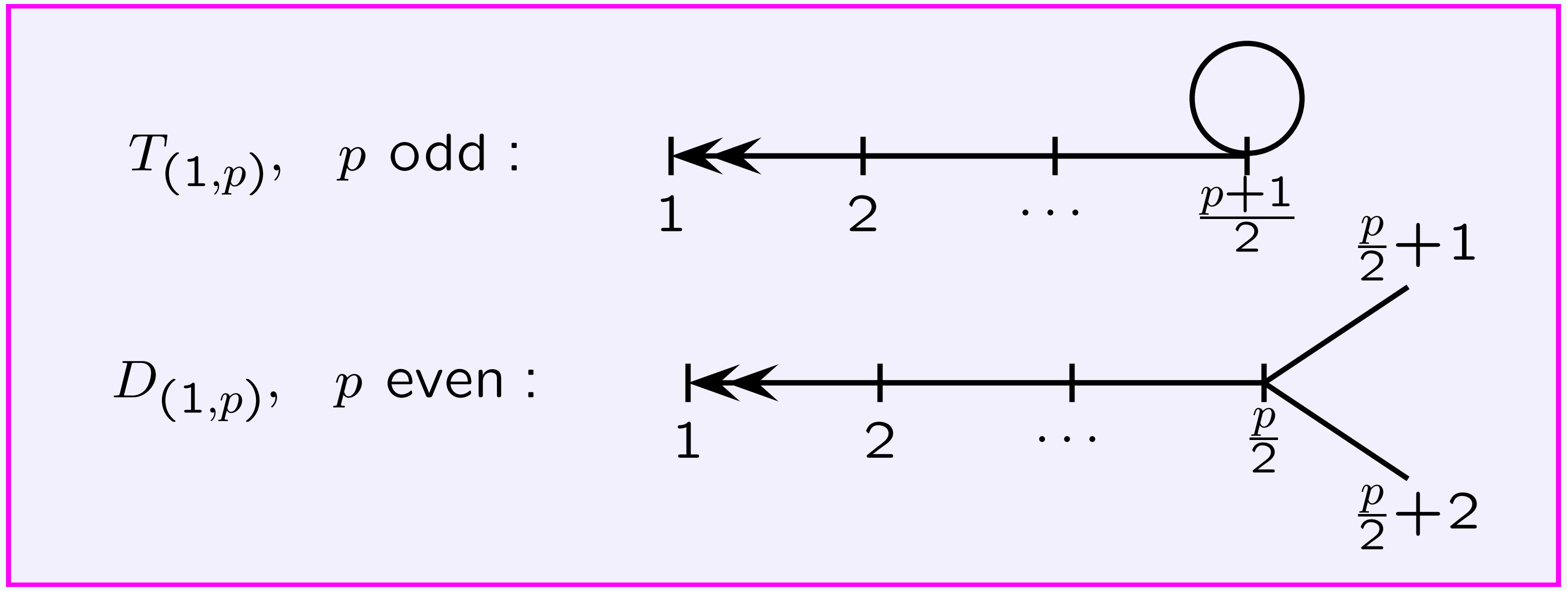


$$N_2 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 4 & 0 & 2 & 2 & 0 & 4 \\ 0 & 1 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 2 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 4 & 0 & 4 & 4 & 0 & 4 \\ 0 & 2 & 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 & 2 & 0 \\ 2 & 0 & 2 & 2 & 0 & 2 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$$= N_2 + N_5$$

A-D-E-T

- A $\mathbb{Z}_2$ folding or orbifold of the $A_{(1,p)}$ graphs gives T or D type graphs:



- Indeed, Feigin et al (2006) have found A, D and E_6 modular invariant sesquilinear forms in the characters $\varkappa_k(q) = \varkappa_{r,s}(q)$.

- This leads to some intriguing open questions:

1. Is there an A-D-E classification of these logarithmic Verlinde fusion graphs a la Behrend, Pearce, Petkova and Zuber?

2. Is there a corresponding A-D-E classification of the logarithmic modular invariant sesquilinear forms a la Cappelli, Itzykson and Zuber?

3. Is there a logarithmic coset construction a la Goddard, Kent and Olive?

4. What are the corresponding D and E logarithmic minimal models on the lattice?

Summary

- Representation Content:

Reps	Dense Polymers/ Symp Fermions	Percolation
Vir	∞	∞
$\mathcal{W}$	6	26
Proj	4	12
Proj Grothendieck	3	6

- Empirical Virasoro fusion rules for $\mathcal{LM}(p,p')$:

Checks:
1. $\mathcal{LM}(p,p')$ fusion rules agree with level-by-level fusion rules of Eberle and Flohr (2006) using the Nahm (1994) algorithm.
2. Vertical sub-fusion algebras agree with Read and Saleur (2007).
3. Associativity.

- Inferred $\mathcal{W}$-algebra fusion rules for $\mathcal{WLM}(p,p')$:

Checks:
1. $\mathcal{WLM}(1,p')$ fusion rules agree with Gaberdiel and Kausch (1996) and Gaberdiel and Runkel (2008).
2. $\mathcal{WLM}(p,p')$ characters agree with Feigin et al (2006).
3. Associativity.

- Projective Grothendieck ring and Verlinde formulas for $\mathcal{PG}(p,p')$:

Checks:
1. Projective characters agree with Feigin et al (2006).
2. Feigin et al modular S matrix diagonalizes our projective Grothendieck fusion rules!
3. Resulting Verlinde formulas and graph fusion algebras are not ugly!!

Chiral Symplectic Fermions (Kausch 1995)

- The central charge of symplectic fermions is $c = -2$ and the stress-energy tensor is

$$T(z) = \sum_{n\in\mathbb{Z}} L_n \, z^{-n-2} = \tfrac{1}{2} d_{\alpha\beta} :\chi^\alpha(z)\chi^\beta(z):$$

where $d_{\alpha\beta}$ is the inverse of the anti-symmetric tensor $d^{\alpha\beta}$ with $\alpha, \beta = \pm$.

- The chiral algebra $\mathcal{W}$ is generated by a two-component fermion field

$$\chi^\alpha(z) = \sum_{n\in\mathbb{Z}} \chi^\alpha_n \, z^{-n-1}, \qquad \alpha = \pm$$

of conformal weight $\Delta = 1$. The modes satisfy the anticommutation relations

$$\{\chi^\alpha_m, \chi^\beta_n\} = m \, d^{\alpha\beta} \, \delta_{m,-n}$$

- Alternatively, the extended symmetry algebra $\mathcal{W}$ is generated by the Virasoro modes L_n and the modes of a triplet of weight 3 fields W^a_n.

Logarithmic Ising and Yang-Lee Kac Tables

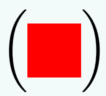
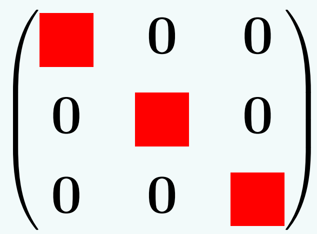
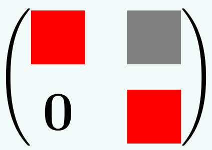
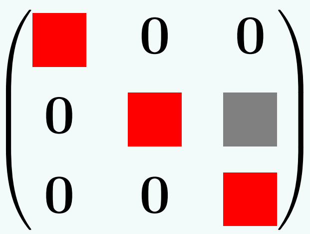
s							
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
10	$\frac{225}{16}$	$\frac{161}{16}$	$\frac{323}{48}$	$\frac{65}{16}$	$\frac{33}{16}$	$\frac{35}{48}$	$\cdots$
9	11	$\frac{15}{2}$	$\frac{14}{3}$	$\frac{5}{2}$	1	$\frac{1}{6}$	$\cdots$
8	$\frac{133}{16}$	$\frac{85}{16}$	$\frac{143}{48}$	$\frac{21}{16}$	$\frac{5}{16}$	$-\frac{1}{48}$	$\cdots$
7	6	$\frac{7}{2}$	$\frac{5}{3}$	$\frac{1}{2}$	0	$\frac{1}{6}$	$\cdots$
6	$\frac{65}{16}$	$\frac{33}{16}$	$\frac{35}{48}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{35}{48}$	$\cdots$
5	$\frac{5}{2}$	1	$\frac{1}{6}$	0	$\frac{1}{2}$	$\frac{5}{3}$	$\cdots$
4	$\frac{21}{16}$	$\frac{5}{16}$	$-\frac{1}{48}$	$\frac{5}{16}$	$\frac{21}{16}$	$\frac{143}{48}$	$\cdots$
3	$\frac{1}{2}$	0	$\frac{1}{6}$	1	$\frac{5}{2}$	$\frac{14}{3}$	$\cdots$
2	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{35}{48}$	$\frac{33}{16}$	$\frac{65}{16}$	$\frac{323}{48}$	$\cdots$
1	0	$\frac{1}{2}$	$\frac{5}{3}$	$\frac{7}{2}$	6	$\frac{55}{6}$	$\cdots$
	1	2	3	4	5	6	r

s							
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
10	$\frac{27}{5}$	$\frac{91}{40}$	$\frac{2}{5}$	$-\frac{9}{40}$	$\frac{2}{5}$	$\frac{91}{40}$	$\cdots$
9	4	$\frac{11}{8}$	0	$-\frac{1}{8}$	1	$\frac{27}{8}$	$\cdots$
8	$\frac{14}{5}$	$\frac{27}{40}$	$-\frac{1}{5}$	$\frac{7}{40}$	$\frac{9}{5}$	$\frac{187}{40}$	$\cdots$
7	$\frac{9}{5}$	$\frac{7}{40}$	$-\frac{1}{5}$	$\frac{27}{40}$	$\frac{14}{5}$	$\frac{247}{40}$	$\cdots$
6	1	$-\frac{1}{8}$	0	$\frac{11}{8}$	4	$\frac{63}{8}$	$\cdots$
5	$\frac{2}{5}$	$-\frac{9}{40}$	$\frac{2}{5}$	$\frac{91}{40}$	$\frac{27}{5}$	$\frac{391}{40}$	$\cdots$
4	0	$-\frac{1}{8}$	1	$\frac{27}{8}$	7	$\frac{95}{8}$	$\cdots$
3	$-\frac{1}{5}$	$\frac{7}{40}$	$\frac{9}{5}$	$\frac{187}{40}$	$\frac{44}{5}$	$\frac{567}{40}$	$\cdots$
2	$-\frac{1}{5}$	$\frac{27}{40}$	$\frac{14}{5}$	$\frac{247}{40}$	$\frac{54}{5}$	$\frac{667}{40}$	$\cdots$
1	0	$\frac{11}{8}$	4	$\frac{63}{8}$	13	$\frac{155}{8}$	$\cdots$
	1	2	3	4	5	6	r

Virasoro Representations and L_0

- In the continuum scaling limit, the transfer matrices give rise to a representation of the Virasoro algebra. Only L_0 is readily accessible from the lattice

$$\boldsymbol{D}(u) \sim e^{-u\mathcal{H}}, \qquad -\mathcal{H} \mapsto L_0 - \frac{c}{24}, \qquad Z_{r,s}(q) = \mathrm{Tr}\,\boldsymbol{D}(u)^P \mapsto q^{-c/24}\,\mathrm{Tr}\,q^{L_0} = \chi_{r,s}(q)$$

Type	Irreducible	Fully Reducible	Reducible yet Indecomposable	Decomposable
L_n				
L_0	Diagonalizable	Diagonalizable	Jordan Cells of Rank ≥ 2	Jordan Cells

- **Rational Theories:**

Irreducible representations are the building blocks for fusion. Fusion closes on the irreducible representations.

- **Logarithmic Theories:**

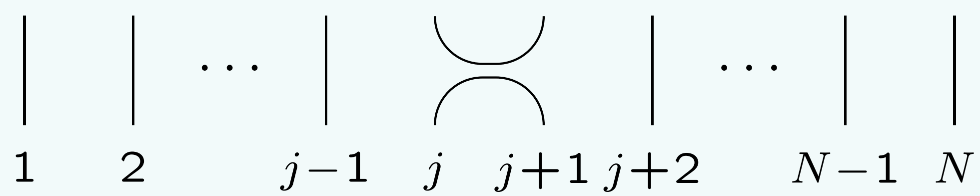
Kac representations are the building blocks for fusion. Higher rank indecomposable representations arise from fusing Kac representations.

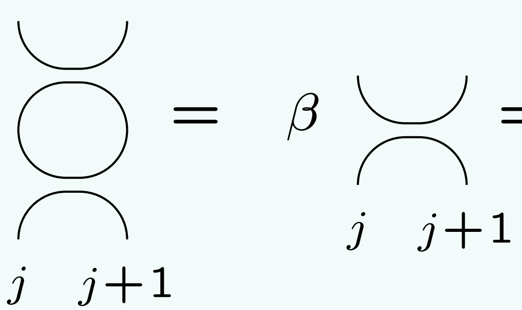
Linear Temperley-Lieb Algebra

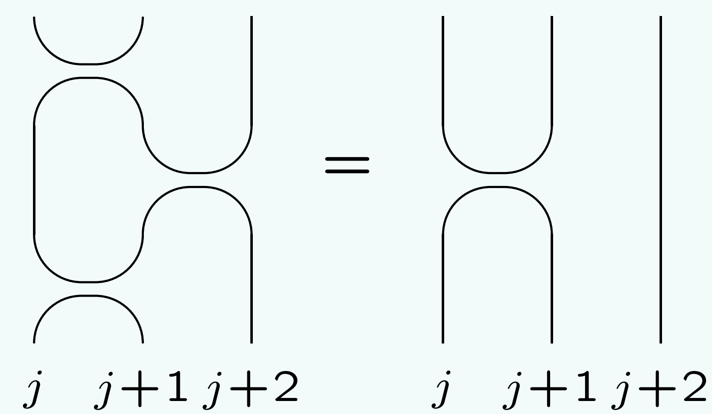
- The linear TL algebra is generated by $e_1, \ldots, e_{N-1}$ and the identity I acting on N strings

$$\begin{cases} e_j^2 = \beta\, e_j, & \\ e_j\, e_k\, e_j = e_j, & |j-k| = 1, \qquad j,k = 1,2,\ldots,N-1; \qquad \beta = 2\cos\lambda \\ e_j\, e_k = e_k\, e_j, & |j-k| > 1 \end{cases}$$

- The TL generators e_j are represented graphically by *monoids*

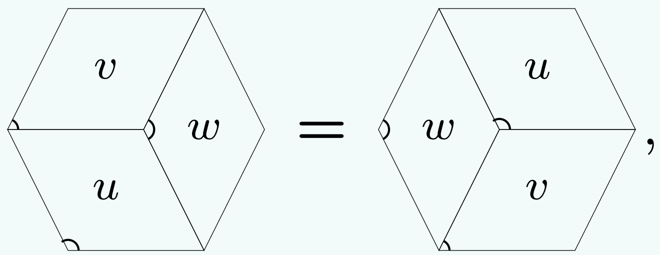
$e_j =$ 

$e_j^2 =$  $= \beta\, e_j,$

$e_j e_{j+1} e_j =$  $= e_j$

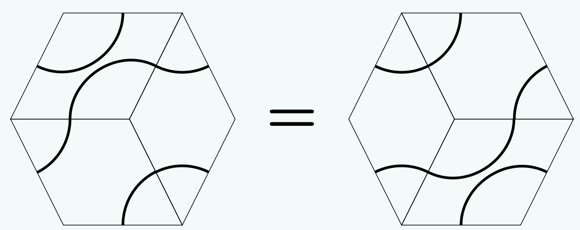
Integrability I: Yang-Baxter Equation (YBE)

- The YBE express the equality of two planar 3-tangles $(w = v - u)$

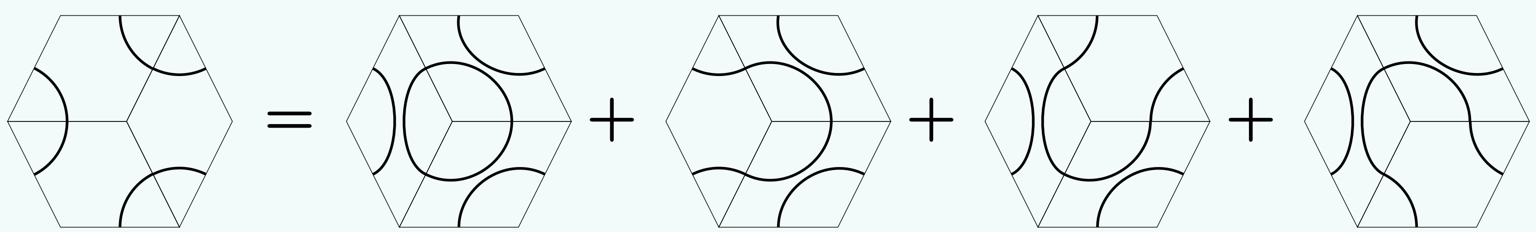


$$X_j(w)X_{j+1}(u)X_j(v) \;=\; X_{j+1}(v)X_j(u)X_{j+1}(w)$$

- The five possible connectivities of the external nodes give the diagrammatic equations



$\times\ 3\quad (120^\circ$ rotations$)$



$\times\ 2\quad (180^\circ$ rotations$)$

- The first equation is trivial. The second equation follows from the identity

$$\begin{aligned} s_1(-u)s_0(v)s_1(-w) \;&=\; \beta\, s_0(u)s_1(-v)s_0(w) \;+\; s_0(u)s_1(-v)s_1(-w) \\ &\quad + \; s_1(-u)s_1(-v)s_0(w) \;+\; s_0(u)s_0(v)s_0(w) \end{aligned}$$

$$s_r(u) = \frac{\sin(u + r\lambda)}{\sin\lambda}, \qquad \beta \;=\; 2\cos\lambda \;=\; \text{loop fugacity}$$