On an application of the Ordinary Differential Equation / Integrable Model Correspondence

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with Patrick Dorey (Durham)

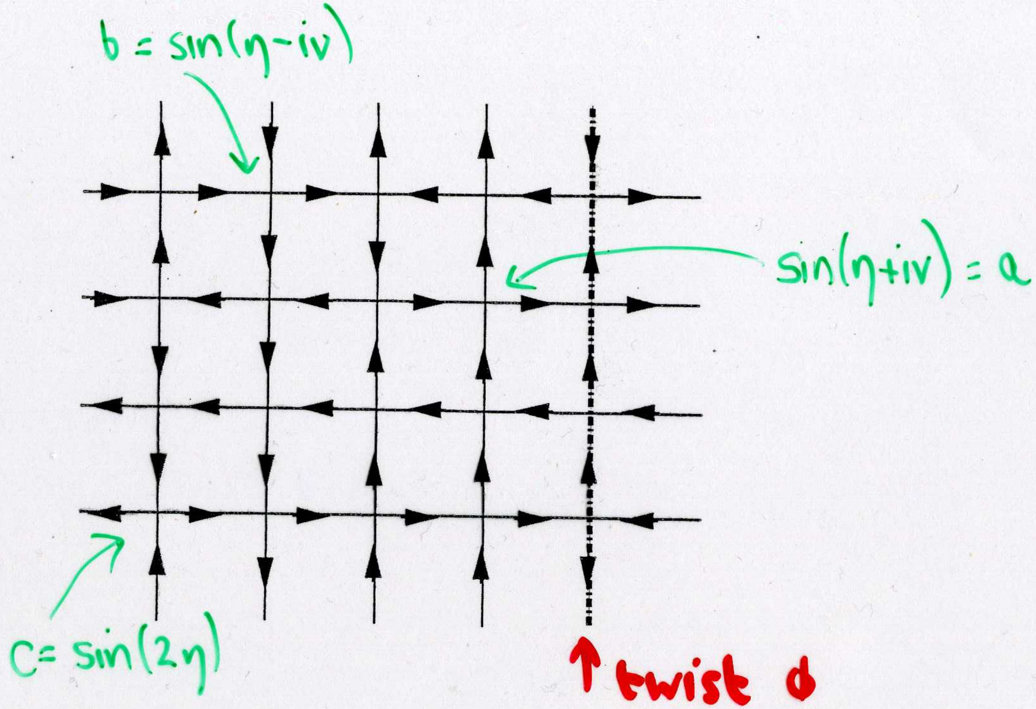
Anna Lishman (Durham)

and Roberto Tateo (Torino)

to appear.

Integrable model - 6 vertex model (p.b.c)

in continuum limit



$$Z = \sum \prod_{\text{spin sites}} +$$

$$T = \sum \text{++++}$$

$$\text{free energy} = -\frac{1}{NN'} \log Z = -\frac{1}{NN'} \log \operatorname{Tr} T^N \sim -\frac{1}{N} \log t_0$$

transfer matrix

largest eigenvalue

One way to calculate $f\,(t_0)$:

Baxter's TQ relation

$$t_0(\nu)\, q_0(\nu) = e^{i\phi}\, q_0(\omega^2 \nu) + e^{-i\phi}\, q_0(\omega^{-2} \nu)$$

unknown entire function

$\omega = e^{-2i\eta}$

If $Q(\nu) = \prod_{l=1}^{\infty} 1 - \frac{\nu}{\nu_i}$ then $\prod_{l=1}^{\infty} \frac{\nu_l - e^{4i\eta}\,\nu_i}{\nu_l - e^{-4i\eta}\,\nu_i} = -e^{-2i\phi}$ BAE

ODEs

Schrödinger Equation

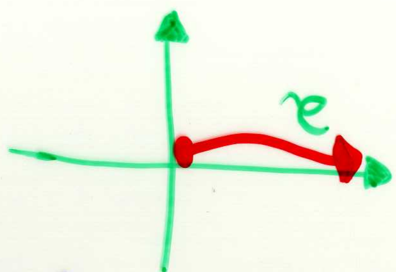
$$\left(-\frac{d^2}{dx^2} + V(x)\right)\psi(x) = E\,\psi(x)$$

Boundary condition $\quad \psi(x) \in L^2(\mathcal{C})$

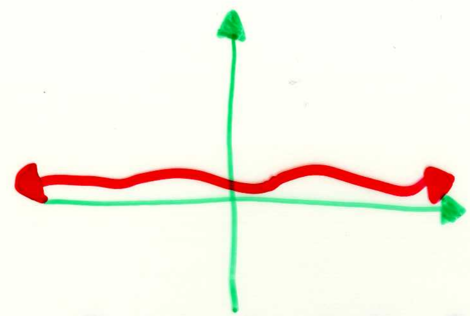
contour in complex plane
usually $\mathbb{R}$ or half line

(Trivial) examples

$$V(x) = x^2$$



or

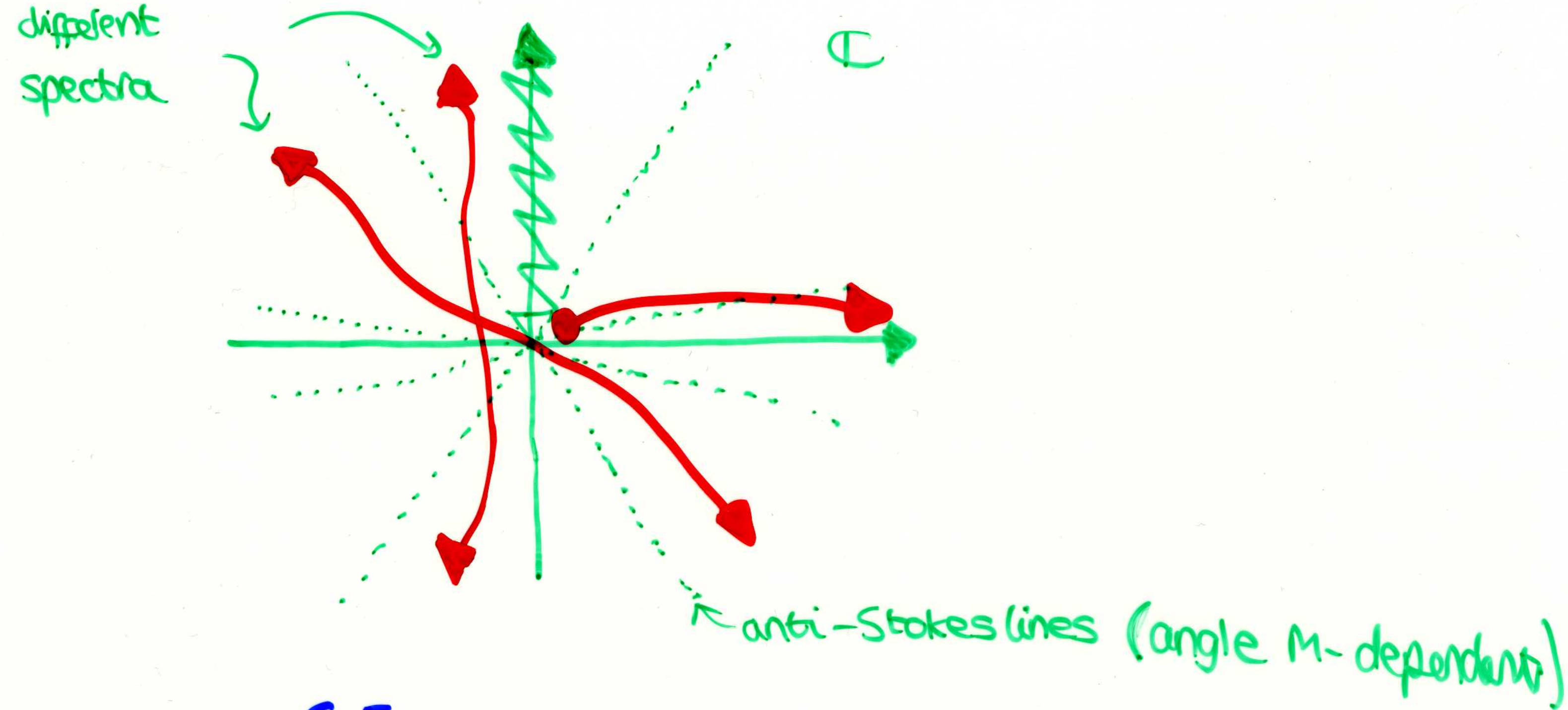


(D) $\quad \psi(0) = 0 \qquad E = 3, 7, 11, \dots$

(N) $\quad \psi'(0) = 0 \qquad E = 1, 5, 9, \dots$

$E = 1, 3, 5, 7, 9, \dots$

'Boundary' conditions can be more exotic:



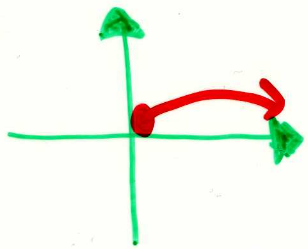
for, say, S.E.

$$\left(-\frac{d^2}{dx^2} - (ix)^{2M}\right)\psi(x) = E\,\psi(x) \qquad \begin{pmatrix} M>0 \\ M\in\mathbb{R} \end{pmatrix}$$

- Any pair of non-adjacent sectors can be chosen

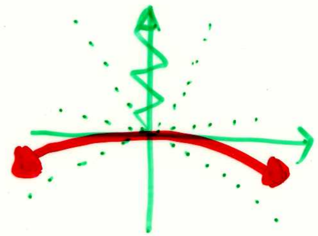
Two spectral problems:

$$\left[-\frac{d^2}{dx^2} + x^{2M} + \frac{l(l+1)}{x^2}\right]\psi(x) = E\psi(x)$$



$$\psi\big|_{x\to 0} \sim x^{l+1}$$

$$\left[-\frac{d^2}{dx^2} - (ix)^{2M} + \frac{l(l+1)}{x^2}\right]\psi(x) = E\psi(x)$$



Two spectral determinants

$$D(E,l) = \prod_{i=0}^{\infty} 1 - \frac{E}{e_i}$$

$$T(E,l) = \prod_{i=0}^{\infty} 1 + \frac{E}{E_i}$$

which together turn out to satisfy

$$C(E,l)\,D(E,l) = \omega^{-(\frac{1}{2}+l)}\, D(\omega^{-2}E,l) + \omega^{(\frac{1}{2}+l)}\, D(\omega^{2}E,l)$$

Stokes multipliers

$\boxed{\text{Baxter's TQ relation}}$

ODE/IM correspondence (Dorey, Tateo 98)

- 6-vertex t_0 and q_0 are equal to these particular spectral determinants C, D

- Bethe ansatz roots are the Schrödinger eigenvalues

- Can study ODEs using IM techniques and vice versa

Why study this non-hermitian problem? (other than in connection with IM/ODE)

~'92 Bessis/Zinn-Justin $H = p^2 + ix^3$

97 Bender/Boettcher $H = p^2 - (ix)^{2M}$

99 Dorey / Tateo (BLZ) $H = p^2 - (ix)^{2M} + \frac{l(l+1)}{x^2}$

01 Dorey / TCD / Tateo (S) $H = p^2 - (ix)^{2M} - \alpha(ix)^{M-1} + \frac{l(l+1)}{x^2}$

Reality of spectrum non-trivial

$H_{M,\alpha,l}$ not 6-vertex model but Perk-Schultz $U_q(\hat{gl}(2|1))$ (Suzuki) and perturbed hairpin model of boundary interaction (Fateev, Lukyanov)

(ODE/IM now extended to various models)

$$H_{M,\alpha,\ell} = p^2 - (ix)^{2M} - \alpha(ix)^{M-1} + \frac{\ell(\ell+1)}{x^2} \qquad \psi \in L^2(\mathcal{C})$$

- H invariant under $P: x \to -x \quad p \to -p$ (PT symmetric quantum mechanics)

  $T: x \to x \quad p \to -p \quad i \to -i$

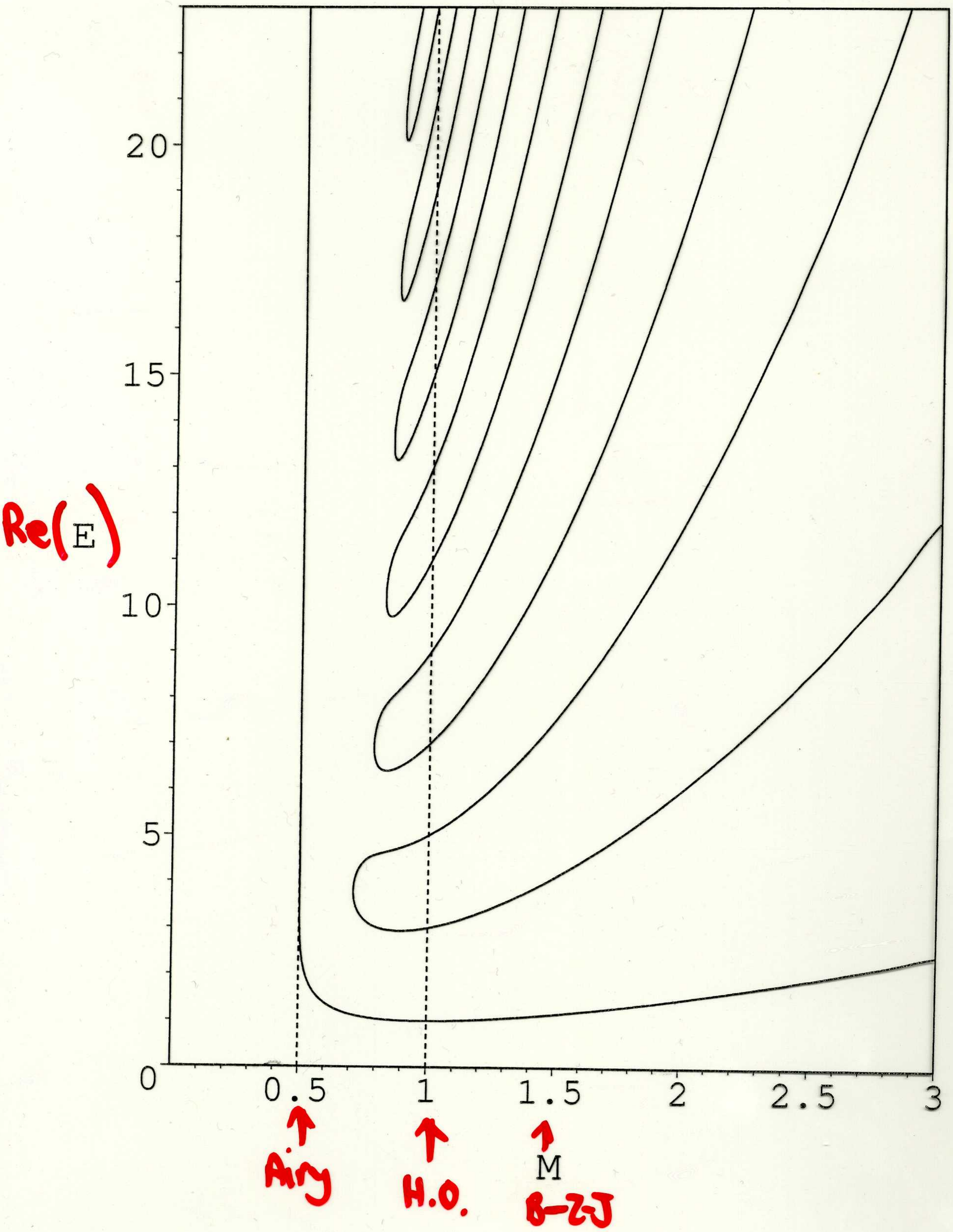
- So eigenvalues form complex-conjugate pairs or are real

- There are regions where spectrum is entirely <u>real</u>

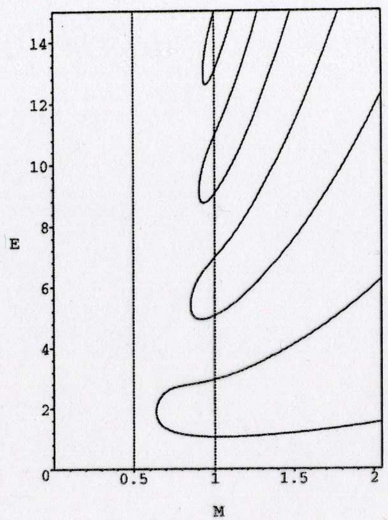
$\boxed{Q}$ can we prove this? Yes, but not with standard ODE techniques.

Spectrum of

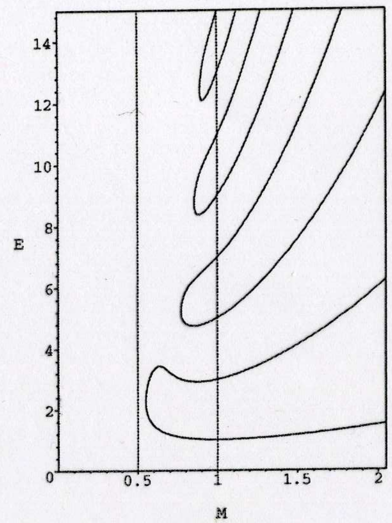
$p^2 - (ix)^{2M}$ (BB)



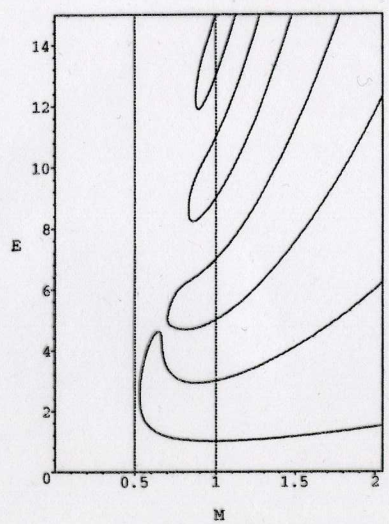
Spectrum of $p^2 - (ix)^{2M} + \frac{\ell(\ell+1)}{x^2}$ (DT)



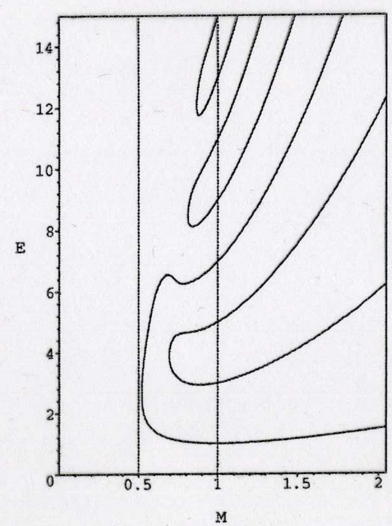
(*f*) $l = -0.025$



(*d*) $l = -0.005$



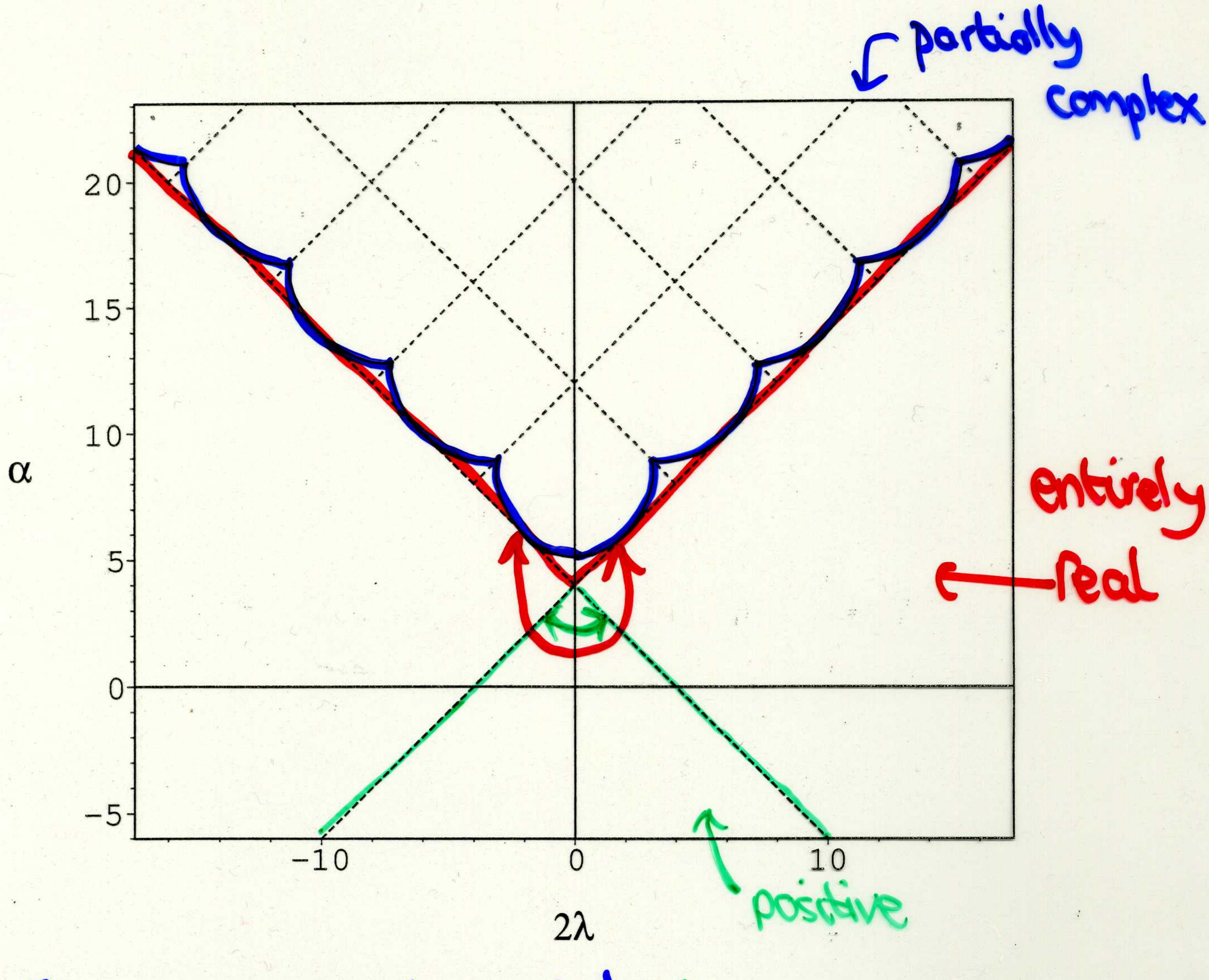
(*c*) $l = -0.0025$



(*e*) $l = -0.0015$

Real eigenvalues of $p^2 - (\mathrm{i}x)^{2M} + l(l+1)/x^2$ as functions of M, for various values of l.

Domain of (un) reality for M=3 (DDT '01)



Reality result (DDT '01) (proof via IM/ODE)

Reality $M>1$ and $\alpha < M+1+|2l+1|$

Positivity $M>1$ and $\alpha < M+1-|2l+1|$

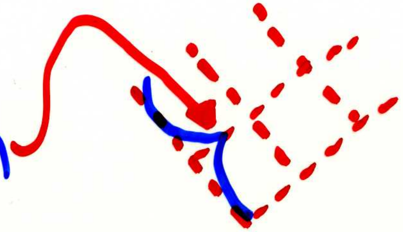
- Line across which 1 complex pair of e-values appear is unexpectedly 'cuspy'

- Dotted lines are where there is a zero-energy level for all M: $\alpha \pm 2\lambda = M+1\,(1+2n)$ $\quad n \in \mathbb{Z}^+$

- For $M=3$ only, S.E. is QES on dotted lines
  ↑ $2n+1$ e-values known exactly (& e-functions)

- Where does 2nd complex pair appear?

~~Open~~ questions

- Are cusps precisely on  ?

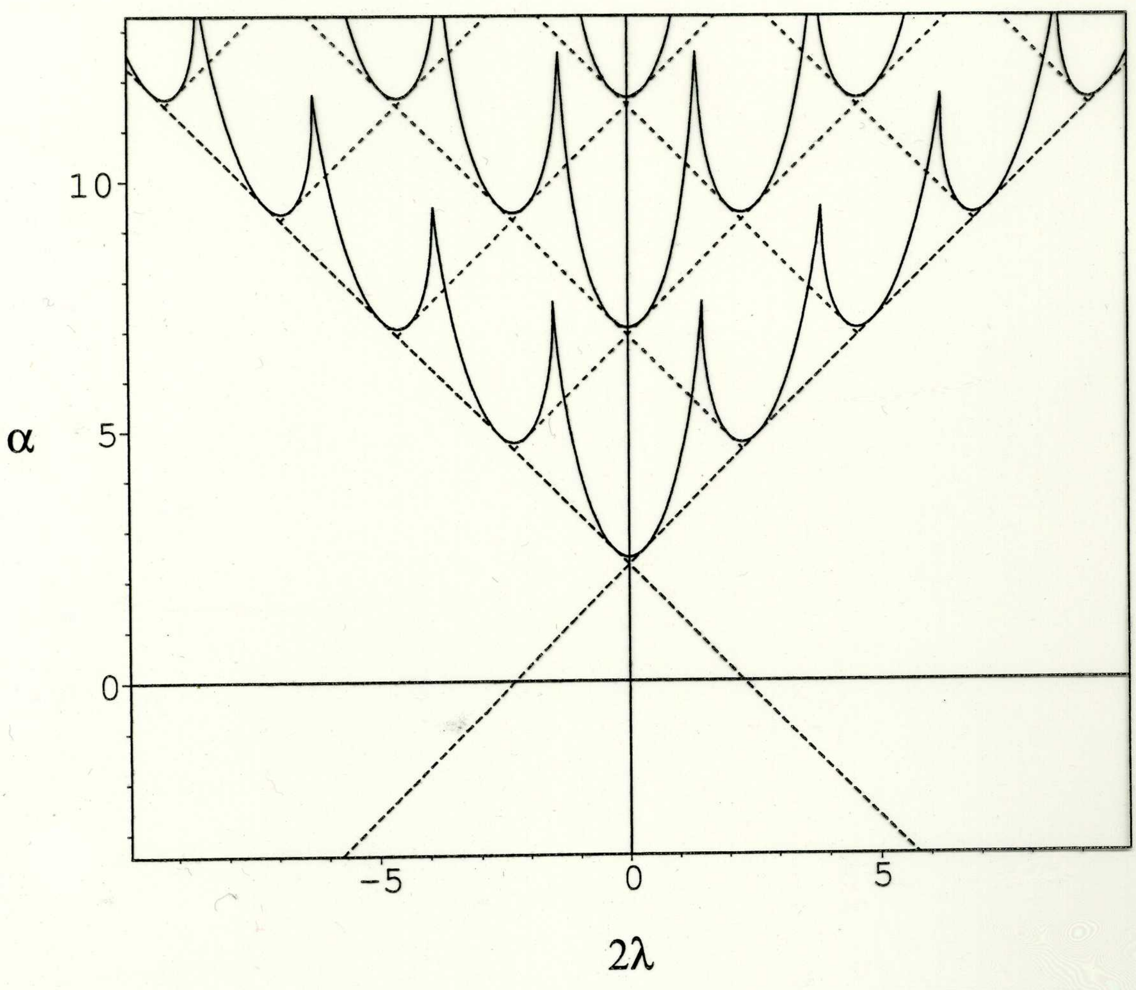
- If so, why? → because of E=0 ?

  → because of QES ?

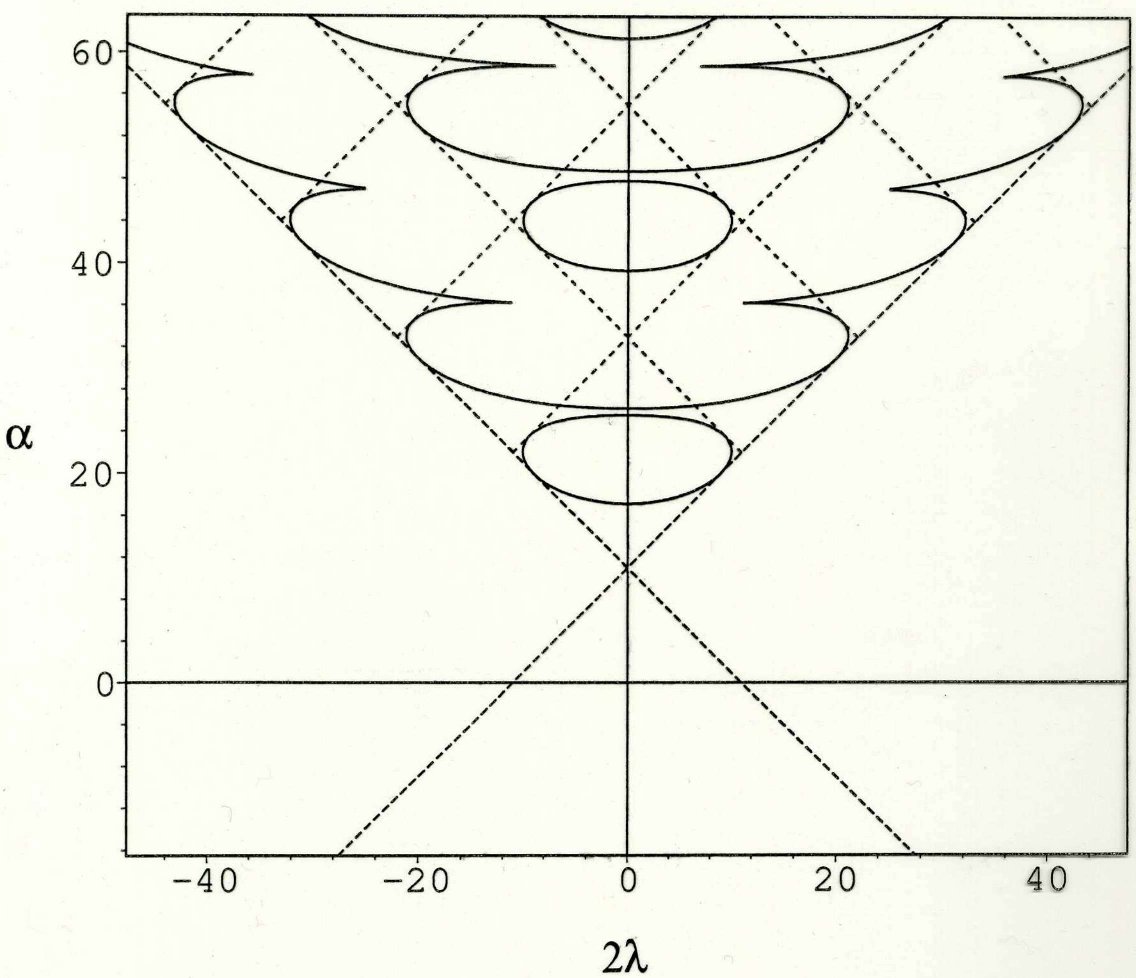
- What happens for other M?

  Do cusps stay on lines? Not QES but do have E=0 still.

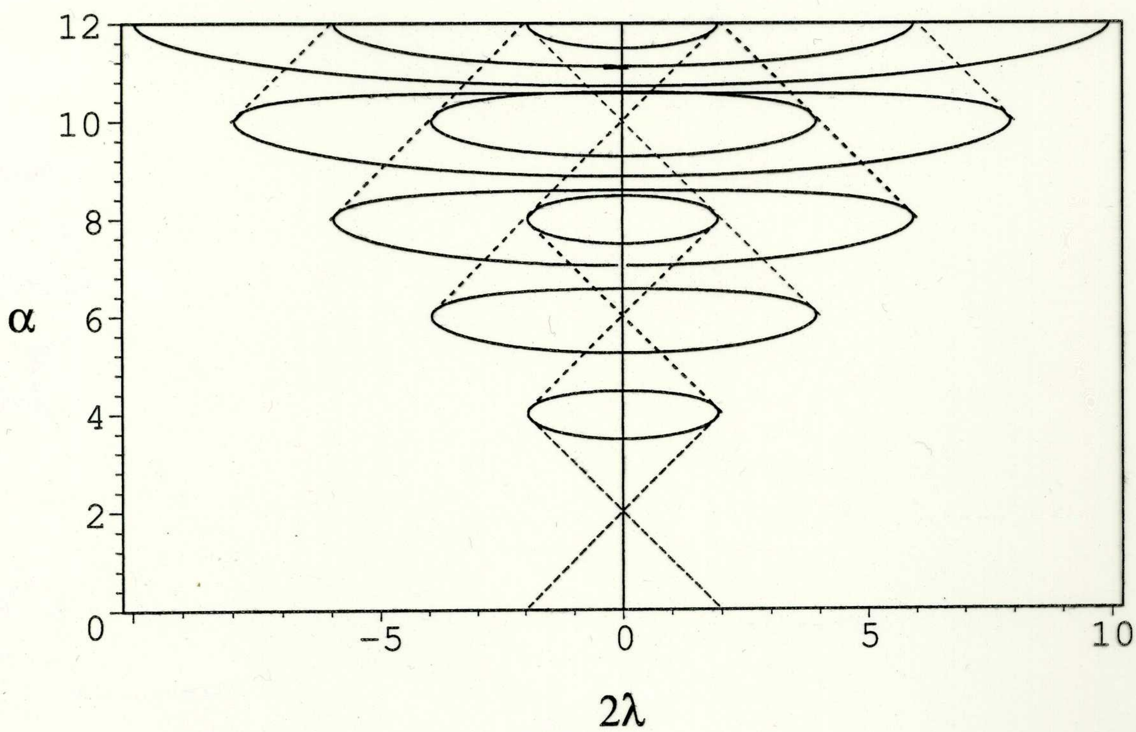
M = 1.3



m = 10



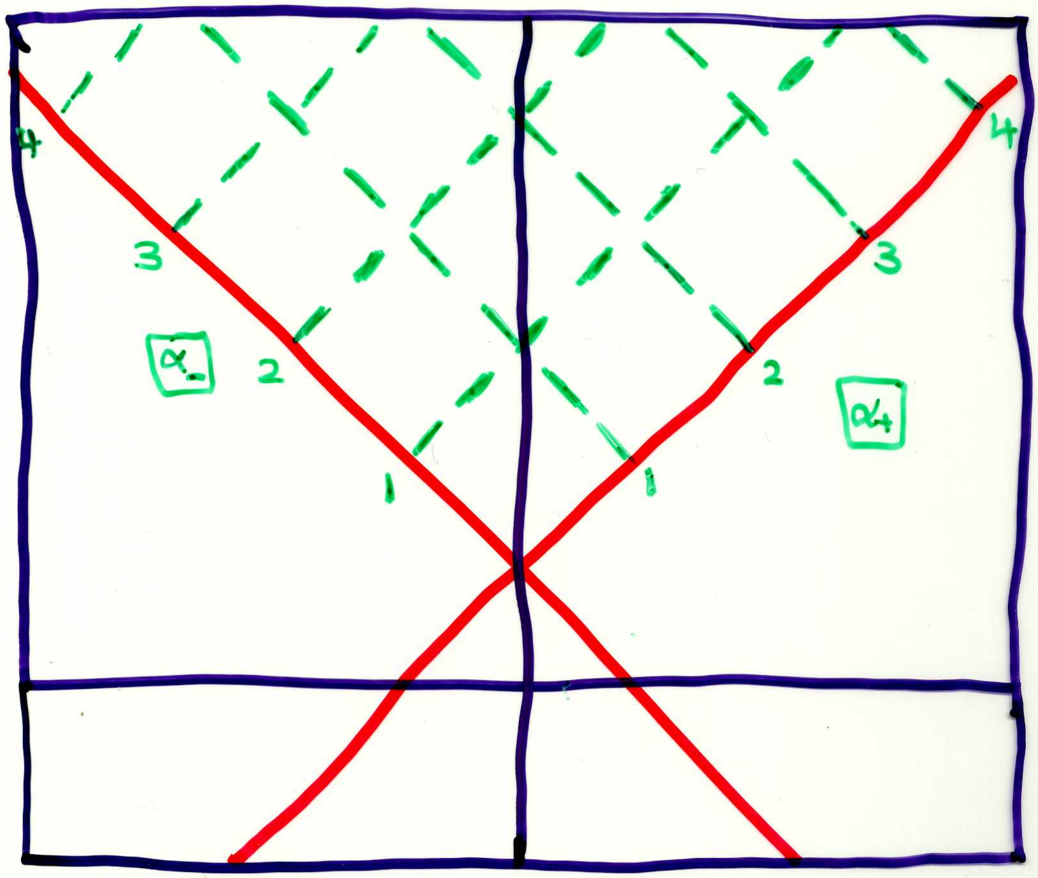
m = 30



Alternative coordinates

$$\alpha_\pm = \frac{1}{2(m+1)}\left\{\alpha - (m+1) \pm 2\lambda\right\}$$

- these are zero-energy lines

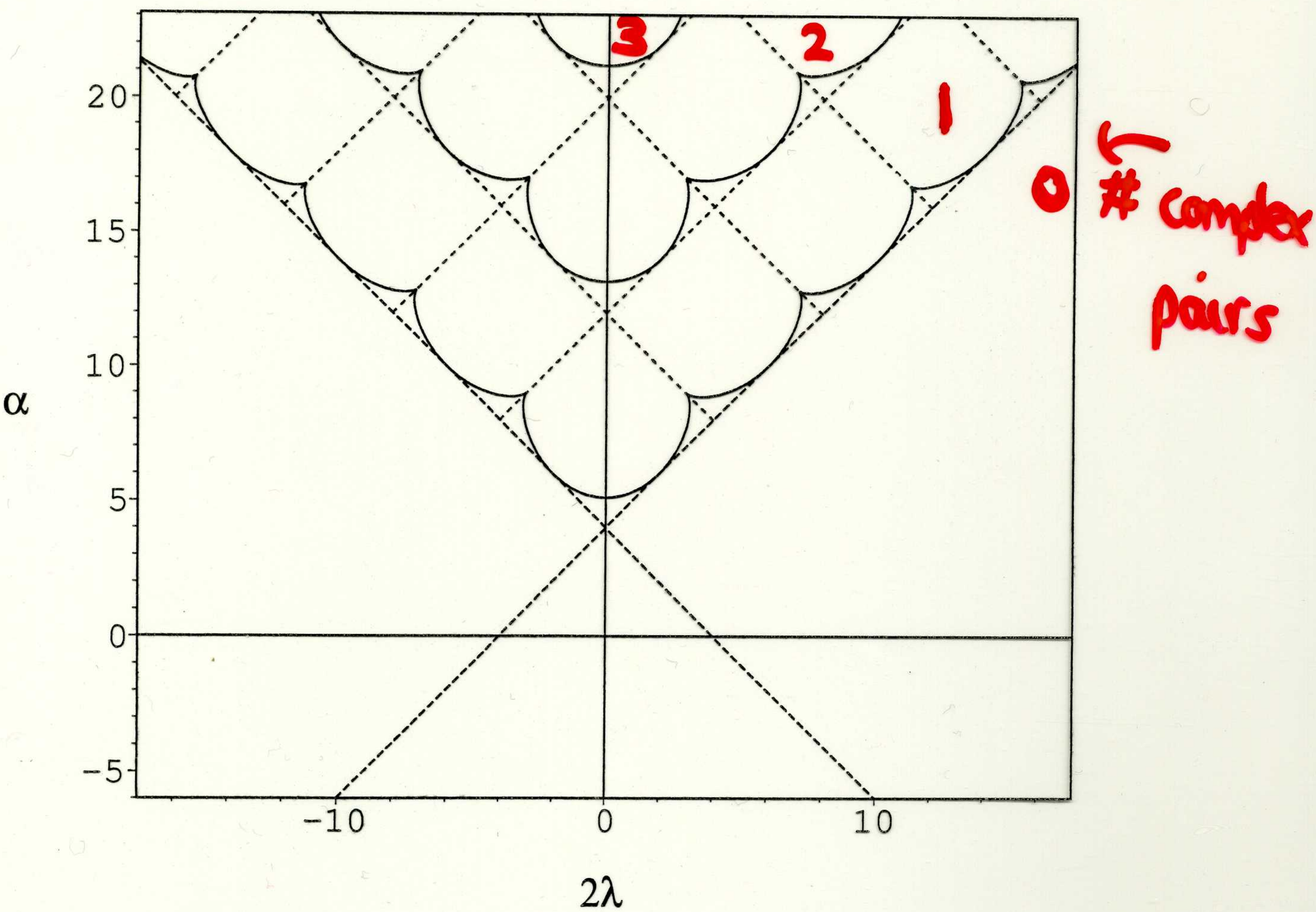


α

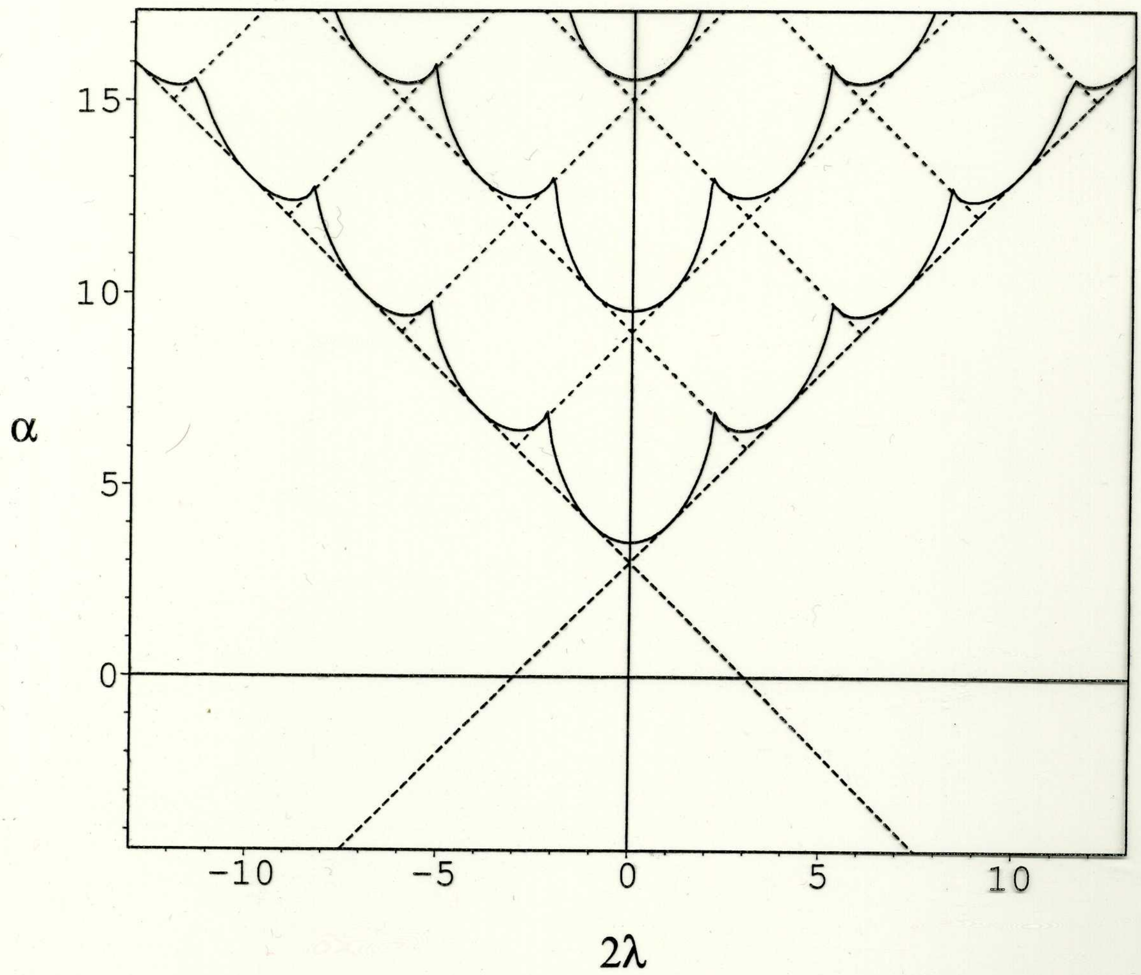
$2\lambda = 2\ell + 1$

$$H = p^2 - (ix)^{2m} - \alpha(ix)^{m-1} + \frac{\lambda^2 - \frac{1}{4}}{x^2}$$

m = 3



M=2



Exceptional points, Jordan blocks and self-orthogonality

Real E's collide to form complex-conjugate pairs

$E_0 = E_1 = \ldots = E_j$ but also $\psi_0 = \psi_1 = \ldots = \psi_j$ (no 'real' degeneracy)

Hamiltonian has Jordan block at this exceptional point

$$\begin{pmatrix} 0 & 1 & 0 & & \\ & 0 & 1 & 0 & \\ & & \ddots & \ddots & \\ & & & & 1 \\ & & & & 0 \end{pmatrix}$$

Jordan chain spans eigenspace

$$(H - E_0)\,\phi^k = \phi^{k-1} \qquad \phi^{-1} = 0 \qquad \phi^0 = \psi_0$$

$$\rightsquigarrow \{\phi^{j-1}, \phi^j, \ldots, \phi^0\}$$

$(f, Hg) = (Hf, g)$

- For a **symmetric** inner product $(f,g) = \int_{\mathcal{C}} f(z)\,g(z)\,dz$

  $\psi_0 = \phi^0$ is **self-orthogonal** $(\psi_0, \psi_0) = 0.$

  $$\left[\ (\psi_0, \psi_0) = ((H - E_0)\psi_1, \psi_0) = (\psi_1, (H - E_0)\psi_0) = 0\ \right]$$

- Jordan block occurs when $(\psi, \psi) = 0$

- Converse also true, so use this $(\cdot, \cdot)$ to hunt for exceptional points

Exceptional points when $E=0$

(for $E\neq 0$ we don't know ψ in general)

Zero-energy eigenfunction when $\alpha_- = n$, $n\in \mathbb{Z}^+$

$$\psi = (ix)^{\frac{1}{2}+\lambda}\, L_n^{\left(\frac{2\lambda}{M+1}\right)}\left(-\frac{2(ix)^{M+1}}{M+1}\right) e^{\frac{(ix)^{M+1}}{M+1}}$$

Then

pochhamer $(x)_k = x(x+1)\dots(x+k-1)$

$$(\psi,\psi) = \frac{1}{\Gamma\left(1-\frac{2(1+\lambda)}{M+1}\right)} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \left(1-\frac{2}{M+1}-k\right)_n \left(\frac{2\lambda+2}{M+1}\right)_k \left(1+\frac{2\lambda}{M+1}+k\right)_{n-k}$$

EPs:

$\left(\alpha_+ = n+m-\frac{2}{M+1}, \alpha_-=n\right)$ $\quad m\in\mathbb{N}$

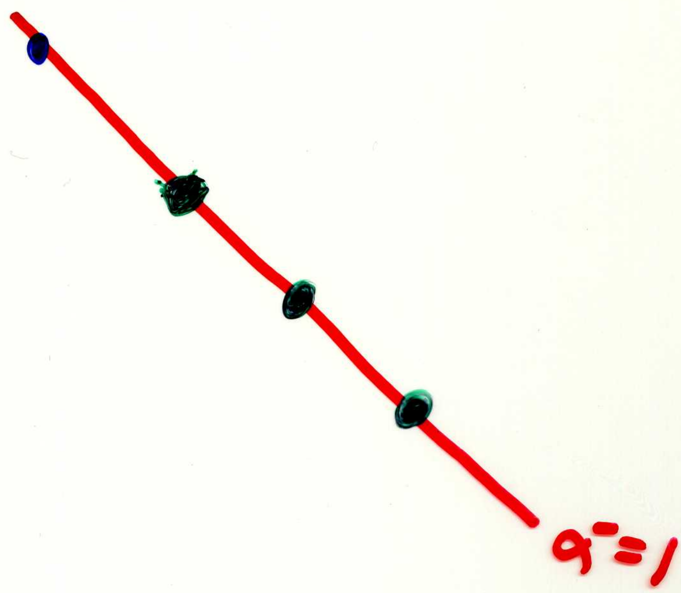
$\left(\alpha_+ = \frac{2(m-1)}{(M+1)k}, \alpha_-=1\right), \dots \left(\alpha_+=-, \alpha_-=n\right)$

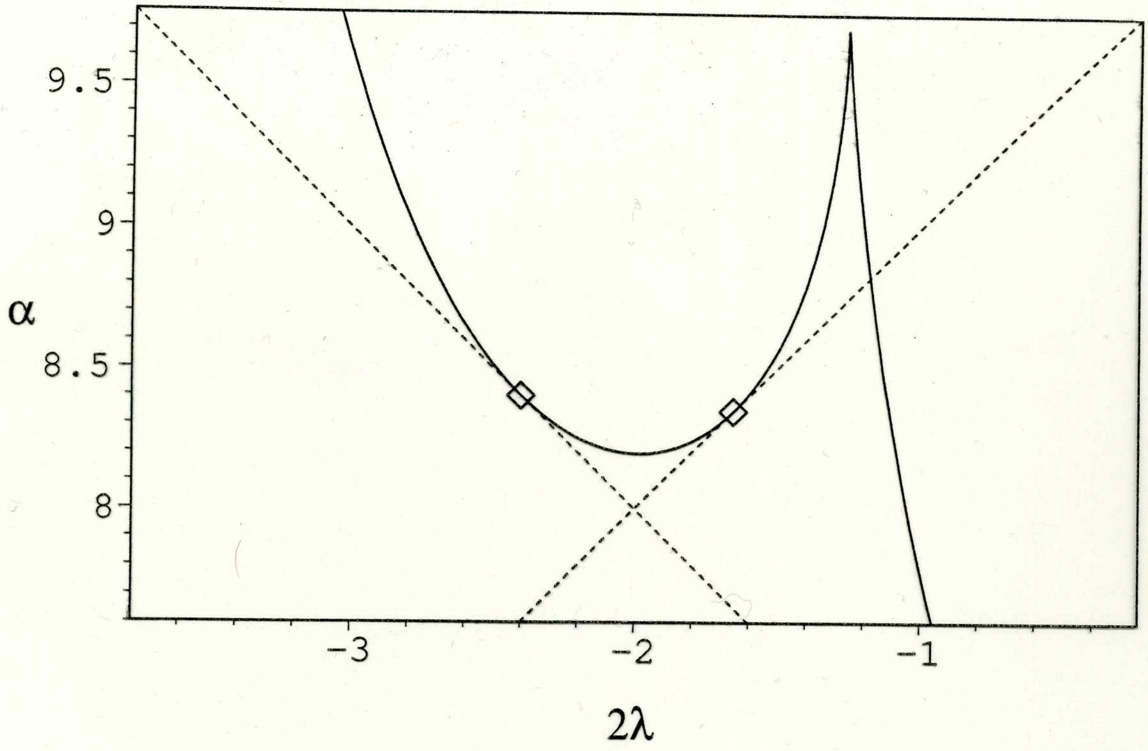
- Same degree-n polynomial as that for exceptional points with $M=3$

- Shows there are exactly n cusps on QES lines $\alpha_\pm = n$, matching our numerical observations.

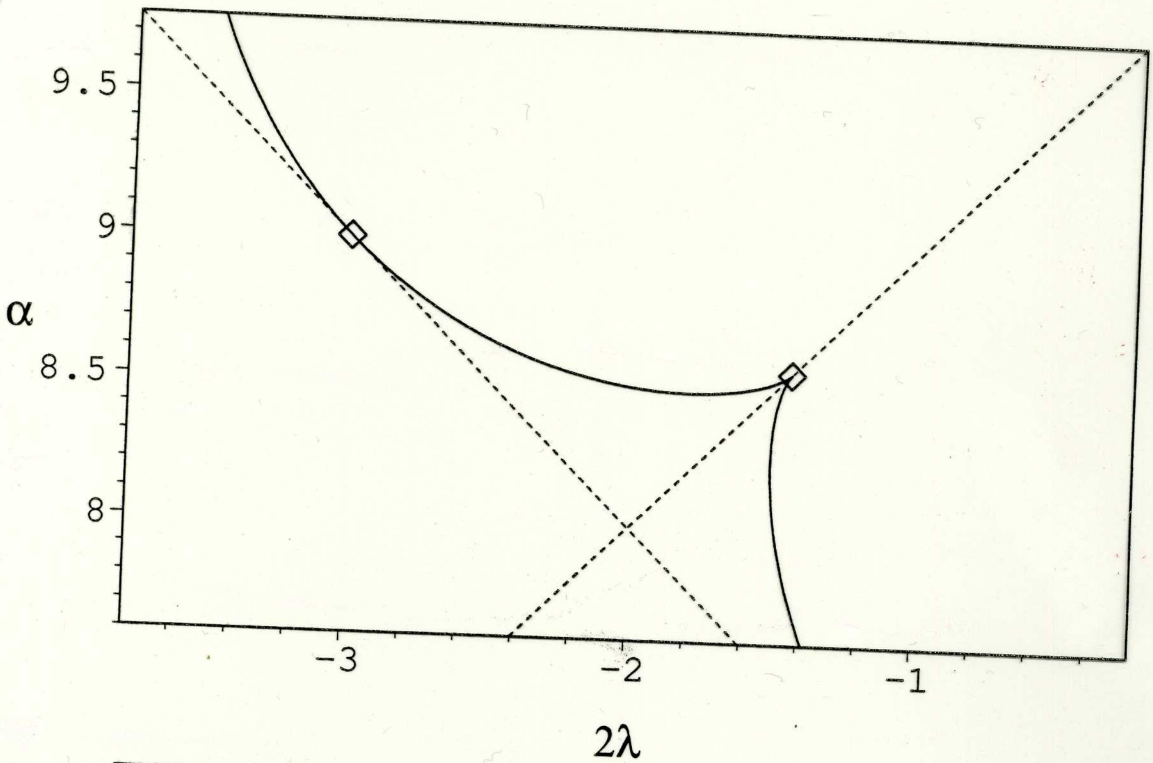
- We can say a bit more at $M=3$ on QES lines, but not today, see soon-to-appear paper. (Jordan blocks, # complex levels along QES line, ...)

- ∞-many EPs from L-function
- N EPs from polynomial

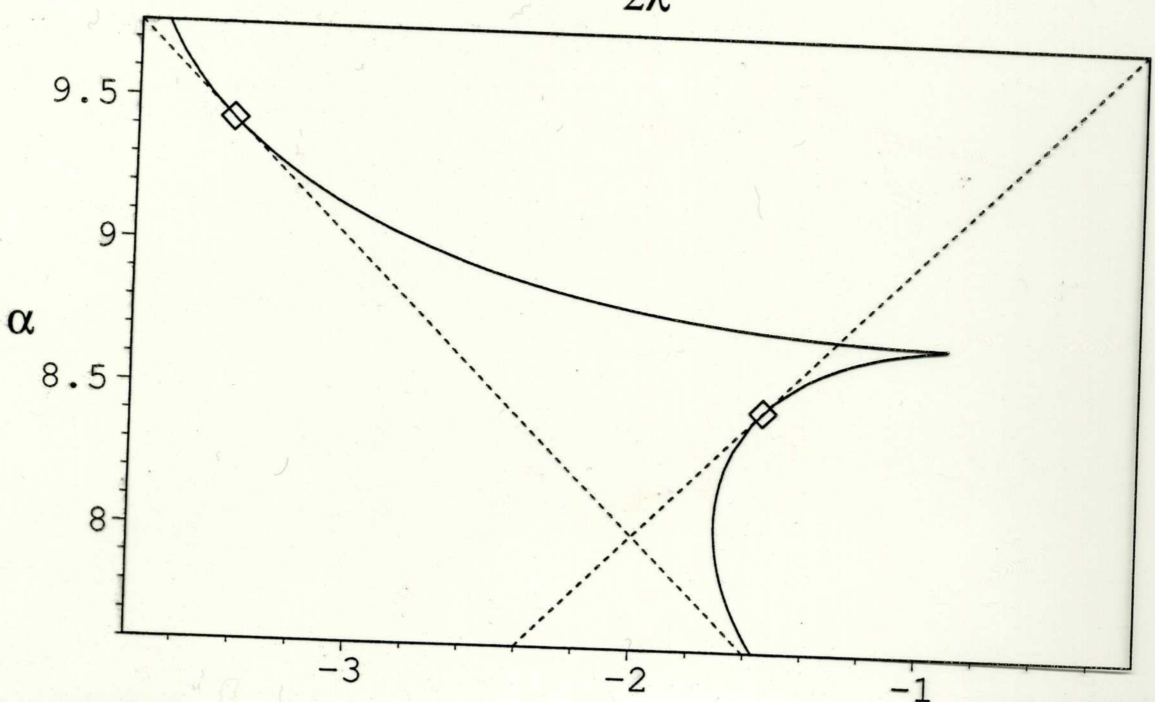




M=1.5



M=3



M=6

Back to $M=3$

- Zero-energy lines do not constrain cusps to lie on them at general M.

Result

- At $M=3$, the quasi-exact solvability does.

Quasi-exact solvability

$$H = p^2 + x^6 + \alpha x^2 + \frac{\ell(\ell+1)}{x^2} \qquad \lambda^2 - \frac{1}{4}$$

When $\alpha = 4J + \lambda$

$$\psi(x) = e^{\frac{(ix)^4}{4}} (ix)^{\lambda + \frac{1}{2}} \sum_{n=0}^{\infty\,(J-1)} \left(-\frac{1}{4}\right)^n \frac{1}{n!\,\Gamma(n+\lambda+1)}\, p_n(E,\lambda,J)\,(ix)^{2n}$$

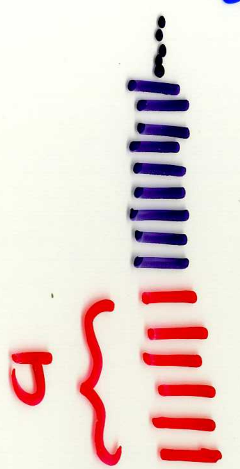
with

$$p_n = E\,p_{n-1} + 16\underbrace{(J-n-1)}_{\text{when } J-n-1=0 \text{ and } \sum_{n=0}^{\infty} \text{ truncates}}(n-1)(n-1+\lambda)\,p_{n-2}\,, \quad p_0 = 1$$

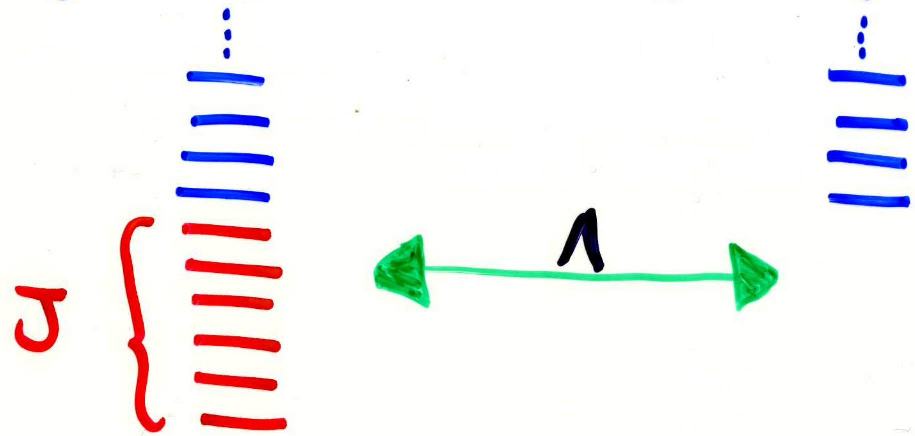
Bender - Dunne polynomial

$$p_J(E,\lambda) = p_J(E,\lambda,J) = 0$$

$\Rightarrow$ J QES eigenvalues and eigenfunctions

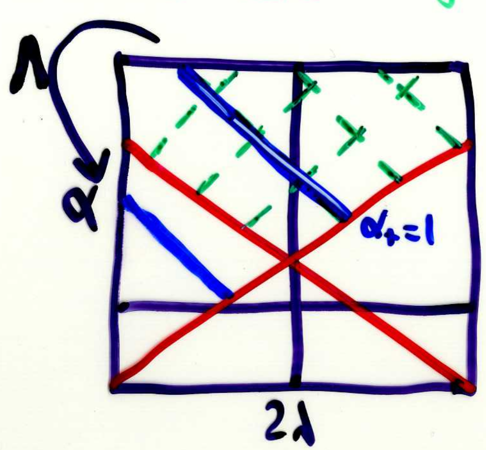


- Along QES line, complex eigenvalues are QES (use spect. equivalence)



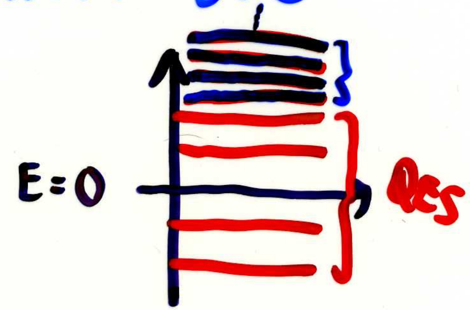
$\left(\alpha_+ = \frac{J-1}{2}, \alpha_-\right)$ $\left(\alpha_+ = -\frac{(J+1)}{2}, \alpha_- = -\frac{J}{2}\right)$

(Proof of spectral equivalence uses higher-order supersymmetry DDT '01)



- M=2 - see unproven Bender et. al similar result

- QES eigenvalues are symmetric about $E=0$



- Complex eigenvalues arise by 2 or 3 eigenvalues coinciding at $E=0$.

- Find cusps by locating triple zeros of Bender Dunne polynomials $P_{\underbrace{2n+1}_{=J}}$ at $E=0$:

zeros of $$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \left(\tfrac{1}{2}-k\right)_n \left(\tfrac{1}{2}+\tfrac{J}{2}\right)_k \left(1+\tfrac{J}{2}+k\right)_{n-k}$$

$M=1$ exact solution

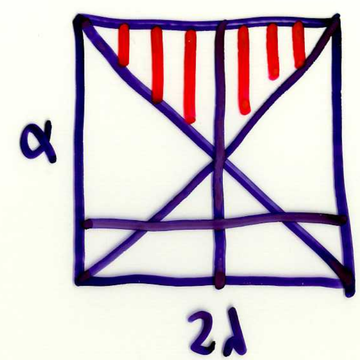
$$H = p^2 + x^2 + \frac{\lambda^2 - \frac{1}{4}}{x^2} + \alpha \qquad \psi \in L^2(\mathbb{R})$$

Solution: $E_n^{\pm} = -\alpha + 4n + 2 \pm 2\lambda$

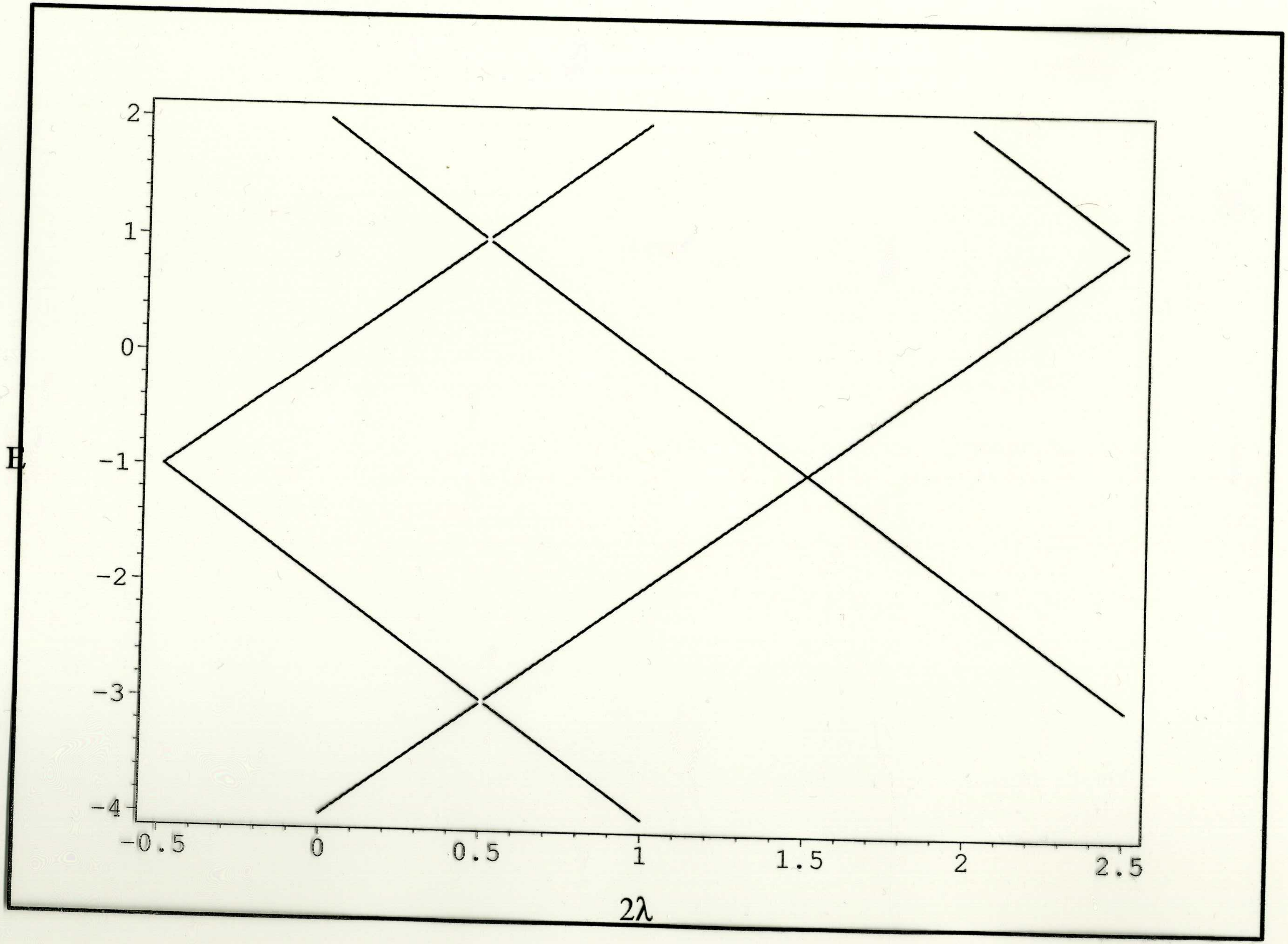
$$\psi_n^{\pm} = x^{\frac{1}{2}+\lambda} e^{-\frac{x^2}{2}} L_n^{\pm\lambda}(x^2)$$

- As M increases, complex eigenvalues appear from

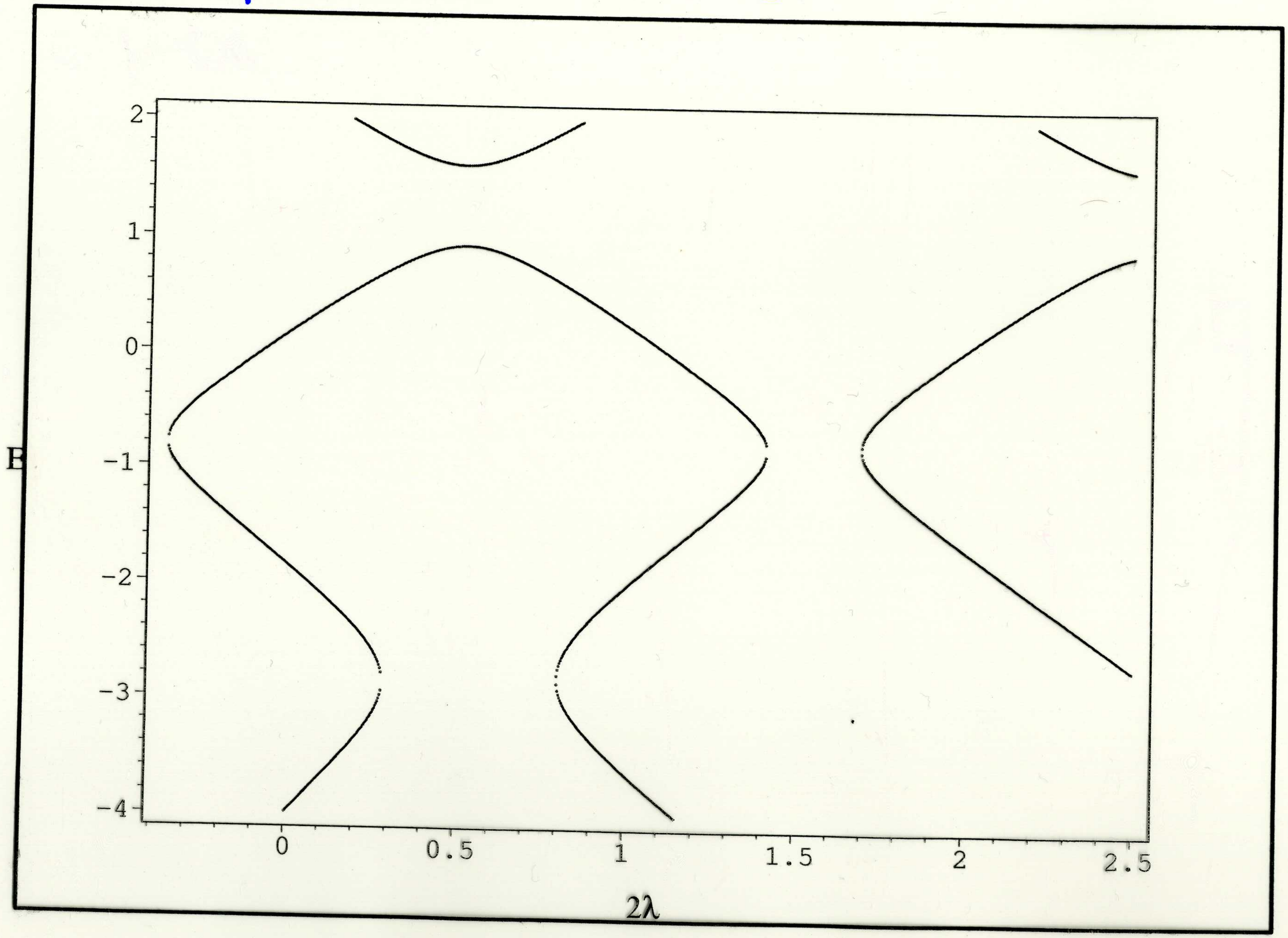
$$E_n^{+} = E_m^{-} \Rightarrow \lambda = n - m$$



Spectrum at $l = M$



Spectrum at M = 1.05



M near to 1   perturbative solution

- Take $M = 1 + \varepsilon$ ← small
- Truncate Hamiltonian to two levels $\underbrace{E_p, E_{q+p}}_{\text{degenerate}}$ $(p = 0, 1, \ldots$; $q = 0, 1, \ldots)$

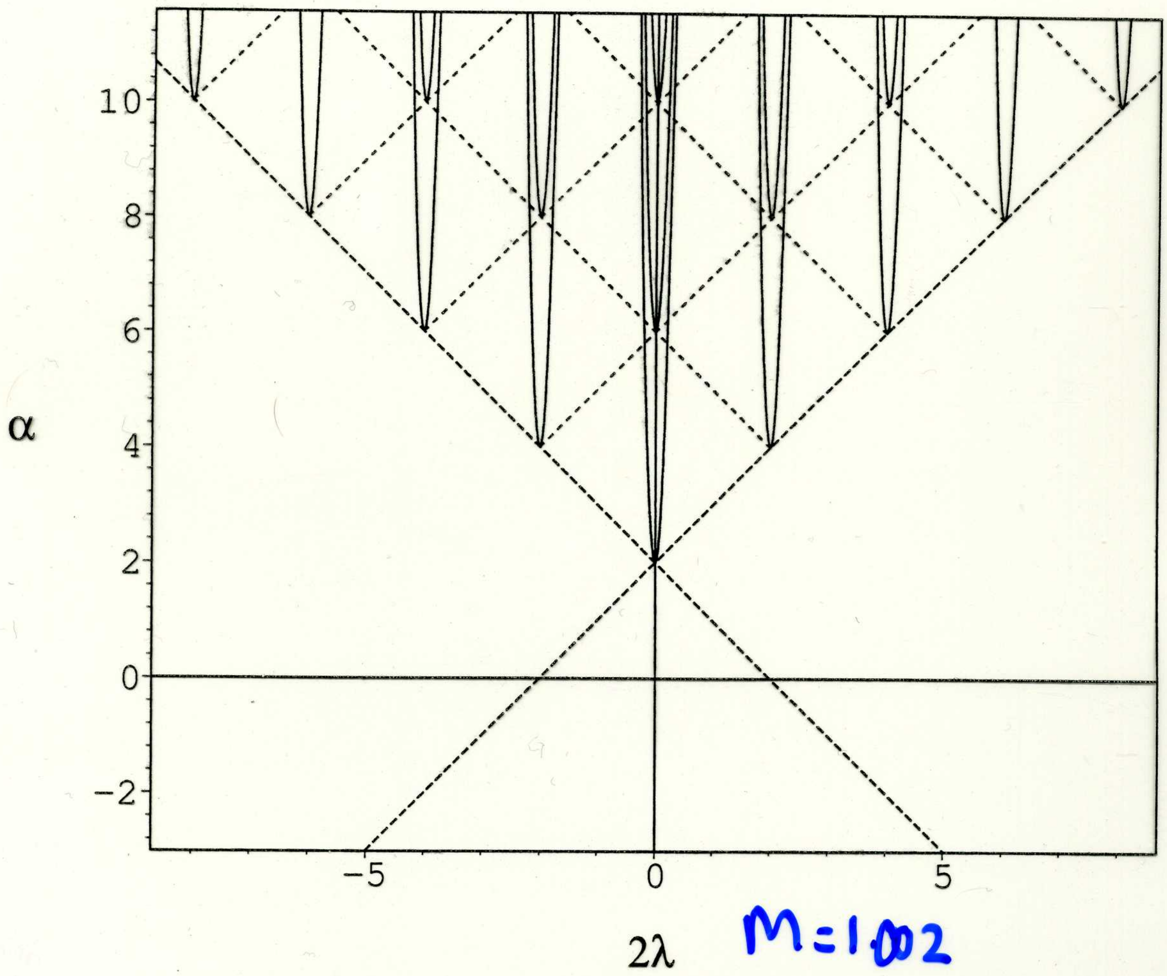
  by tuning $\lambda = q + \eta$ ← small

- Result:

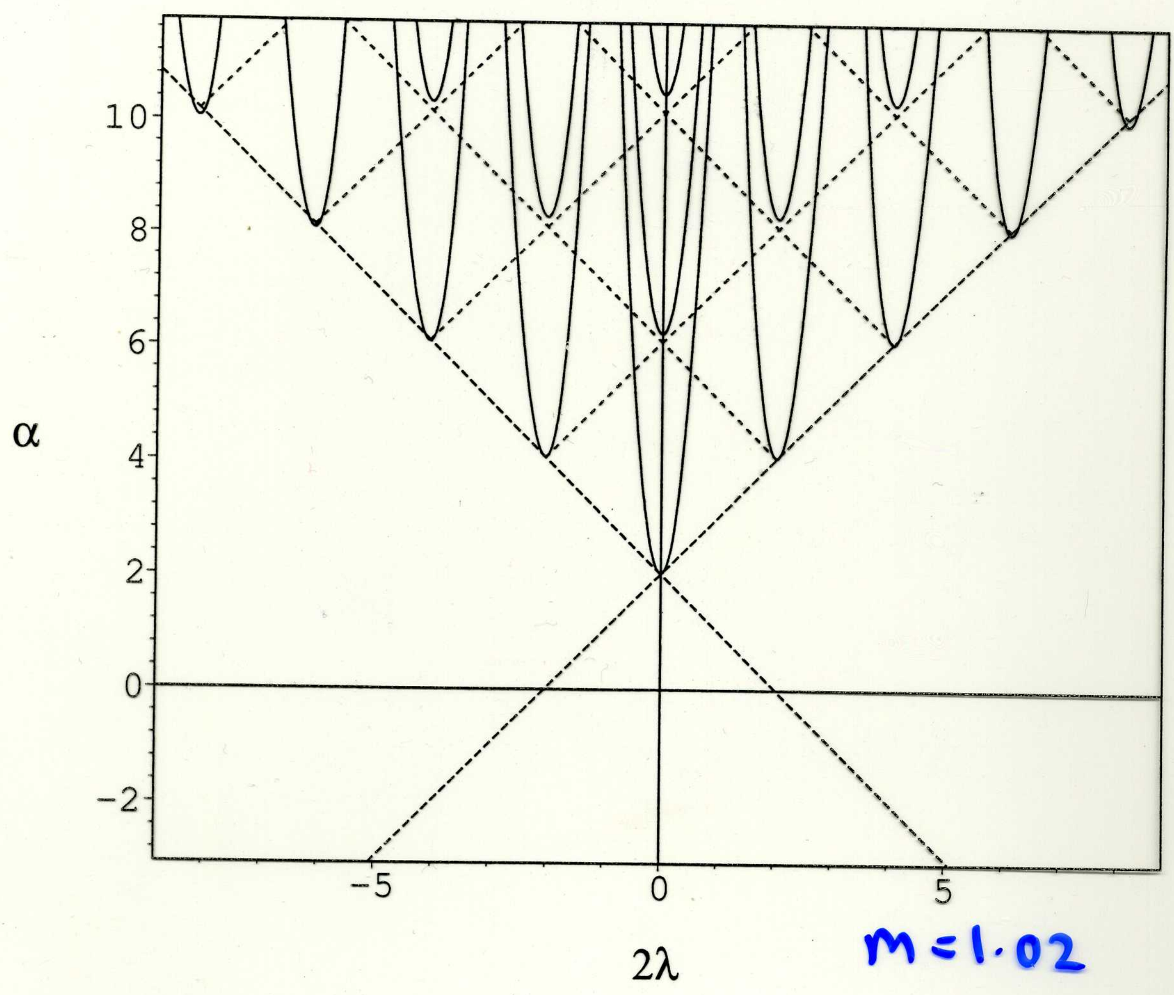
$$E_{\pm} \approx E_p \pm \left(a_1 \varepsilon \pm \left(a_2 \varepsilon + a_3 \varepsilon^2 + a_4 \varepsilon \eta + a_5 \eta^2 \right)^{1/2} \right)$$

$= 0$ gives curve in (α, λ) plane of exceptional points

parabola



$m = 1.002$



$m = 1.02$

<u>$M=\infty$</u> also exact solution via variable change

$$H_\infty = p^2 + x^2 - \bar{E}(-ix)^{2\bar{M}} + \frac{\bar{\lambda}^2 - \frac{1}{4}}{x^2} - \bar{\alpha}$$

with $\bar{M} = -1 + \frac{2}{M+1}$ $\quad \bar{E} = \left(\frac{2}{M+1}\right)^{\frac{2M}{M+1}}$ $\quad \bar{\lambda} = \frac{2\lambda}{M+1}$ $\quad \bar{\alpha} = \frac{2\alpha}{M+1}$

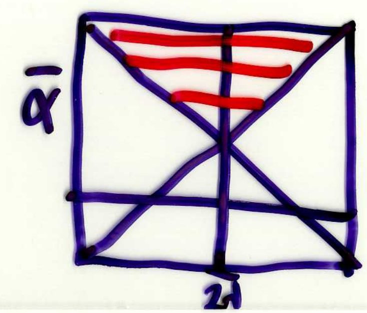
Solution:

$$\bar{E}_n^{\pm} = -\bar{\lambda}^2 + \left(2n+1-\frac{\hat{\alpha}}{2}\right)^2$$

$$\psi_n^{\pm} = x^{\frac{1}{2} \pm \sqrt{\bar{\lambda}^2 + \bar{E}_n^2}}\, e^{-x^2/2}\, L_n^{\pm\sqrt{\bar{\lambda}^2 + \bar{E}_n^2}}(x^2)$$

- As M decreases / complex eigenvalues appear from

$$\bar{E}_n^{+} = \bar{E}_m^{-} \quad \Rightarrow \quad \hat{\alpha} = 2(n-m)$$



M near to infinity perturbative solution

- Take $\bar{M} = -1 + \frac{2}{1+M} = -1 + \varepsilon$ ← small

- Truncate Hamiltonian to two levels $\underbrace{\bar{E}_p, \bar{E}_{q-p}}_{\text{degenerate}}$ $q = 1, 2, \dots$; $p = 0, 1, \dots \left[\frac{q}{2} - \frac{1}{2}\right]$

  by tuning $\bar{\alpha} = 2q + 2 + \eta$ ← small

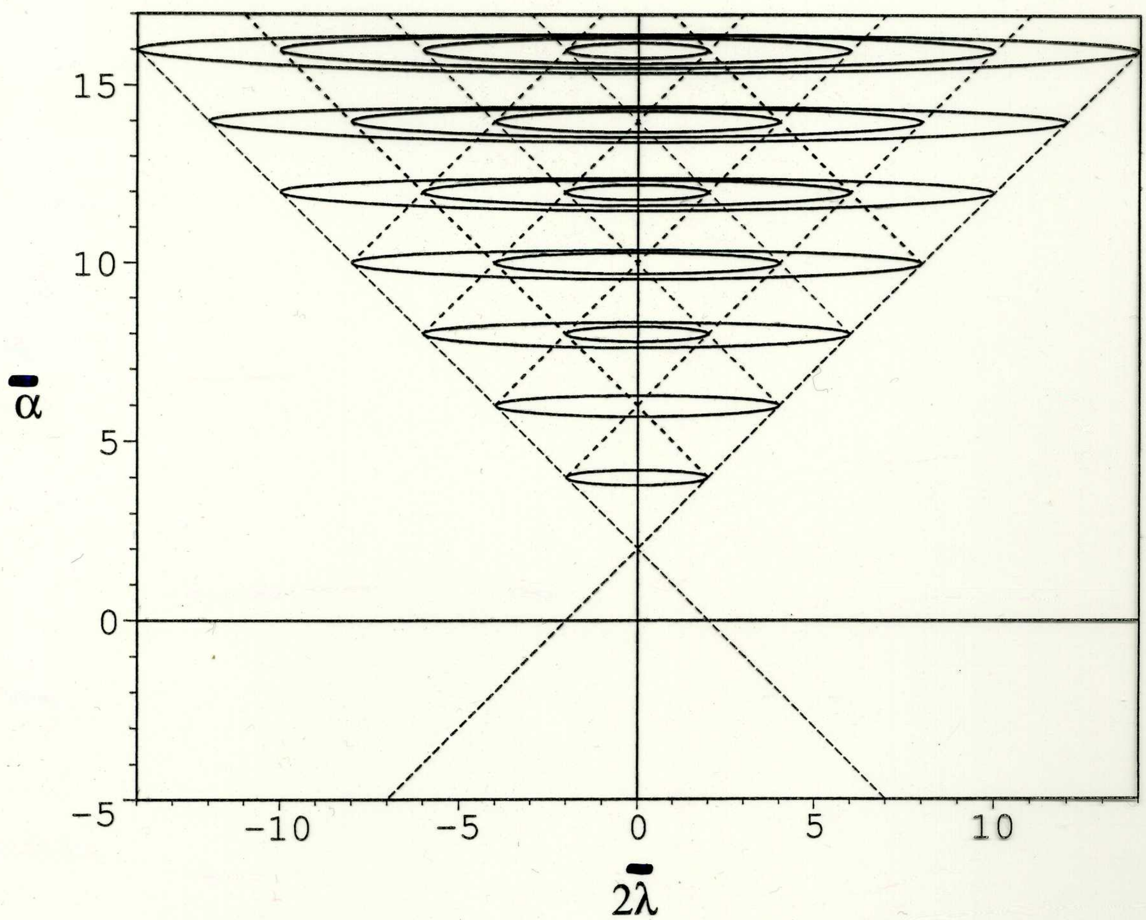
- Result:

$$\bar{E}_{\pm} \approx \bar{E}_p + b_1 \varepsilon \pm \left(b_2 \varepsilon + b_3 \varepsilon^2 + b_4 \varepsilon \eta + b_5 \eta^2 \right)^{1/2}$$

$=0$ gives curve in (α, λ) plane of EPs

ellipse

M = 180



To conclude

- Phase diagram of spectral unreality mapped out and understood for all $M \geq 1$.

- Lots of interesting mathematics (spectral equivalences, SUSY, Jordan blocks, exceptional points, quasi-exact solvability, dualities, ...) and methods

- Surely, nothing more to be done...

  Ah, but $M < 1$ is also interesting

  Goes from finitely many to ∞-many complex eigenvalues

M = 0.98