Finite temperature density matrix elements

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in collaboration with

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Many-body system I

- Free Fermion one body Green's function

$$\langle a^\dagger(x)a(0)\rangle \sim \int dk \frac{e^{ikx}}{1+e^{\beta\varepsilon_0(k)}}$$

$\varepsilon_0(k)$ bare 1 body energy(– chemical potential)

- Interacting theory (perturbation theory)

$$\langle a^\dagger(x)a(0)\rangle \sim \sum_n \int f(k_1,\cdots,k_n)e^{ik_1x}\prod_i^n dk_i \frac{1}{1+e^{\beta\varepsilon_0(k_i)}}$$

Wick theorem is vital.

Many-body system II

- Interacting theory (Form Factors)
  1 pt function (Mussardo-LeClair, hep-th/9902075) ;

$$\langle O \rangle \sim \sum_n \int f_{2n}(\theta_1, \cdots, \theta_n) \prod_i^n d\theta_i \frac{1}{1 + \mathrm{e}^{\beta \varepsilon(\theta_i)}}$$

1. ε is not bare but solves TBA .
2. T dependency comes from $\beta\varepsilon$, not from f_{2n}.

(See also Takacs, e.g., arXiv.0804.4096)

c.f. g factors (Dorey et al (hepth/0404014))

$$\ln g_\alpha(\ell) = \int \Theta(\theta) \ln(1 + \mathrm{e}^{-\varepsilon_a(\theta)}) d\theta + \sum_n \int \Phi(\theta_1, \cdots, \theta_n) \prod_i^n \frac{d\theta_i}{1 + \mathrm{e}^{\beta \varepsilon(\theta_i)}}$$

Motivation of the present study

1. Any insight from the recent progress on the study of correlation functions at $T = 0$?
   - Jimbo-Miwa (hep-th 9601135) qKZ equation $T = h = 0$
   - Lyon group (hep-th/0201045) $T = 0$, arbitrary h
   - Boos et al (hep-th/0104008) Factorization of multiple integral formula by , Takahashi group (cond-mat/0302564)
   - Boos et al (hep-th/0412191) reduced qKZ approach
2. Any physically interesting observations?

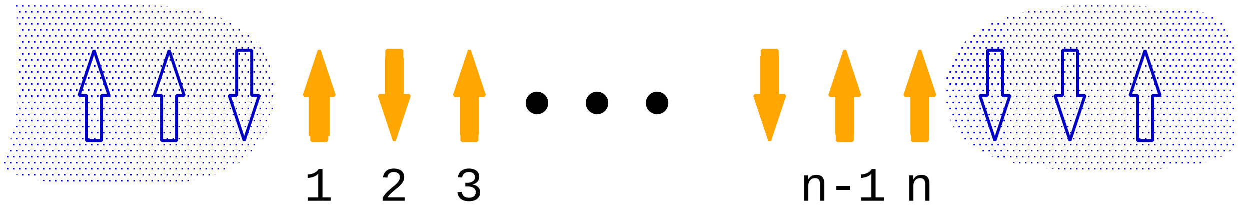
Outline of the talk

1. Density matrix elements $T > 0$ - QTM, multiple integral representation.
2. some observation of factorization.
3. Adjoint action and Grassmanian variables
4. Quantum-Classical cross-over $-1 < \Delta < 0$
5. strange lumps $0 < \Delta < 1, h \neq 0$
6. summary and future problems

Definition of the density matrix

$$H = J \sum_i \left(\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \Delta (\sigma_i^z \sigma_{i+1}^z - 1) \right)$$

- density matrix of a finite segment n



$$D_n = \sum_{\{\alpha\},\{\beta\}} E_{\alpha_1}^{\beta_1} \otimes E_{\alpha_2}^{\beta_2} \otimes \cdots \otimes E_{\alpha_n}^{\beta_n} \left(D_n \right)_{\beta_1,\cdots,\beta_n}^{\alpha_1,\cdots,\alpha_n}$$

$$\left(D_n \right)_{\beta_1,\cdots,\beta_n}^{\alpha_1,\cdots,\alpha_n} := \langle E_{\beta_1}^{\alpha_1} E_{\beta_2}^{\alpha_2} \cdots E_{\beta_n}^{\alpha_n} \rangle$$

Quantum Transfer Matrix

• QTM (M. Suzuki, Inoue, Koma, JS-Akutsu-Wadati, Klümper etc)
Map :

1D quantum system at $\beta = \frac{1}{k_B T}$ with size $L \to \infty$

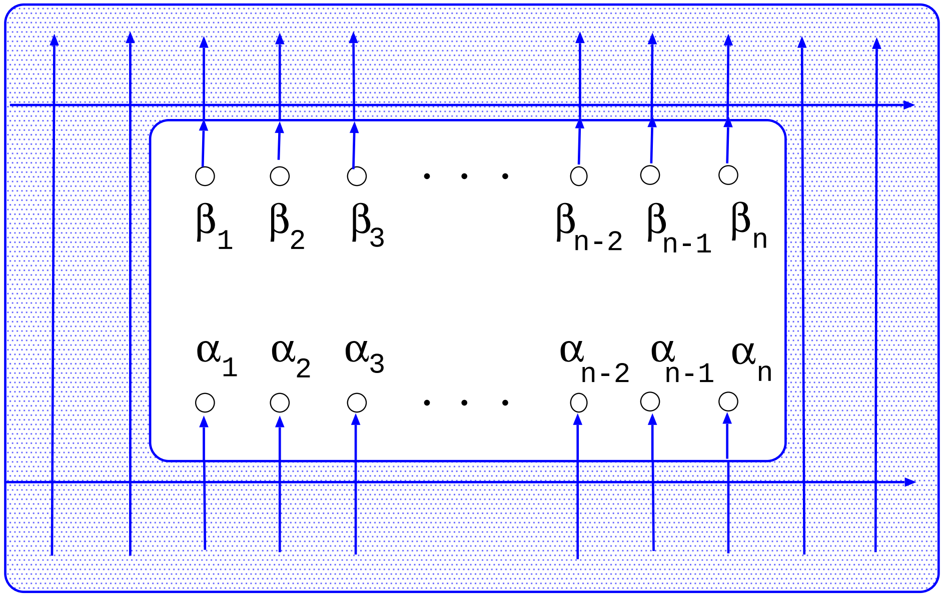
$$\Updownarrow$$

2D classical system, size $L \times N$ (both ∞)   with (inhomogeneous) spectral parameter $u = \frac{J\beta}{N}$

$$\lim_{L} \ln Z_{1\mathrm{D}}(\beta) \sim \lim_{L} \ln \mathrm{tr} e^{-\beta H_{1\mathrm{D}}} \sim \lim_{L,N} \ln Z_{2\mathrm{D}}(u)$$

D_n picture 1

The density matrix is represented graphically,



$\xi_i \to 0$: inhomogeneities in the spectral parameter
or by wrapping around

D_n picture 1

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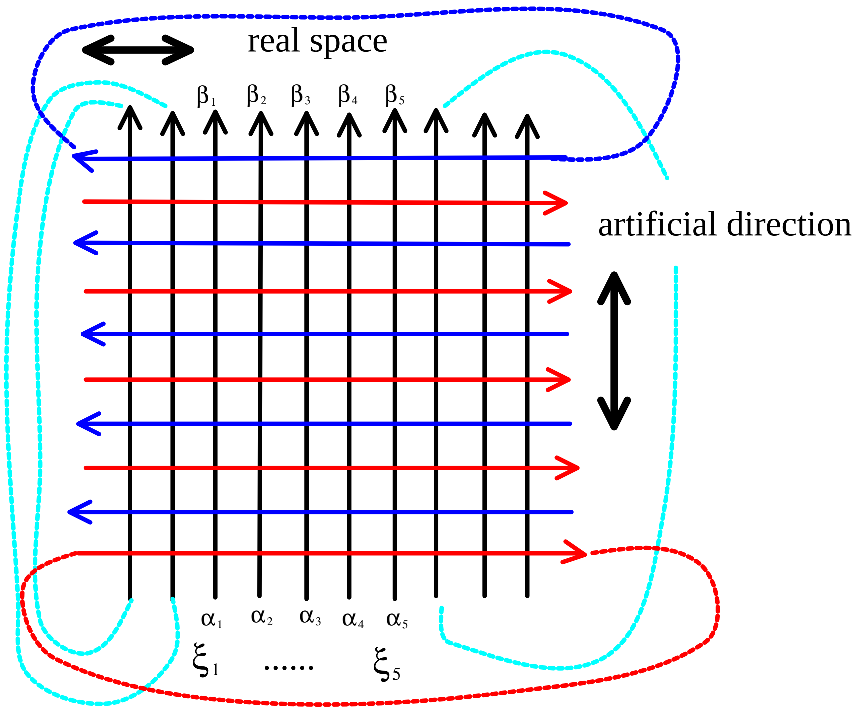
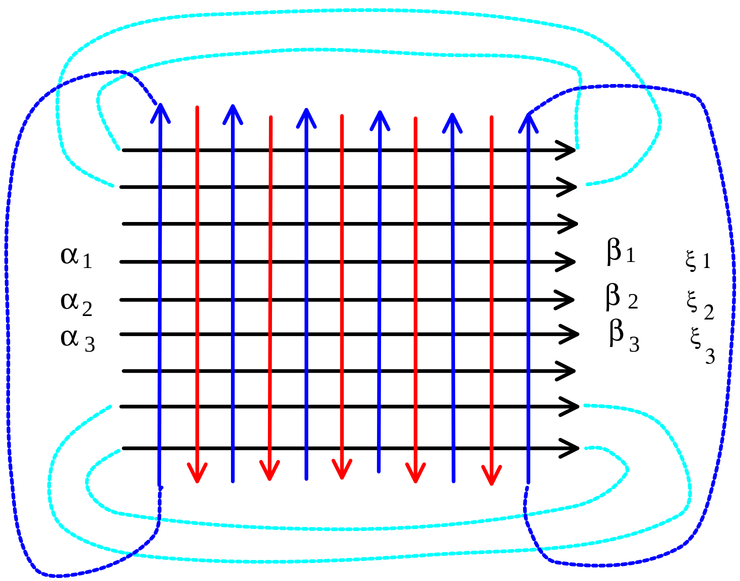


D_n picture II

If you rotate 90°, there is a gap in the spectrum of the eigenvalue of the (quantum) transfer matrix (M. Suzuki). Only needs the largest eigenvalue for the free energy: no summation needed. The limit $N \to \infty$ needs care as u depends on aritificial system size.



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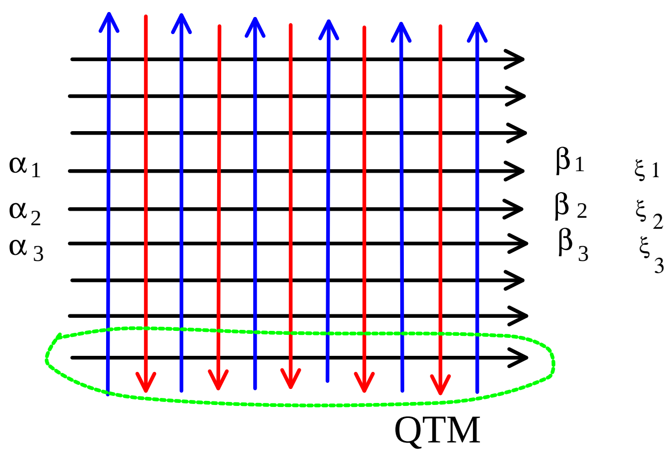

This representation is also useful for D_n.

D_n in terms of QTM

The Quantum Transfer Matrix formulation yields

$$\Big(D\Big)^{\alpha_1,\cdots,\alpha_n}_{\beta_1,\cdots,\beta_n}(\xi_1,\cdots,\xi_n) = \frac{\langle\{\mu\}|T^{\alpha_1}_{\beta_1}(\xi_1)\cdots T^{\alpha_n}_{\beta_n}(\xi_n)|\{\mu\}\rangle}{\langle\{\mu\}|t(\xi_1)\cdots t(\xi_n)|\{\mu\}\rangle}$$

where

1. $T^{\alpha}_{\beta}(\xi)$: the (α,β) element of QTM (ξ_j inhomogeneity)
2. $t(\xi) = \sum_\alpha T^\alpha_\alpha(\xi)$
3. $|\{\mu\}\rangle$ the largest eigenvalue state of QTM

D_n in terms of QTM II

$|\{\mu\}\rangle = B(\mu_1)\cdots B(\mu_m)|\text{vac}\rangle$ $\qquad\qquad$ $\{\mu_i\}$ Bethe ansatz roots

example:

$$D^{-+}_{+-}(\lambda_1,\lambda_2) \sim \langle\text{vac}|C(\mu_1)\cdots C(\mu_m)B(\xi_1)C(\xi_2)B(\mu_1)\cdots B(\mu_m)|\text{vac}\rangle$$

On the other hand, by Slavnov's formula

$$\langle\text{vac}|C(\mu_1)\cdots C(\mu_m)B(\mu'_1)\cdots B(\mu'_m)|\text{vac}\rangle = \det m \times m$$

known explicitly. (μ'_i not necessary BAE roots)

The standard QISM (or Faddeev-Zamolodchikov) algebra
$\rightarrow$ algebraic formula for D_n.

Evaluation of D_n I

- How to evaluate algebraic representation with $\{\mu_i\}$
  1. Solve BAE with fixed N and find $\{\mu_i\}$
  2. Substitute roots into the algebraic expression with $\frac{N}{2} \times \frac{N}{2}$ determinant.
  3. Take the limit $N \to \infty$ with fine tuning $u = \frac{J\beta}{N} \to$ subtle!.
- Multiple integral representation for D_n
  Göhmann, Klümper and Seel (JPA 37(2004) 7625), Göhmann, Hasenclever and Seel (JSTAT (2005)
  1. No need to know $\{\mu_i\}$ explicitly
  2. The limit $N \to \infty$ is taken analytically.

Multiple integral representation for D_n

Three fundamental parts:

1. auxiliary function $a(\lambda)$ (s.t. $a(\mu_j) = -1$) satisfying NLIE
   (Kluemper, Batchelor and Pearce (JPA 24 (1991) 3111), Destri-de Vega (NPB438 (1995) 413)

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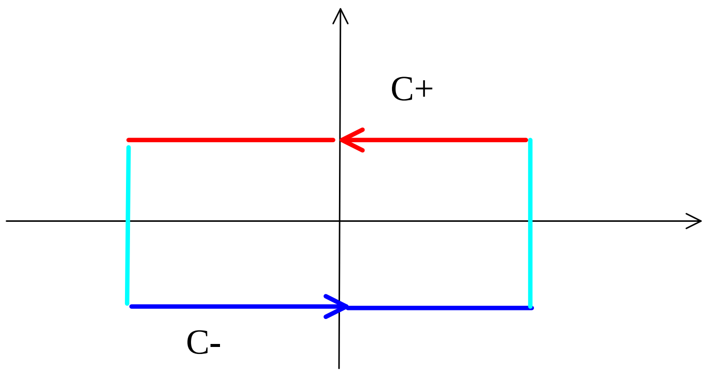
$$\ln a(\lambda) = -\beta h - 2\frac{\beta J \operatorname{sh}^2\eta}{\operatorname{sh}\lambda \operatorname{sh}(\lambda+\eta)} - \int_C \frac{dw}{2\pi i} \frac{\operatorname{sh}\eta \ln A}{\operatorname{sh}(\lambda - w + \eta)\operatorname{sh}(\lambda - w - \eta)}$$

$$q = \mathrm{e}^{\eta} \quad A(\lambda) := 1 + a(\lambda) \qquad \bar{A}(\lambda) := 1 + \frac{1}{a(\lambda)}$$



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1. auxiliary function $a(\lambda)$ (s.t. $a(\mu_j) = -1$) satisfying NLIE
2. $G(\lambda, \xi)$ satisfying linear integral equation: analogue of root density reduces the ratio of $m(\to\infty) \times m(\to\infty)$ determinant to $n \times n$ determinant.

$$G(\lambda, \xi) = -\frac{\operatorname{sh}\eta}{\operatorname{sh}(\lambda - \xi)\operatorname{sh}(\lambda - \xi - \eta)} + \int_C \frac{dw}{2\pi i A(w)} \frac{\operatorname{sh} 2\eta G(w, \xi)}{\operatorname{sh}(\lambda - w + \eta)\operatorname{sh}(\lambda - w - \eta)}$$

Multiple integral representation for D_n

Three fundamental parts:

1. auxiliary function $a(\lambda)$ (s.t. $a(\mu_j) = -1$) satisfying NLIE
2. $G(\lambda, \xi)$ satisfying linear integral equation: analogue of root density
3. ratios of elementary functions

D_n Explicit result

n- fold coupled integrals for D_n

$$D^{\alpha_1\ldots\alpha_n}_{\beta_1\ldots\beta_n}(\xi_1,\ldots,\xi_n) =$$

$$\left[\prod_{j=1}^{|\alpha^+|}\int_C \frac{d\omega_j}{2i\pi A(\omega_j)} \prod_{k=1}^{\widetilde{\alpha_j}^+ -1} \mathrm{sh}(\omega_j-\xi_k-\eta) \prod_{k=\widetilde{\alpha_j}^+ +1}^{n} \mathrm{sh}(\omega_j-\xi_k)\right]$$

$$\left[\prod_{j=|\alpha^+|+1}^{n}\int_C \frac{d\omega_j}{2i\pi A(\omega_j)} \prod_{k=1}^{\widetilde{\beta}_j^- -1} \mathrm{sh}(\omega_j-\xi_k+\eta) \prod_{k=\widetilde{\beta}_j^- +1}^{n} \mathrm{sh}(\omega_j-\xi_k)\right]$$

$$\frac{\det - G(\omega_j,\xi_k)}{\prod_{1\leq j<k\leq n}\mathrm{sh}(\xi_k-\xi_j)\,\mathrm{sh}(\omega_j-\omega_k-\eta)}$$

multiple integral formula: Remarks

1. it reminds us of similar formulae in QF if $a(\omega) \to \varepsilon(\theta)$, (equivalently, $A(\omega) \to 1+\varepsilon(\theta)$) on upper half plane and so on. note ε and a both reduce to the dressed energy function in a limit $T \to 0, h \to 0^+$.
   example (1pt function)

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   example (1pt function)

$$\phi(z) - 1 = 2\int_C \frac{d\omega_1}{2\pi i A(\omega_1)} G(\omega_1, z)$$

$$G(\omega_1, z) = G_0(\omega_1, z) + \int_C \frac{d\omega_2}{2\pi i A(\omega_2)} K(\omega_1 - \omega_2) G(\omega_2, z)$$

$$= \sum_{n=1}^{\infty} \prod_{j=2}^{n} \int_C \frac{d\omega_j}{2\pi i A(\omega_j)} K(\omega_{j-1} - \omega_j) G_0(\omega_n, z)$$

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   example (1pt function)

$$\phi(0) - 1 = 2 \sum_{n=1}^{\infty} \prod_{j=1}^{n} \int_C \frac{dx_j}{2\pi(1+\mathrm{e}^{\beta\varepsilon(x_j)})} f(x_1, \cdots, x_n)$$

$$f(x_1, \cdots, x_n) = \frac{\sin\gamma}{\mathrm{sh}(x_n + i\frac{\gamma}{2})\,\mathrm{sh}(x_n + i\frac{\gamma}{2})}$$

$$\times \left(\prod_{k=1}^{n-1} \frac{-\sin\gamma}{\mathrm{sh}(x_k - x_{k+1} + i\gamma)\,\mathrm{sh}(x_k - x_{k+1} + i\gamma)} \right)$$

$$a(x + i\frac{\gamma}{2}) = \beta\varepsilon(x) \qquad a(x - i\frac{\gamma}{2}) \to \infty \qquad \eta = i\gamma$$

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1. it reminds us of similar formulae in QF if $a(\omega) \to \varepsilon(\theta)$, (equivalently, $A(\omega) \to 1+\varepsilon(\theta)$) on upper half plane and so on. note ε and a both reduce to the dressed energy function in a limit $T \to 0, h \to 0^+$.

2. As $T \to 0$

$$\begin{cases} a(\overline{a}) \sim 0 & \text{on } C_+(C_-) \\ |a|(|\overline{a}|) >> 1 & \text{on } C_-(C_+) \\ G(\omega,\xi) \sim -\frac{1}{\sinh\frac{\omega-\xi}{\eta}} & \end{cases}$$

use them then Jimbo-Miwa formula (or Lyon, if $h \neq 0$) recovered.

difficulty with the formula

- The coupled integrals too difficult to treat in quantative study.
- However, Boos and Takahashi groups decoupled the integrals at $T = 0$. is this a hope?
- The basic technique, shifts in integration contours, can not be applied for $T > 0$ as measures are not constant now, $\frac{d\omega}{1+a(\omega)}$ etc.
- The case studies, unexpectedly, show that the factorization is possible even for $T > 0, h \neq 0$.
- Even-more strinking similarlity to $T = 0$!

Similarity to $T = 0$

$T = h = 0, n = 2$ (Boos et al.)

$$h_2(\lambda_1, \lambda_2) = \begin{pmatrix} D^{++}_{++} \\ -D^{+-}_{+-} \\ D^{+-}_{-+} \\ D^{+-+}_{+--} \\ -D^{-+}_{-+} \\ D^{--}_{--} \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ -1 \\ 1 \end{pmatrix} - \frac{\omega(\lambda_{12})}{(q-q^{-1})^2}\left(((\zeta+\zeta^{-1})A + (q+q^{-1})B\right)$$

$$- \frac{\tilde{\omega}(\lambda_{12})}{(q-q^{-1})^2}\left(\frac{q^2-q^{-2}}{\zeta-\zeta^{-1}}A + (q-q^{-1})\frac{\zeta+\zeta^{-1}}{\zeta-\zeta^{-1}}B\right)$$

$$A = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, B = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \zeta = q^{\lambda_{12}}$$

Similarity to $T = 0$

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ω : spinon-spinon S matrix etc at $T = 0$

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ω : spinon-spinon S matrix etc at $T=0$

The observation from the case studies at $T>0$, algebraic parts remain the same: one only has to replace ω by its finite T analogue !

finite T ω

$$\omega(\mu_1,\mu_2,\alpha) \sim -\psi(\mu_1,\mu_2,\alpha) + \text{elementaryfunction}$$

$$\psi(\mu_1,\mu_2,\alpha) = \int_C dw \frac{G(w,\mu_1)}{\pi i A(w)} \left(-\coth(w-\mu_2) + q^{\alpha}\coth(w-\mu_2-\eta)\right)$$

$$\phi(\mu) = 1 + \int_C \frac{G(w,\mu)}{\pi i A(w)}$$

$$\omega(\mu_1,\mu_2) = \omega(\mu_1,\mu_2,0)$$

$$\tilde{\omega}(\mu_1,\mu_2) = \frac{d}{d\alpha}\omega(\mu_1,\mu_2,\alpha)|_{\alpha=0}$$

at $T > 0$ they are no longer the function of $\zeta = q^{\lambda_{12}}$

Two representations

- multiple integral representation :
  proved but difficult to analyse
- The “exponential formula” (Boos at al. hep-th/0606280, 0702086, 0801.1176) :
  solves “reduced”q KZ equation, valid only at $T = 0$,but gives us algebraic parts. It can explain the intrinsic reason of the factorization

No direct proof of the equivalence (mutiple integral = exponential formula) even at $T = 0$.
Our strategy: believe in the equivalence and obtain algebraic parts using the latter at $T > 0$. check by high T expansions.

Exponential formula: only result

- define $D_n(O) = \langle O \rangle_{T,h}$
- define: $\mathrm{Tr} := \frac{1}{2}\mathrm{tr}_1 \frac{1}{2}\mathrm{tr}_2 \cdots$
- 3 operators :"Grassmanian" b, c and $\mathbb{H}$ to be defined later.

> Conjecture:
> $D_n(O) = \mathrm{Tr} e^{\Omega}(O)|_{\xi_i=0}$

where

$$\Omega = \Omega_1 + \Omega_2$$

$$\Omega_1 = -\lim_{\alpha=0} \int \int \frac{d\mu_1}{2\pi i} \frac{d\mu_2}{2\pi i} \omega(\mu_1, \mu_2; \alpha) b(\zeta_1; \alpha - 1) c(\zeta_2; \alpha)$$

$$\Omega_2 = -\int \lim_{\alpha=0} \frac{d\mu_1}{2\pi i} \phi(\mu_1) \mathbb{H}(\zeta_1; \alpha)$$

$$\zeta_i = \mathrm{e}^{\mu_j}$$

Hidden Fermions

- Strandard Fermions in spin chains
  - Jordan-Wigner 's transformation.
  - Fermion operators act on vectors in the Hilbert space.

- Hidden fermions (Boos,Jimbo,Miwa, Smirnov and Takayama)
  - D_n : operators.
  - Introduce operators act on operators D_n.
  - Adjoint action is useful: $A(O) = A^{-1}OA$
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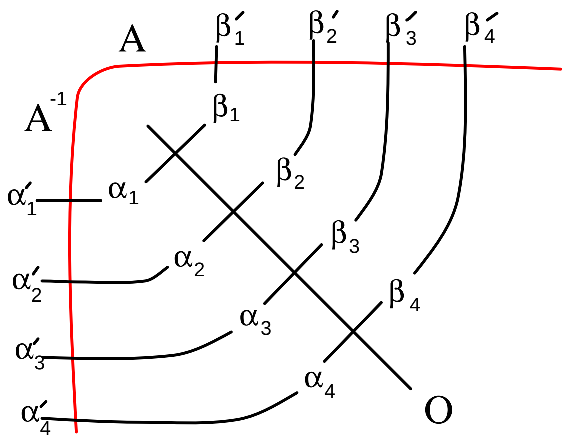

Transfer matrices

1. auxiliary space (2 dim), labelled by a
2. j-th site quantum space (2 dim) , labelled by j
3. $q-$oscillator space (a_A, a_A^*, D_A) labelled by A

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$$q^D a^* q^{-D} = q a^* \qquad q^D a q^{-D} = q^{-1} a$$

$$a^* a = 1 - q^{2D} \qquad a a^* = 1 - q^{2D+2}$$

representations $W^{\pm}$

$$W^+ = \oplus_{k \geq 0} \mathbb{C}|k\rangle \qquad a^*|k-1\rangle = |k\rangle \qquad a|0\rangle = 0 \qquad D|k\rangle = k|k\rangle$$

$$W^- = \oplus_{k \leq -1} \mathbb{C}|k\rangle \qquad a|k+1\rangle = |k\rangle \qquad a^*|-1\rangle = 0 \qquad D|k\rangle = k|k\rangle$$

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Introduce L operators

1. $L_{a,j}$ usual 4×4 matrix.
2. $L^+_{A,j}(\zeta)$ acting on also $q-$ oscillator space.

$$L^+_{A,j}(\zeta) = \begin{pmatrix} 1 & -\zeta a_A^* \\ -\zeta a_A & 1 - \zeta^2 q^{2D_A+2} \end{pmatrix}_j \begin{pmatrix} q^{D_A} & 0 \\ 0 & q^{-D_A} \end{pmatrix}_j$$

Fusion of transfer matrices

Introduce $L^{+}_{(A,a),j}$:Fusion of $L_{a,j}$ and $L_{A,j}$, triangular form

$$L^{+}_{(A,a),j}(\zeta)=\begin{pmatrix}(\zeta q-\zeta^{-1}q^{-1})L^{+}_{A,j}(\zeta q^{-1}) & 0\\ * & (\zeta-\zeta^{-1})L^{+}_{A,j}(\zeta q)\end{pmatrix}_a$$

and $L^{-}_{(A,a),j}(\zeta)=\sigma^1_a\sigma^1_j L^{+}_{(A,a),j}(\zeta)\sigma^1_a\sigma^1_j$

“Local" $T-Q$ relation.

Adjoint fused transfer matrix

- define fused transfer matrices

$$T^{\pm}_{(A,a)}(\zeta) = L^{\pm}_{(A,a),1}(\zeta)\cdots L^{\pm}_{(A,a),N}(\zeta)$$

- define adoint action $\mathbb{T}$ on $X \in M_N$

$$\left(\mathbb{T}^{\pm}_{(A,a)}(\zeta)\right)^{-1}(X) = (T^{\pm}_{(A,a)}(\zeta))^{-1}(1_A \times X)T^{\pm}_{(A,a)}(\zeta)$$

Now triangular forms,

$$\left(\mathbb{T}^{+}_{(A,a)}(\zeta)\right)^{-1} = \begin{pmatrix} \mathbb{A}^{+}(\zeta) & 0 \\ \mathbb{C}^{+}(\zeta) & \mathbb{D}^{+}(\zeta) \end{pmatrix}_a \quad \left(\mathbb{T}^{-}_{(A,a)}(\zeta)\right)^{-1} = \begin{pmatrix} \mathbb{A}^{-}(\zeta) & \mathbb{B}^{-}(\zeta) \\ 0 & \mathbb{D}^{-}(\zeta) \end{pmatrix}_a$$

Grassmannian objects

- Hidden fermionic operators (Boos at al.)

$$\mathbf{c}(\zeta,\alpha) \sim \mathrm{tr}^+_{\mathrm{A}} q^{2\alpha D_A} \mathbb{C}^+_A \qquad \mathbf{b}(\zeta,\alpha) \sim \mathrm{tr}^-_{\mathrm{A}} q^{-2\alpha(D_A+1)} \mathbb{B}^-_A$$

satisfying

$$\{b(\zeta_1), b(\zeta_2)\} = \{c(\zeta_1), c(\zeta_2)\}$$

- magnetic objects

$$\mathbb{H}(\zeta,\alpha) \sim \mathrm{tr}^+_A q^{2\alpha D_A} a^*_A \mathbb{C}^+$$

also prepare residue operators,

$$b_j(\alpha) \sim \mathrm{res}_{\lambda=\xi_j} b(q^\lambda,\alpha) \quad c_j(\alpha) \sim \mathrm{res}_{\lambda=\xi_j} c(q^\lambda,\alpha)$$

$$h_j(\alpha) \sim \mathrm{res}_{\lambda=\xi_j} \mathbb{H}(q^\lambda,\alpha)$$

Exponential formula reminder

$$\langle O_{1,\cdots,n}\rangle_{T,h} = \mathrm{Tr} e^{\Omega_1+\Omega_2} O_{1,\cdots,n}$$

$$\Omega_1 = -\lim_{\alpha=0}\int \frac{d\zeta_1^2}{2\pi i\zeta_1^2}\frac{d\zeta_2^2}{2\pi i\zeta_2^2}\omega(\lambda_1,\lambda_2)b(\zeta_1,\alpha)c(\zeta_2,\alpha-1)$$

$$\Omega_2 = -\lim_{\alpha=0}\int \frac{d\zeta_1^2}{2\pi i\zeta_1^2}\phi(\lambda_1,\lambda_2)\mathbb{H}(\zeta_1,\alpha)$$

By residual calculation,

$$\Omega_1 = \sum_{1\le i<j\le n}\left(\Omega_{ij}^{+}\omega_{ij} + \Omega_{ij}^{-}\widetilde{\omega}_{ij}\right) \qquad \Omega_2 \sim \sum_j \phi_j h_j$$

$$\omega_{ij} := \omega(\xi_i,\xi_j) \qquad \widetilde{\omega}_{ij} := \partial_\alpha \omega(\xi_i,\xi_j)|_{\alpha=0}$$

Algebraic relations

- algebraic relations

$$\Omega^{\pm}_{i,j}\Omega^{\pm}_{i,k} = \Omega^{\pm}_{i,j}\Omega^{\mp}_{i,k} = 0$$

$$[\Omega^{\pm}_{i,j}, \Omega^{\pm}_{i',j'}] = 0 \qquad [\Omega^{\pm}_{i,j}, h_k] = 0 \qquad h_i h_j + h_j h_i = 0$$

- consequence
  - the expansion $e^{\Omega_1+\Omega_2} = 1 + \Omega_1 + \Omega_2 + \cdots$ truncates at finite order (nilpotency)
  - explains factorization.

Classical-Quantum Crossover

Fabricius-McCoy (cond-mat/98053379), F-Klümper-M (cond-mat/9812012)

$S_{zz}(n,T,\Delta) = \langle \sigma_0^z \sigma_n^z \rangle$

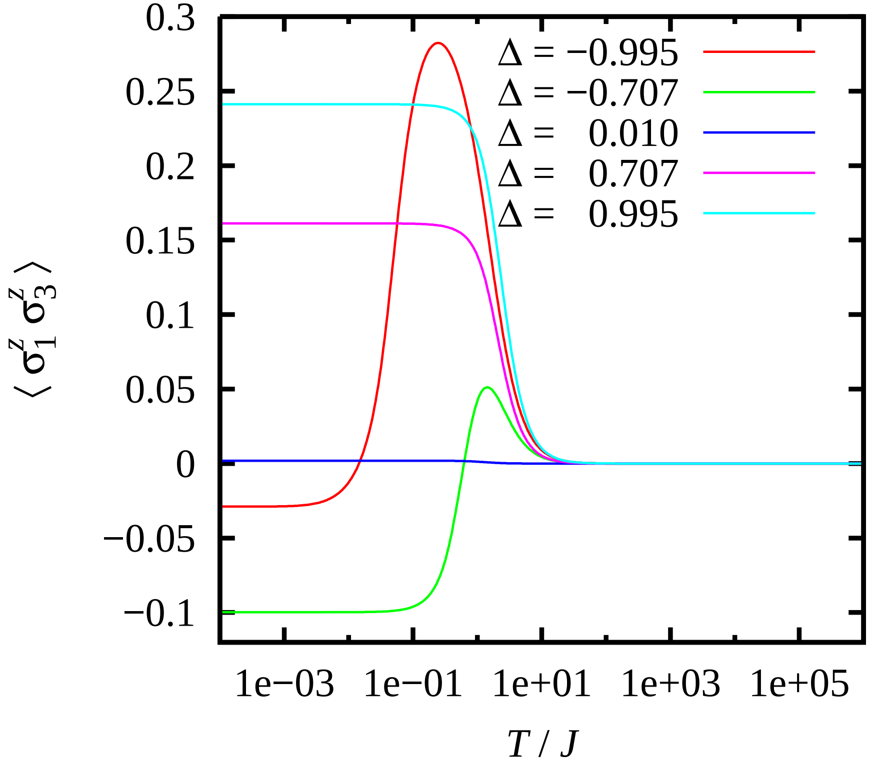
- quantum($T \ll 1$)
  CFT predicts: ($\theta := 1/2 + \frac{1}{\pi}\sin^{-1}\Delta$)

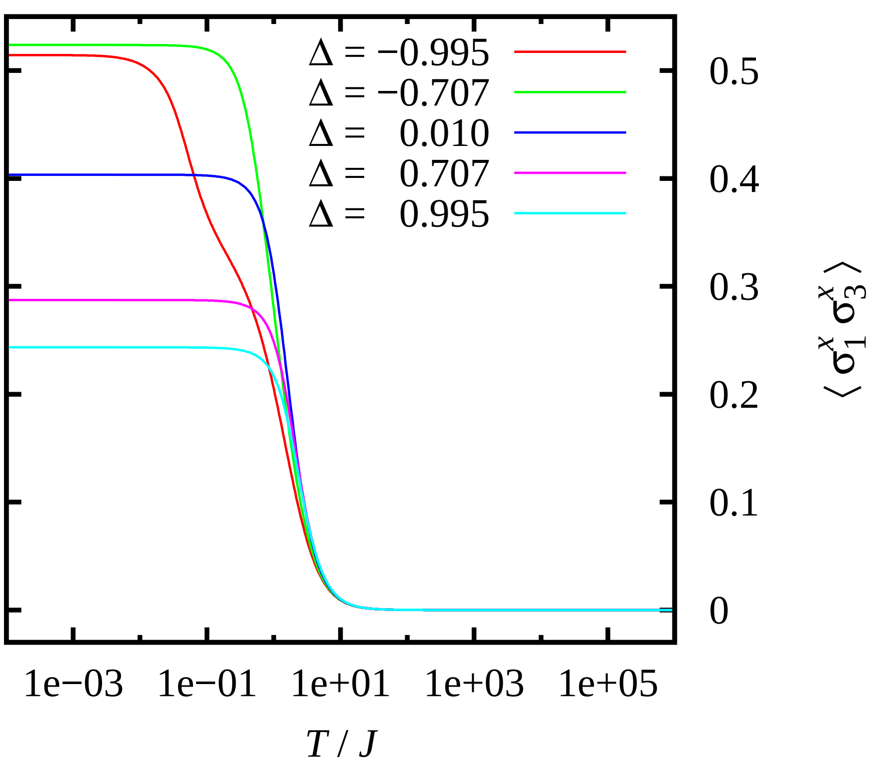
$$S_{zz}(n,T,\Delta) = -\frac{1}{\pi^2 \theta n^2} + (-1)^n \frac{C(\Delta)}{n^{1/\theta}}$$

  - $0 < \Delta < 1 \to \theta > 1/2$: oscillatory
  - $-1 < \Delta < 0 \to \theta < 1/2$:negative for $\forall$ n
- classical ($T \sim 1$)
  $-1 < \Delta < 0$ classically alignment of spins is favored: $S_{zz} > 0$ for $\forall$ n .
  numerical diagonalization(up to $L = 18$ sites, $n \leq 9$) $\to$
  existence of finite T cross-over from classical to quantum

Numerics QC crossover

Now our numerics $L \to \infty$ but small n





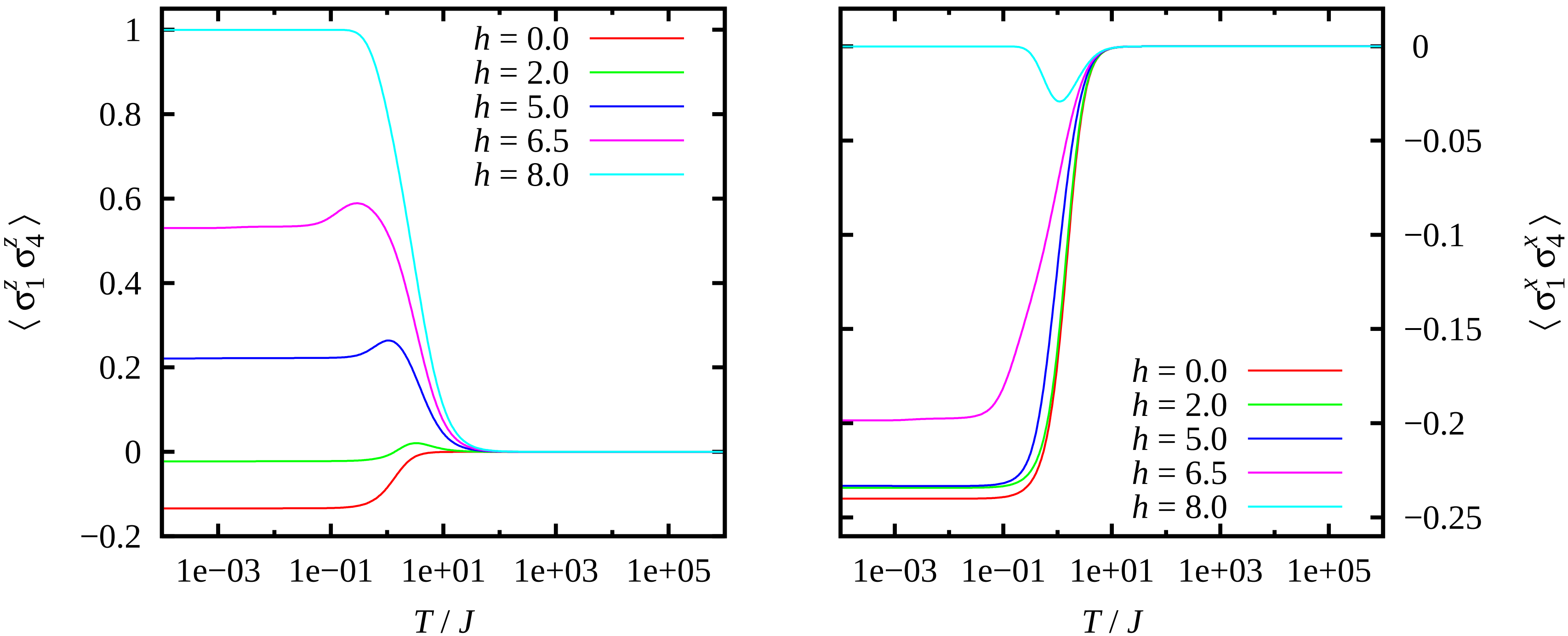
The cross-over temperature

$T_0(n;\Delta)$ for $L = \infty$ and for $L = 18$ (FM)

Δ	n = 2 $L = \infty$	$L = 18$	n = 3 $L = \infty$	L = 18	n = 4 $L = \infty$	L = 18
-0.1	4.96645	4.966	3.32288	3.323	2.56077	2.561
-0.2	2.43157	2.432	1.64332	1.643	1.27520	1.275
-0.3	1.56079	1.561	1.07081	1.071	0.839169	0.839
-0.4	1.10294	1.103	0.771287	0.771	0.611558	0.612
-0.5	0.806967	0.807	0.577718	0.578	0.4641	-
-0.6	0.588818	0.589	0.434179	0.434	0.354030	0.355
-0.7	0.412795	0.413	0.316321	0.318	0.262606	0.264
-0.8	0.262355	0.265	0.211402	0.215	0.179803	0.184
-0.9	0.129195	0.137	0.111127	0.118	0.098055	0.104

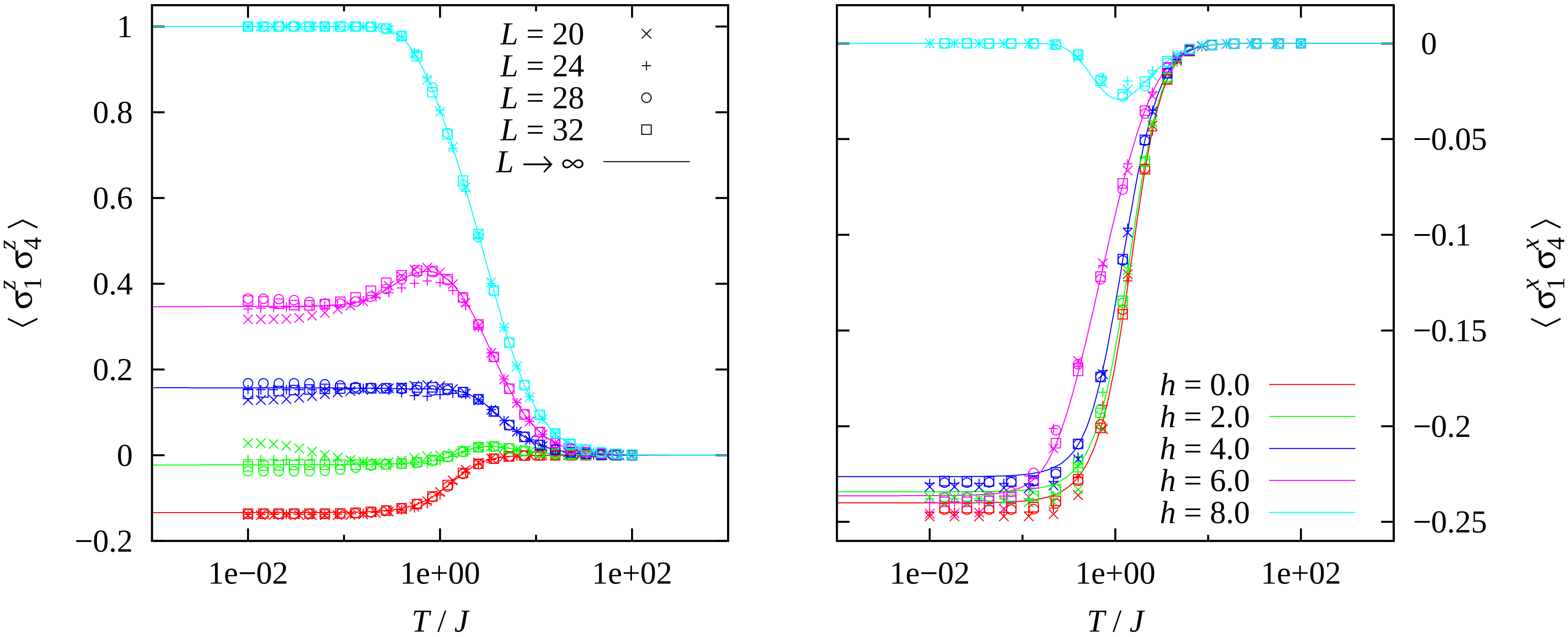
Strange lumps for $\Delta > 0, h \neq 0$

example $\Delta = \cos\frac{\pi}{4}$



competition of 1. paramagnetic order, 2. quantum fluctuation, 3.thermal disorder?

brute force calculation



Summary and future problems

1. Efficient formula for exact evaluation of density matrix element of finite segment
2. Factorization is a consequence of a hidden fermionic nature
3. Valid even $T > 0, h \neq 0$

- higher spins? (in progress)
- $su(n)$ generalizaiton (in progress?)
- Asymptotics?