Non-linear dynamics of interacting electronic systems

$$H = -\sum_i \Big(-\partial_i^2 + \sum_{i \neq j} V(x_i - x_j) \Big)$$

$$H_0 = c^\dagger (-\frac{\nabla^2}{2m} - E_F) c$$

$$H = H_0 + \int c^\dagger(x) c(x) V(x-y) c^\dagger(y) c(y) dx dy$$

$$V(x) = \sum_k V_k e^{ikx}$$

$$V_{k\sim 0} \quad V_{k\sim 2k_F}$$

Lattinger liquid

Model Hamiltonian: Elliptic Calogero-Sutherland model

$$H = \sum_i \Big(-\partial_i^2 + \lambda(\lambda-1)\sum_j \wp(x_i - x_j)\Big)$$

$$V(x) = \wp(x) \to \frac{1}{x^2}, \quad \frac{1}{\sinh^2 x}, \qquad \frac{1}{\sin^2 x}, \qquad \nabla\delta(x)$$

Interpolates between Lattinger liquid and Calogero model - quantum wires, edge states of FQHE

The last and the major unsolved integrable model

Free fermions:

$$\boxed{\dot{\varphi} + (\partial_x \varphi)^2 = 0}$$

$$\rho(x) = -\nabla_x \varphi = \rho_0 + \sum_k a_k e^{ikx}$$

$$[a_k, a_{k'}] = k\delta_{k+k'}, \quad \rho_k = \rho^\dagger_{-k}$$

$$H = \sum_i \Big(-\partial_i^2 + \lambda(\lambda-1) \sum_j \wp(x_i - x_j) \Big)$$

$$\boxed{\dot{\varphi} = \frac{1}{2}(\partial_x \varphi)^2 + \partial_x^2 \tilde{\varphi}}$$

chiral sector

$$\tilde{\varphi} = \frac{1}{\pi L} \int \cot \frac{x - x'}{L} \varphi(x') dx'$$

$$\wp(x + iL) = \wp(x)$$

Chiral sector - long-time asymptotes of non-linear waves

$$\boxed{\dot{\varphi} = \frac{1}{2}(\partial_x \varphi)^2 + (\lambda - 1)\partial^2 \tilde{\varphi}}$$

ILW-equation

$$\tilde{\varphi} = \frac{1}{\pi L} \int \cot \frac{x - x'}{L} \varphi(x') dx'$$

$$L \to 0: \quad \dot{\varphi} = \frac{1}{2}(\partial_x \varphi)^2 + \partial_x^3 \varphi$$

KdV-equation

$$L \to \infty: \quad \dot{\varphi} = \frac{1}{2}(\partial_x \varphi)^2 + \partial_x^2 \int \frac{\varphi(x')}{x - x'} dx'$$

Benjamin-Ono equation

On the relation between Calogero model and CFT

$$\dot{\varphi} = T_{xy}$$

$$T_{xy} = (\partial_x \varphi)^2 + \alpha_0 \partial_x^2 \tilde{\varphi}$$

Flux of energy through the boundary

$$\alpha_0 = \sqrt{\lambda} - 1/\sqrt{\lambda}$$

$$\frac{\lambda(\lambda - 1)}{(x_i - x_j)^2}$$

Period of oscillations is

$$(\text{interaction}) \times (\delta\rho)^{-1} >> k_F^{-1}$$

Quantum Non-linear Equations can be treated semiclassically

Trigonometric Calogero-Sutherland model

$$H = -\sum_i \partial_i^2 + \sum_j \frac{\lambda(\lambda-1)}{\sin^2(x_i - x_j)}$$

Model for edge states of the F

$$H\Psi = E\Psi$$

$$\Psi(x_1,\ldots,x_N) = \prod_{i>j}(e^{ix_i} - e^{ix_j})^\lambda J_Y(x_1,\ldots,x_N)$$

Jack symmetric polynomial

$$\Psi(\ldots x_i \ldots x_j \ldots) = e^{2\pi i\lambda}\Psi(\ldots x_j \ldots x_i \ldots)$$

$\lambda = 0 \ - \ \text{bosons}$ $\lambda = 1 \ - \ \text{fermions}$

$$H = -\sum_i \partial_i^2 + \sum_j \frac{\lambda(\lambda-1)}{\sin^2(x_i - x_j)}$$

$$\dot{\varphi} = \frac{1}{2}(\partial_x \varphi)^2 + (\lambda - 1)\partial^2 \tilde{\varphi}$$

$$\varphi(x) = \sum_k \frac{e^{ikx}}{k} a_k, \qquad [a_k,\ a_{-k'}] = \lambda k \delta_{kk'}$$

$$a_{k>0} = \sum_i e^{ikx_i}$$

Properties of Benjamin-Ono Equation

$$\dot{\varphi} + (\partial_x \varphi)^2 + \nu \partial_x^2 \varphi^H = 0$$

Properties:

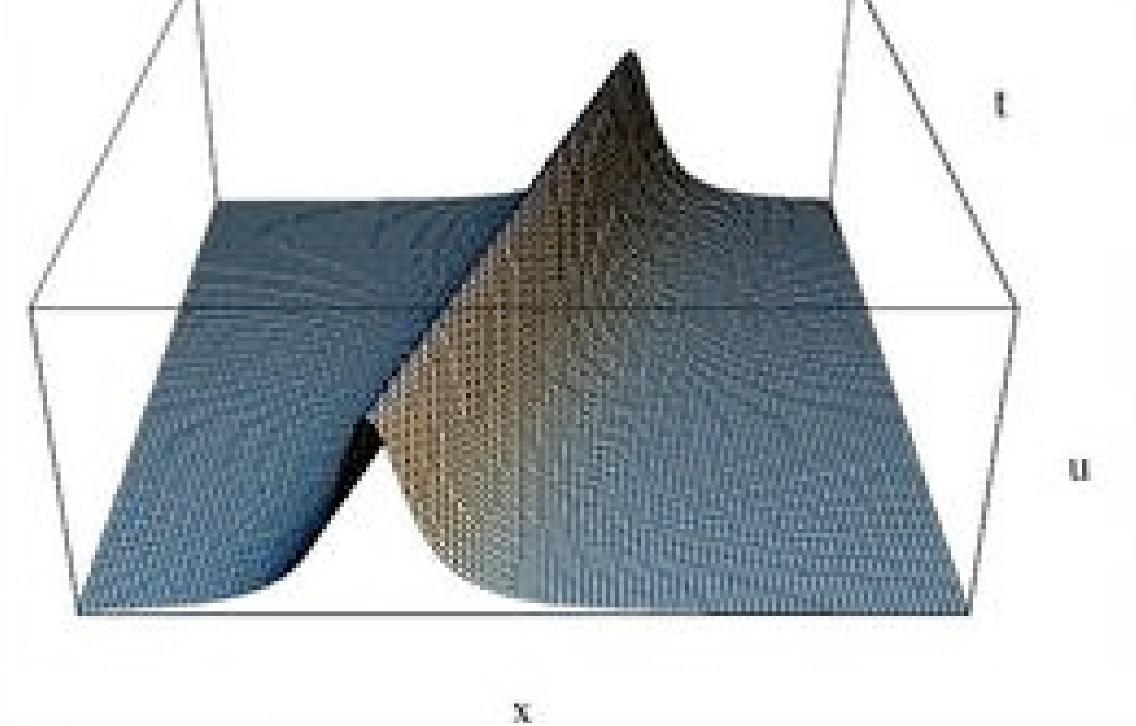
1) Integrable (despite being non-local);

2) Its solitons carry a quantized fractional charge:

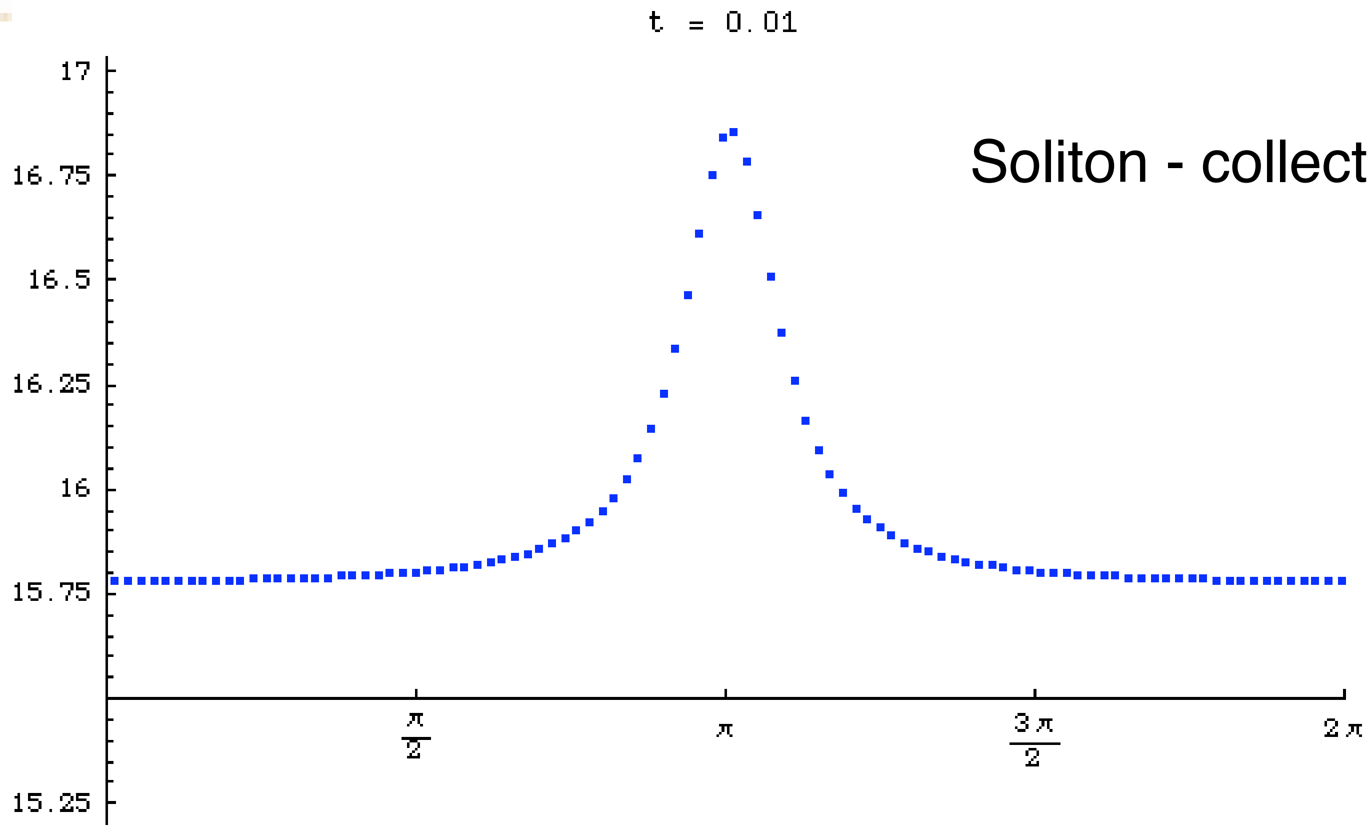
$$\int \rho dx = \int d\varphi = \text{integer} \times \nu$$

3) Solitons have Lorentzian shape:

$$\rho_s(x,t) = \frac{1}{\pi} \frac{v\nu}{v^2(x - vt)^2 + \nu^2}$$



$$u(x,t) = \frac{2V}{V^2(x - x_0 - Vt)^2 + 1}$$

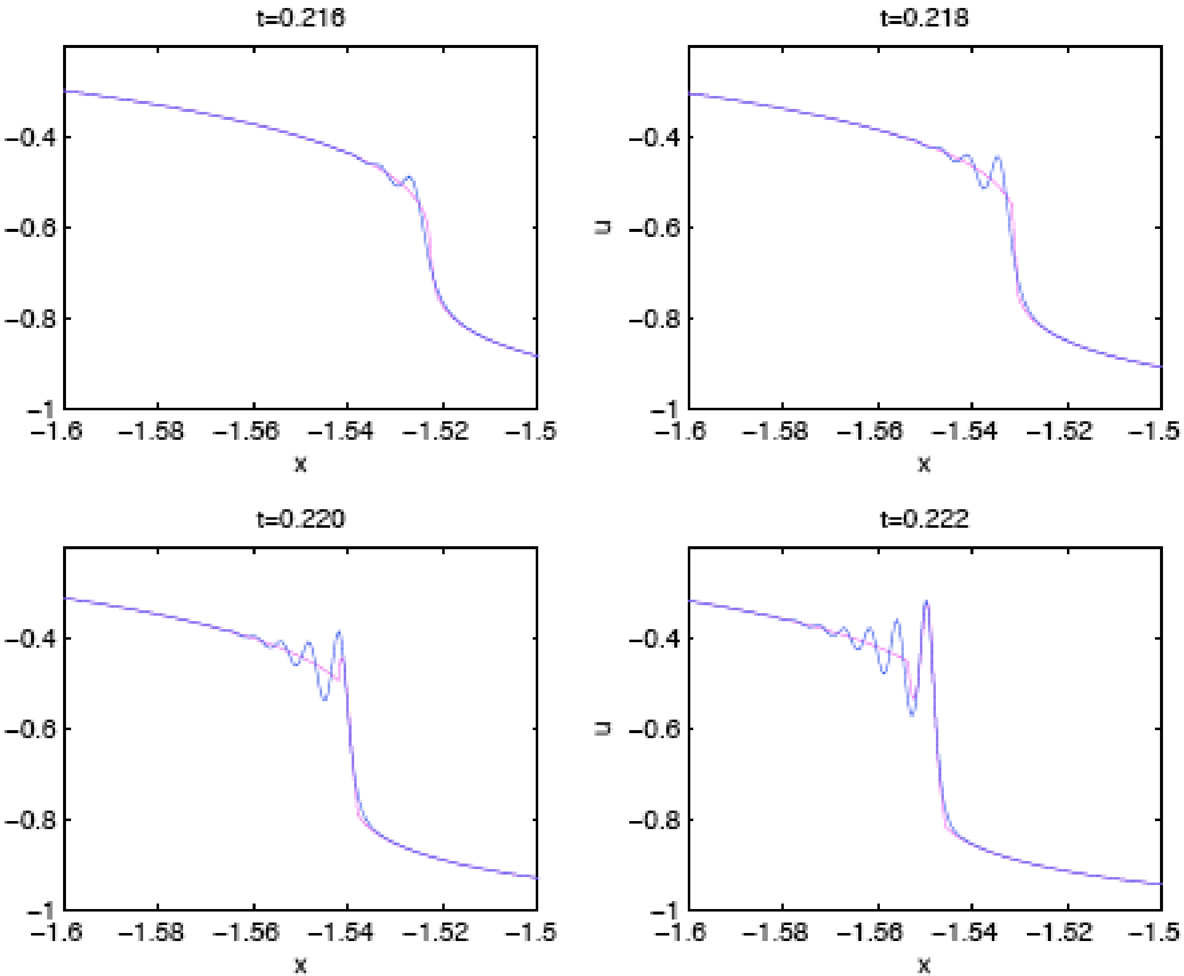


Soliton - collective excitation of particles

Shock wave:

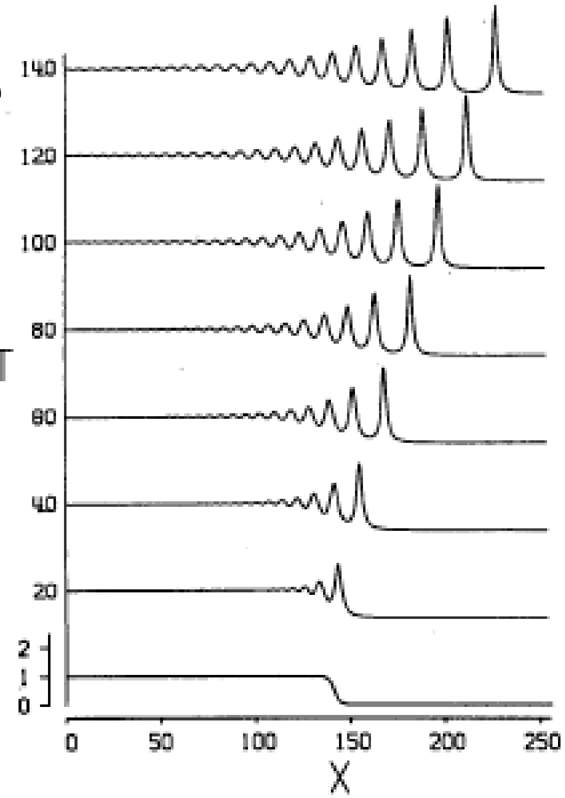
competition between non-linear term and dispersion term

$$\dot{\varphi} + (\partial_x \varphi)^2 + \nu \partial_x^2 \varphi^H = 0$$



Long time asymptote: **Soliton train**

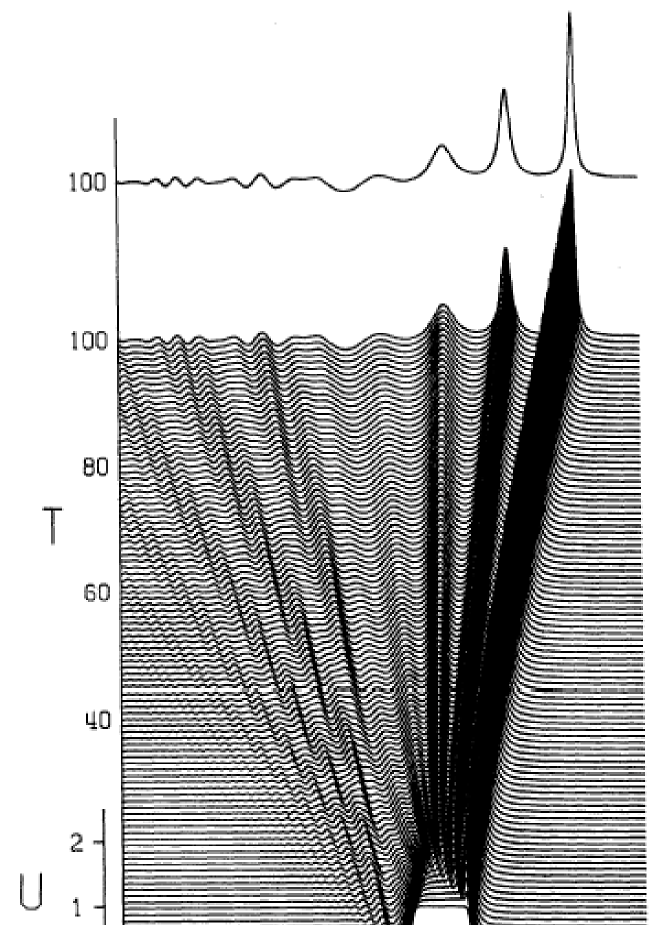
time

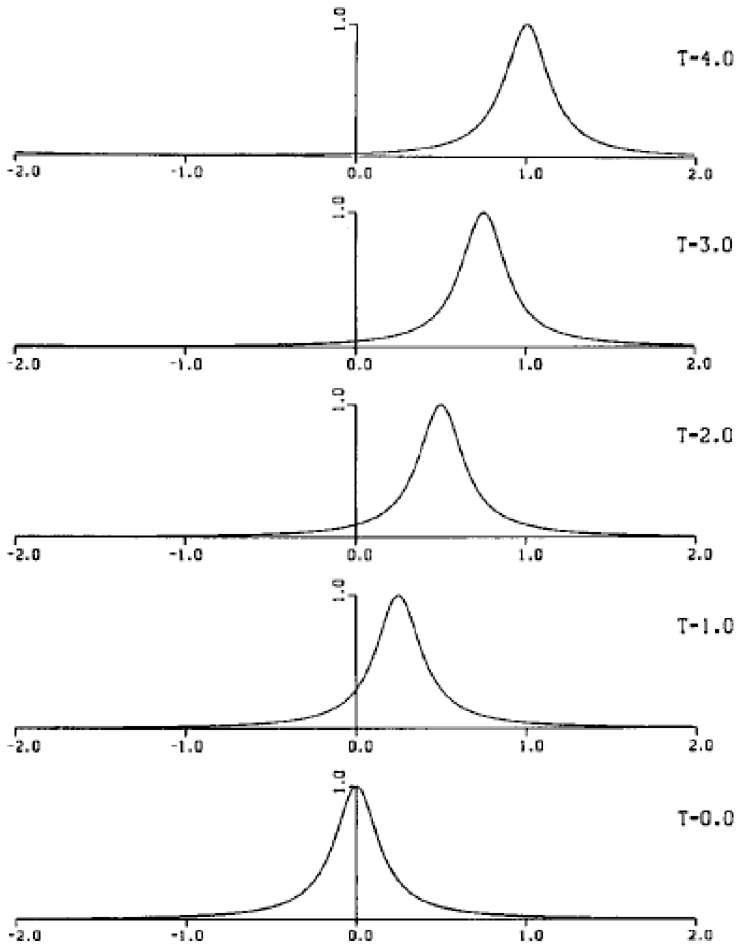


space

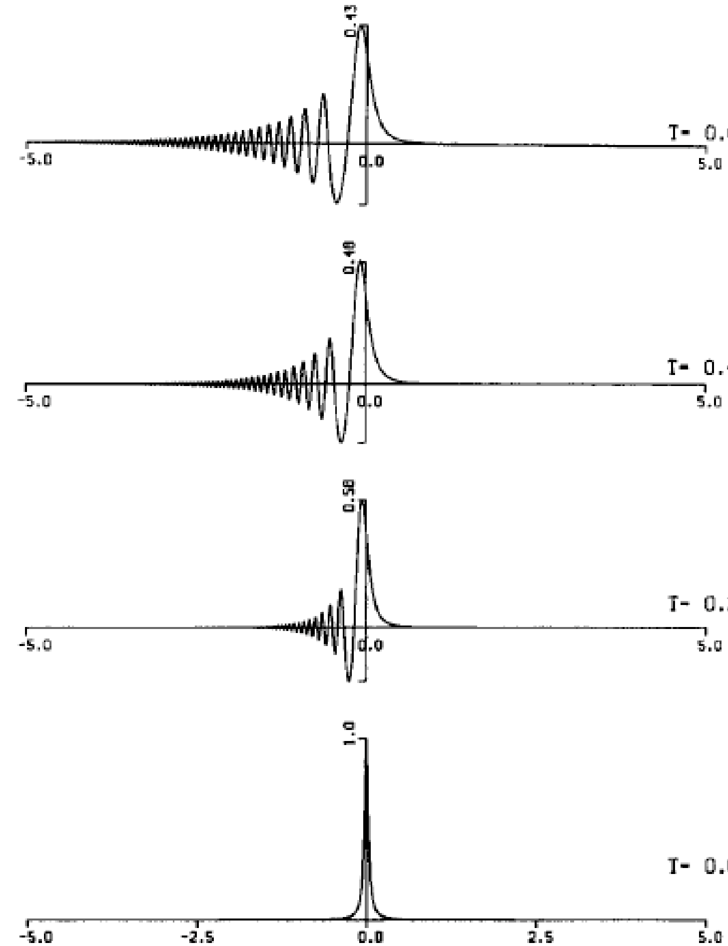
$$x_+ \sim u_0 t, \quad x_- \sim 3 \times 2^{-2/3}(u_0 t)^{1/3}$$

Soliton Train

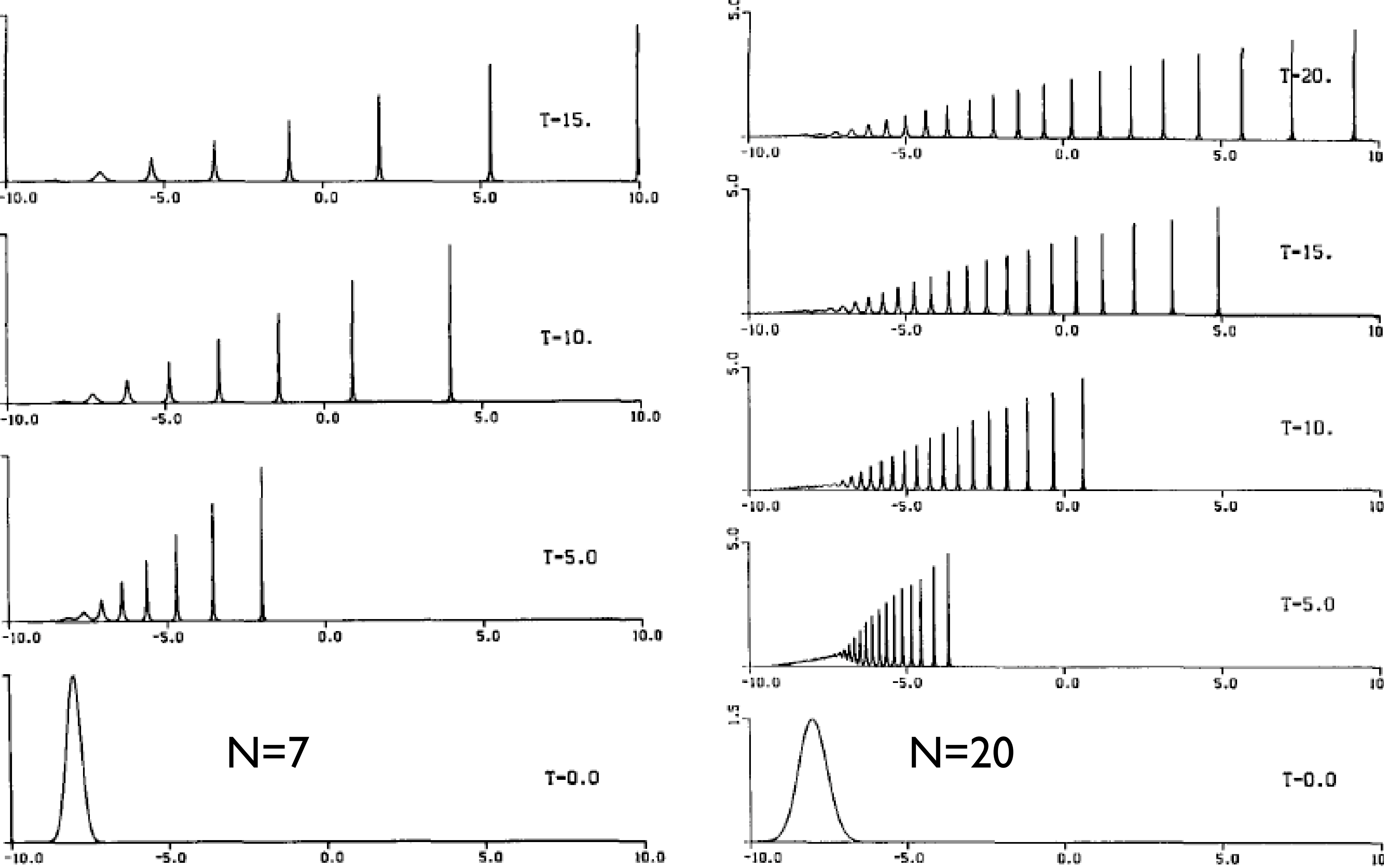




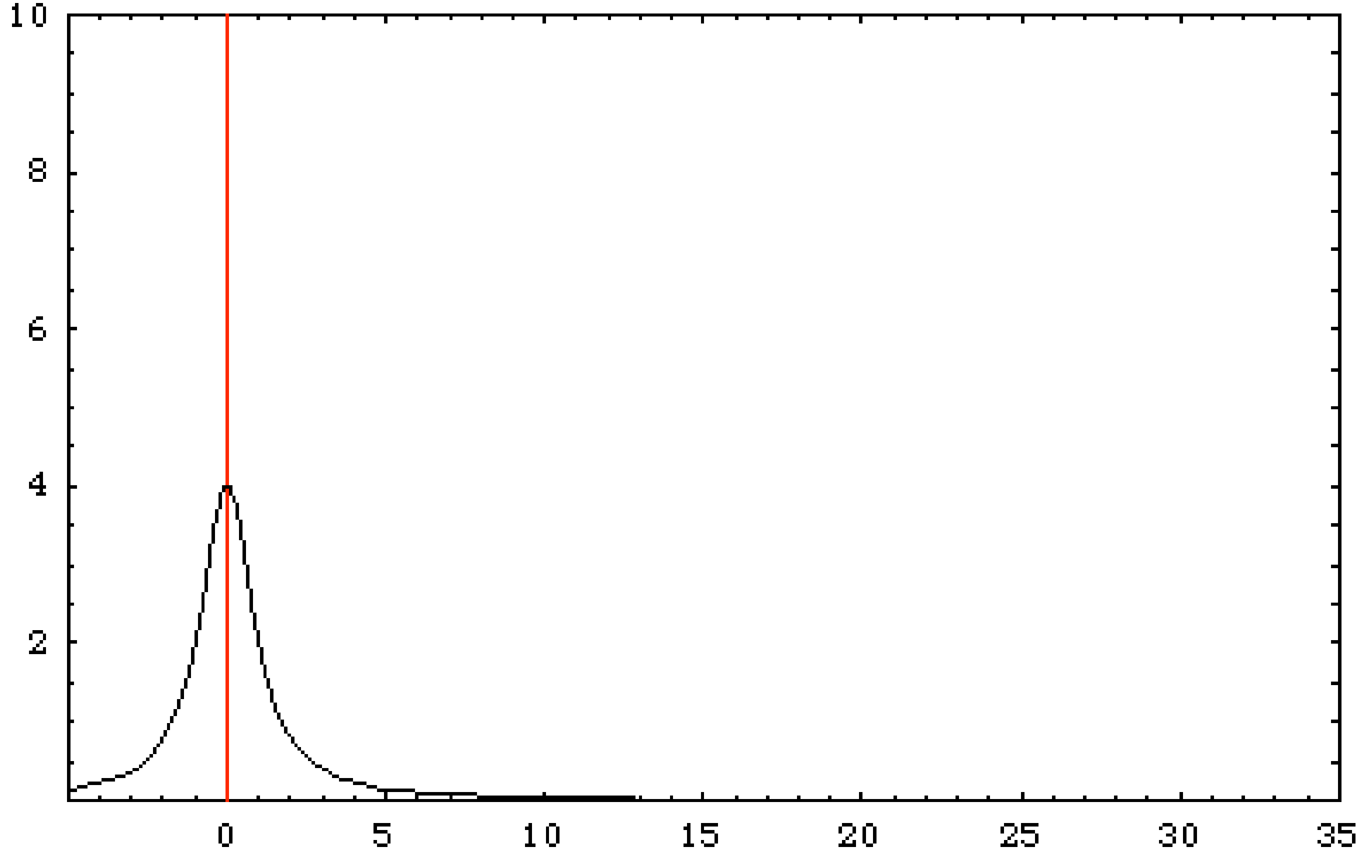
A single soliton
(area is 1/3)

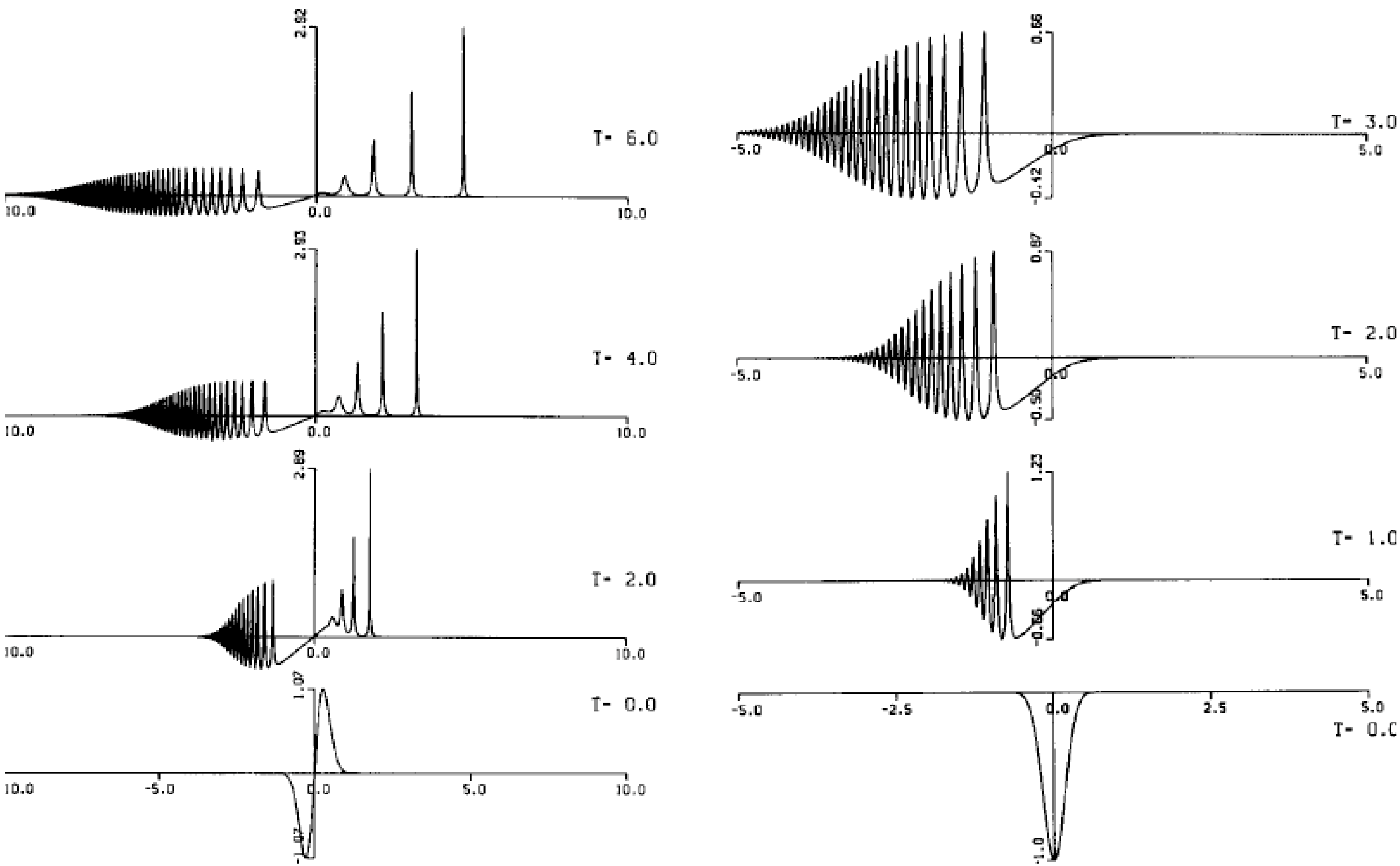


1/4 of quantum
(area is 1/12)



Soliton trains


10
8
6
4
2
0
5
10
15
20
25
30
35



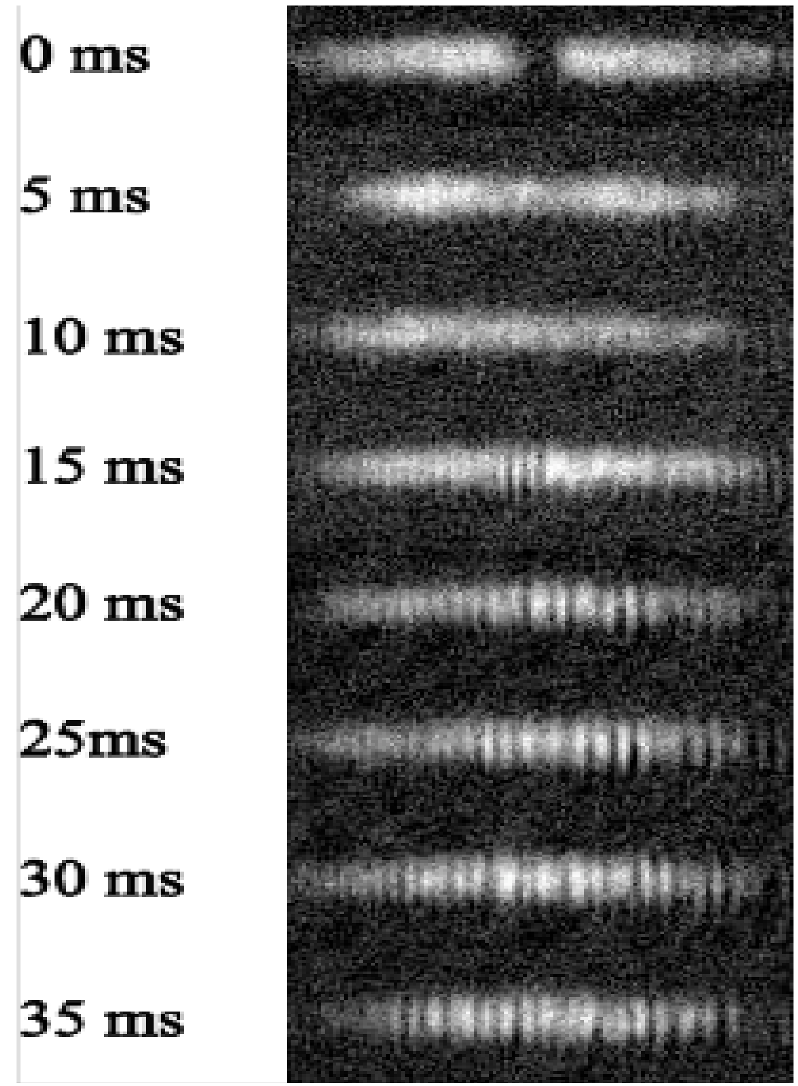
Separation between hole (moving right) and particles (moving left)

Conclusions:

1) Dynamics of the edge state
is essentially non-linear;

2) Solitons of non-linear dynamics
carry fractional charge;

3) A propagation of any front evolves
to a shock wave and further in
a fractionally quantized soliton train.



Quantum shocks
in BEC

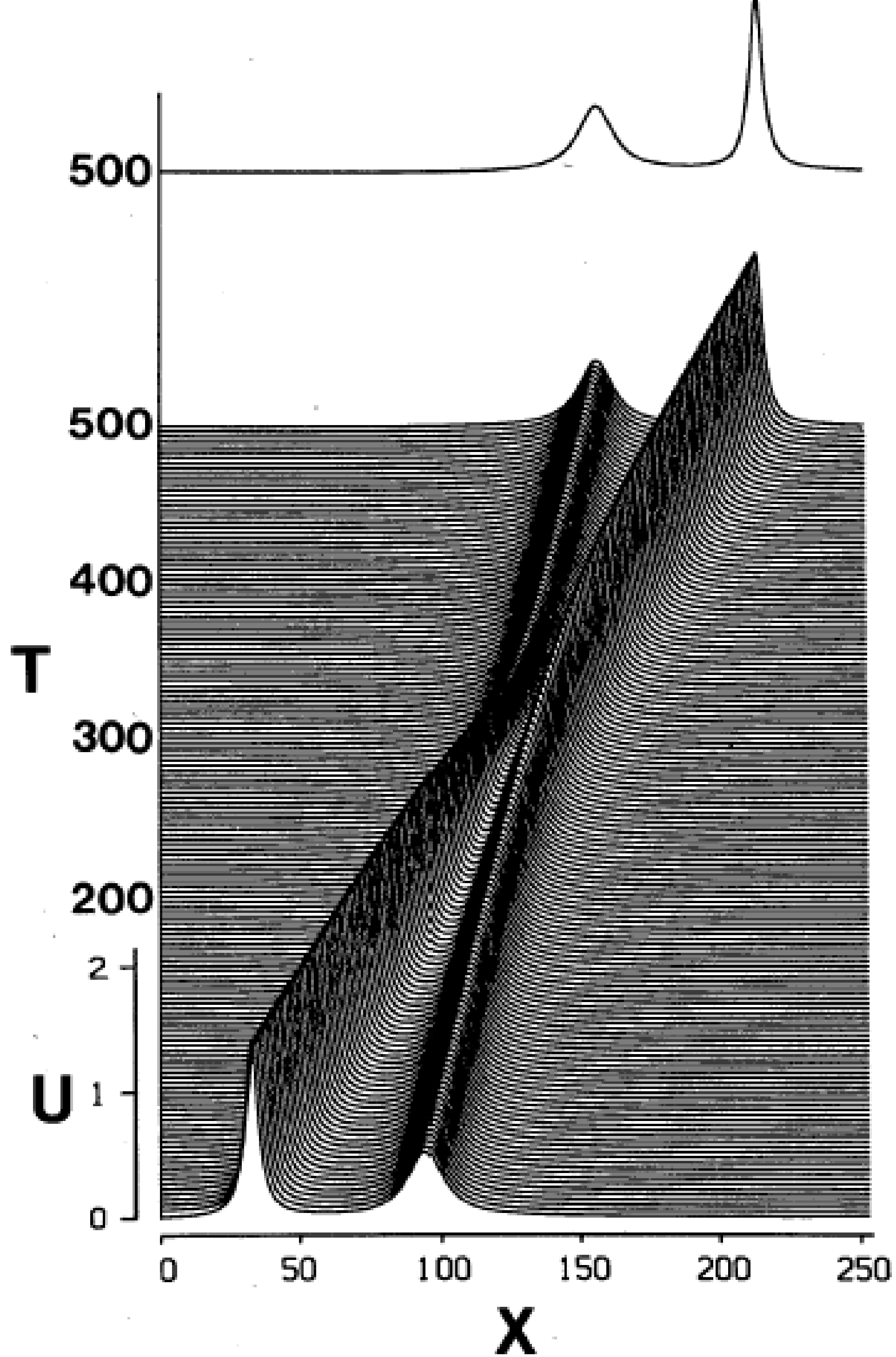


FIG. 6. Interaction of solitary waves governed by the Benjamin–Davis–Ono equation. The last trace has been reproduced separately for clarity. The coordinate system in this illustration and in all subsequent illustrations moves at the critical or linear phase speed. In this frame of reference, solitary waves propagate to the right and subcritical dispersive wave components propagate to the left. Units are nondimensional.

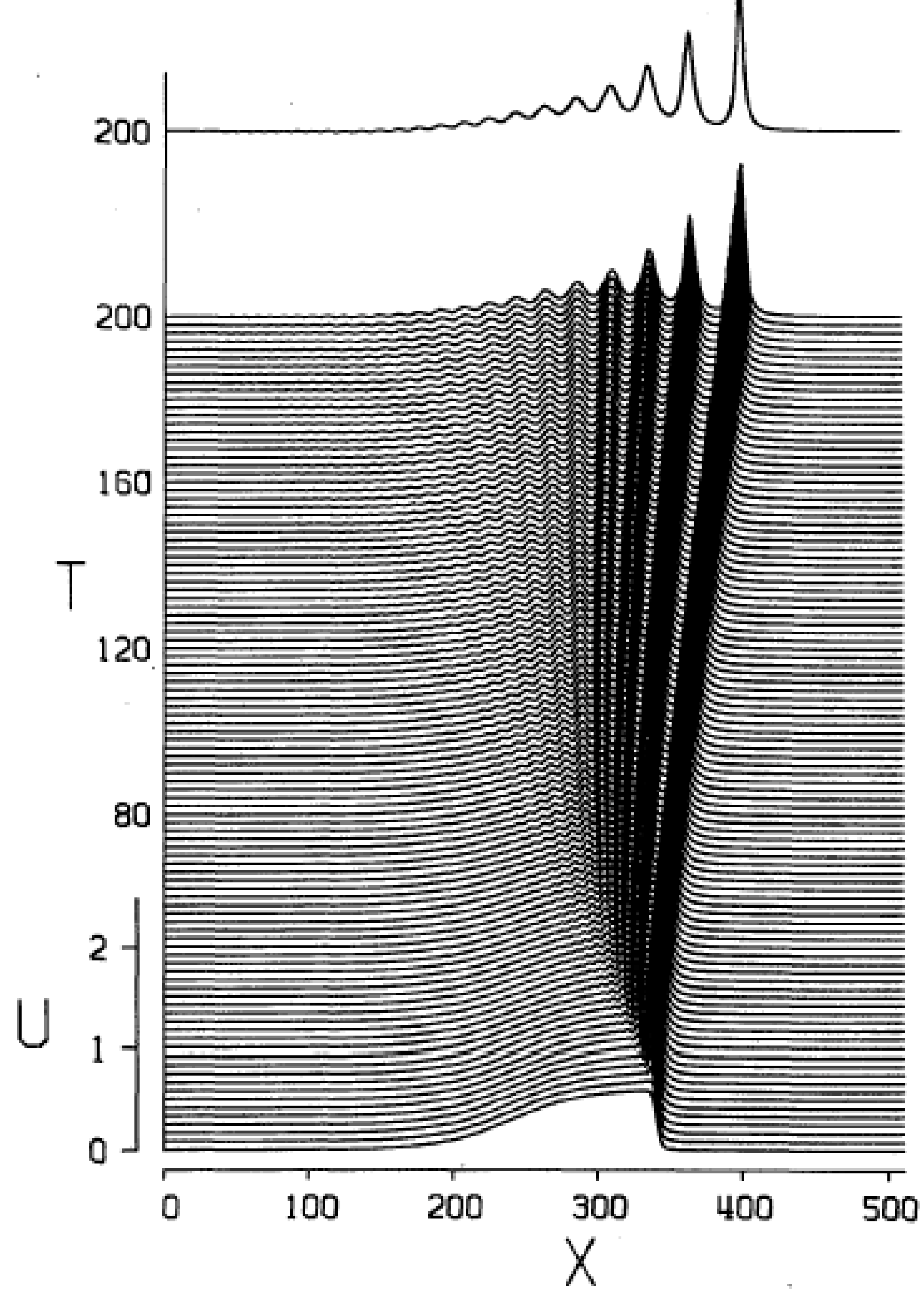


FIG. 11. Solution of the BDO equation for an initial model disturbance for atmospheric waves in the form of a long, but finite-length, wave of elevation. The inversion height behind the leading edge of the disturbance at $T = 0$ decreases slowly, but continuously, to the ambient inversion level.

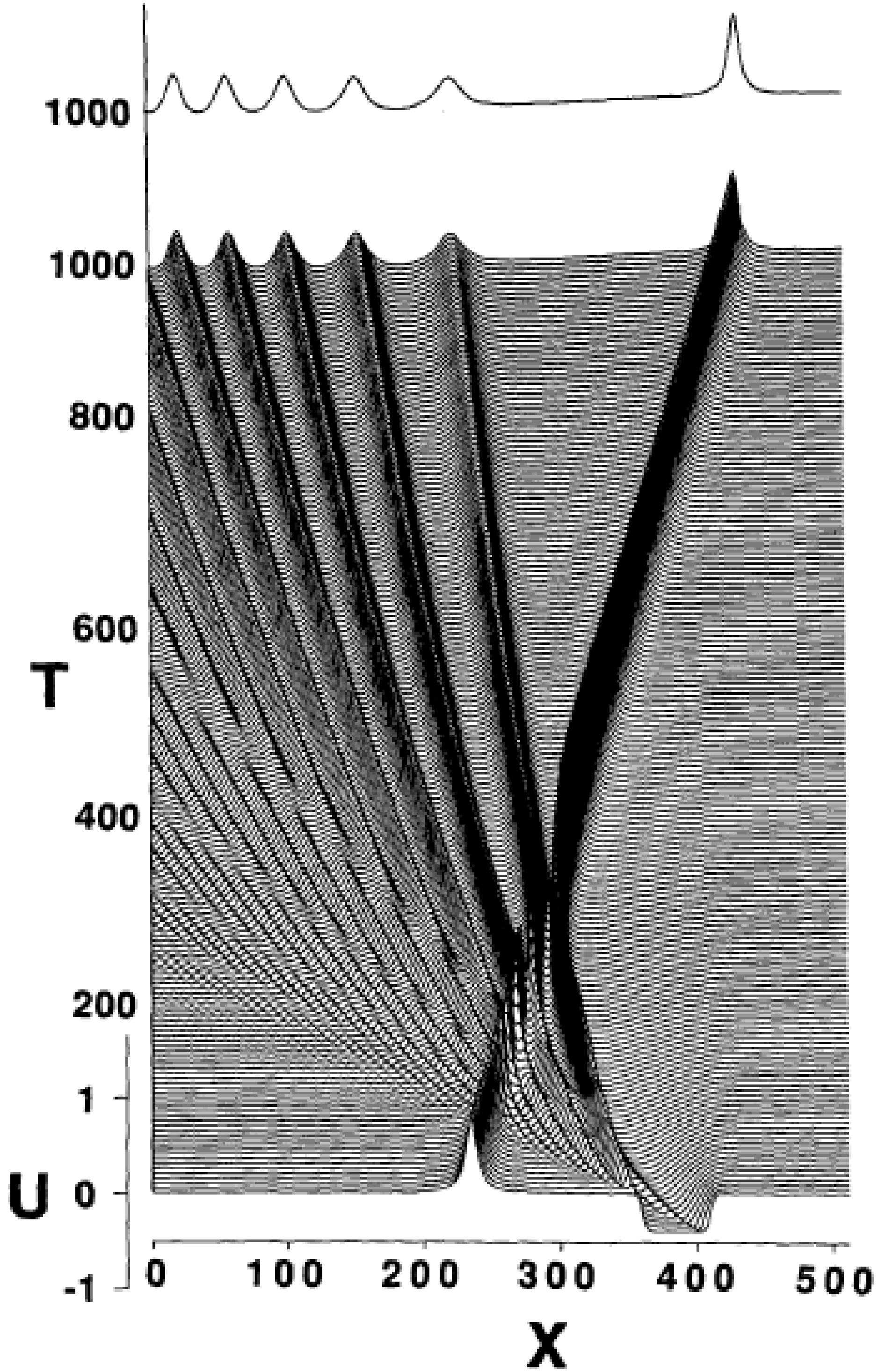


FIG. 7. Numerical integration of the Benjamin–Davis–Ono equation illustrating the stability of a solitary wave under strongly nonlinear interaction with large amplitude subcritical dispersive waves created in the evolution of an initially smooth long wave of depression.

Quantum Hydrodynamics of Calogero-Sutherland model

$$\dot{x}_i = p_i \qquad \dot{p}_i = \sum_j \frac{\lambda(\lambda-1)}{(x_i - x_j)^3}$$

$$p_i = \sum_k \frac{\lambda}{x_i - y_k} - \sum_k \frac{\lambda}{x_i - x_k}$$

$$\dot{x}_i = \sum_k \frac{\lambda}{x_i - y_k} - \sum_k \frac{\lambda}{x_i - x_k}$$

$$-\dot{y}_i = \sum_k \frac{\lambda}{y_i - x_k} - \sum_k \frac{\lambda}{y_i - y_k}$$

$$\dot{x}_i = \sum_k \frac{\lambda}{x_i - y_k} - \sum_k \frac{\lambda}{x_i - x_k}$$

$$-\dot{y}_i = \sum_k \frac{\lambda}{y_i - x_k} - \sum_k \frac{\lambda}{y_i - y_k}$$

$$\varphi(z) = \lambda \sum_i \log(z - x_k) + \lambda \sum_i \log(z - y_k)$$

$$\tilde{\varphi}(z) = \lambda \sum_i \log(z - x_k) - \lambda \sum_i \log(z - y_k)$$

$$\dot{\varphi} = \frac{1}{2}(\partial_x \varphi)^2 + (\lambda - 1)\partial^2 \tilde{\varphi}$$

Density and velocity

$$\partial_x \Big(\varphi(x + i0) - \varphi(x - i0) \Big) = -2\lambda\pi\rho(x)$$

$$\partial_x \Big(\varphi(x + i0) + \varphi(x - i0) \Big) = v - 2i\lambda\partial_x \log \rho$$

$$\dot{\rho} + \partial_x(\rho v) = 0$$

$$\dot{\rho} + \partial_x(\rho v) = 0$$

$$\dot{v} + \partial_x\left(\frac{v^2}{2} + w(\rho)\right) = 0$$

$$w(\rho) = \frac{\lambda^2\pi^2}{2}\rho^2 - \frac{\lambda(\lambda-1)}{2}\frac{1}{\sqrt{\rho}}\partial_x^2\sqrt{\rho} + \pi\lambda(\lambda-1)\partial_x\rho^H$$

$$\rho^H(x) = \int \coth(x-x')\rho(x')dx'$$

Chiral reduction

$$v = \lambda\Big(\pi\rho + \partial_x(\log\sqrt{\rho})^H\Big)$$

two equations become one

Chiral non-linear equation

$$\boxed{\rho_t + \lambda\partial_x\Big(\pi\rho^2 + \rho\partial_x(\log\sqrt{\rho})^H\Big) = 0}$$

$$\rho \approx \rho_0 + u + \dots$$

$$\dot{u} + uu_x + (\lambda - 1)\tilde{u}_{xx} = 0$$

Benjamin-On equation