Hydrodynamic instabilities in quantum liquids

Non-linear quantum hydrodynamics of electronic liquids

- Degenerate Fermi gas in one spatial dimension,
- Edge state in the Fractional Quantum Hall Effect,
- Luttinger Liquid.

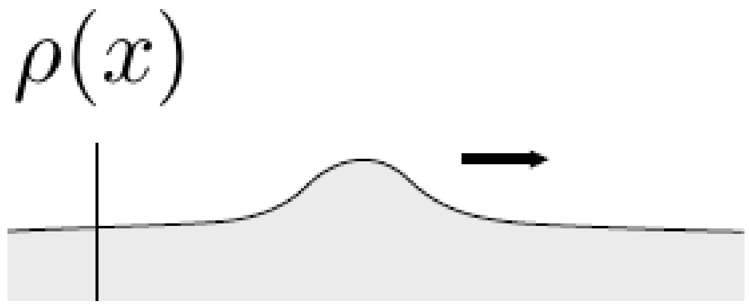
- Why Fermi statistics causes non-equilibrium quantum liquids to be essentially nonlinear,

- Instabilities and singularities arising as a result of nonlinear nature of quantum liquids,

- Mathematical methods for analysis of singularities in dynamics of quantum liquids - Whitham theory,

- Relation to random matrix theory and to perturbed conformal field theory.

Electronic systems in 1D



A smooth bump in density or momenta:

all gradients << Fermi scale

$$< \Psi| =< 0|e^{\sum_{pq} A_{pq} c^\dagger_p c_q}$$

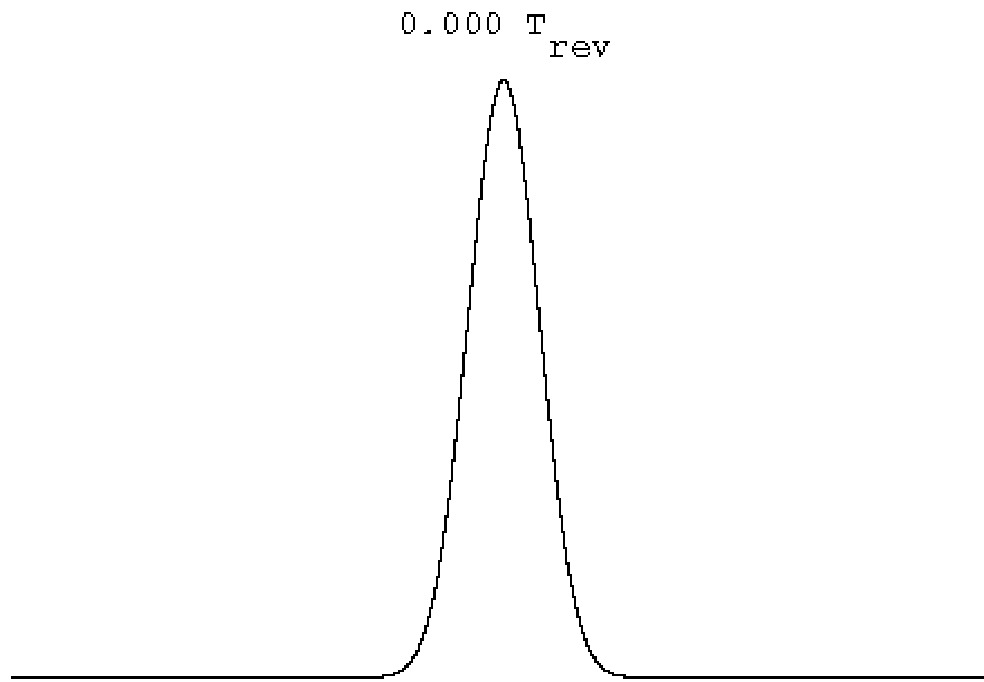
$$< \Psi|e^{-iHt}$$

Wave packet in quantum mechanics <u>diffuses</u>

$$i\partial_t \Psi = \frac{}{2m}\nabla^2 \Psi$$

$$\psi(x, t=0) \sim \frac{1}{\sqrt{\Delta}} e^{-\frac{x^2}{2\Delta}}$$

$$|\psi(x, t > \Delta)| \sim \frac{1}{\sqrt{t\Delta}} e^{-\frac{x^2\Delta}{2t^2}}$$



Does a quantum coherence

(Fermi sea, statistics, interaction)

make an impact?

Quantum hydrodynamics (Landau 1941):

$$N \to \infty \begin{cases} \text{density} \quad \rho(x) = \frac{1}{N}\sum_i \delta(x - x_i), \\ \text{current} \quad j(x) = \frac{1}{N}\sum_i \dot{x}_i \delta(x - x_i), \\ \text{velocity} \quad j(x) = \rho v, \\ [v(x),\, \rho(y)] = -i \ \nabla\delta(x - y). \end{cases}$$

Interaction and/or Fermi statistics

cause the hydrodynamics to be non-linear

and to be a subject of hydrodynamics

instabilities, and singularities, in particular,

shock fronts, and stabilities - solitons;

Free Fermions:

Wigner Function

$$W(x,p,t) = \int \mathrm{tr}\left[\psi^\dagger(x+y/2,t)\psi(x-y/2,t)\hat{\varrho}\right] e^{imyv/\hbar} dy$$

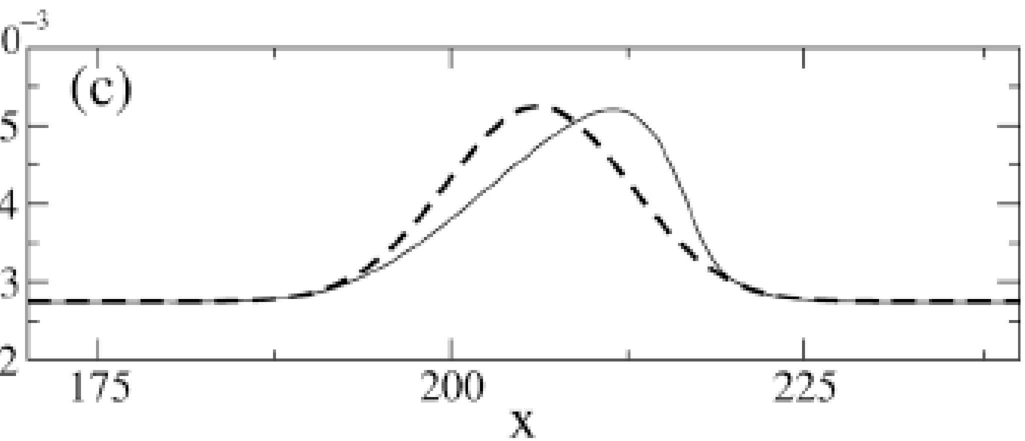
$$(\partial_t + p\partial_x)W(x,p,t) = 0$$

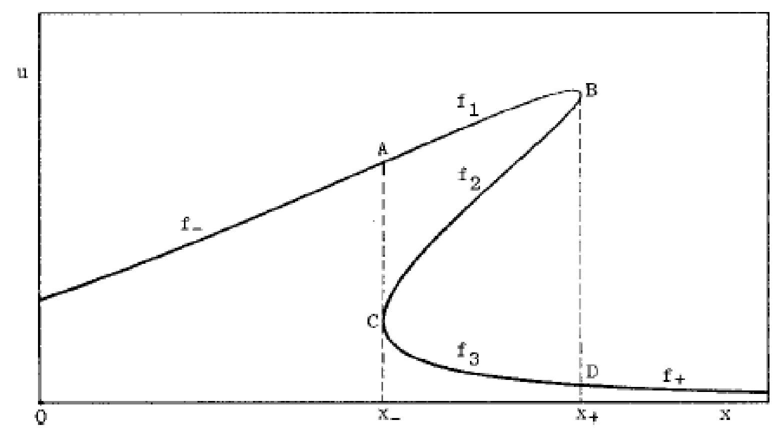
$$W(x,p,t) = n_F\big(p - p(x)\big)$$

Fermi profile: Riemann Equation

$$\partial_t v(x,t) = v(x,t)\partial_x v(x,t)$$

Shock wave:



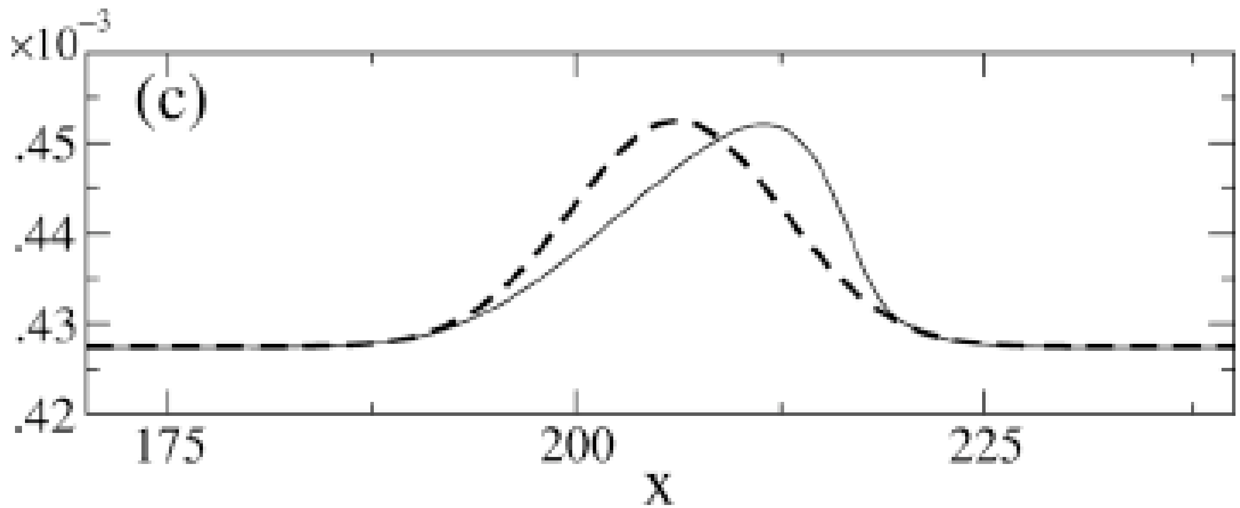


Dispersion - asymmetry between particles and holes

$$E(p) = \frac{p^2}{2m} - E_F = v_F(\pm k + \frac{k^2}{2m})$$

$$p = \pm k_F + k \qquad\qquad v(k) = v_F(1 + \frac{k}{2k_F})$$

p(x)



$$H = -\sum_i \Big(-\partial_i^2 + \sum_{i \neq j} V(x_i - x_j) \Big)$$

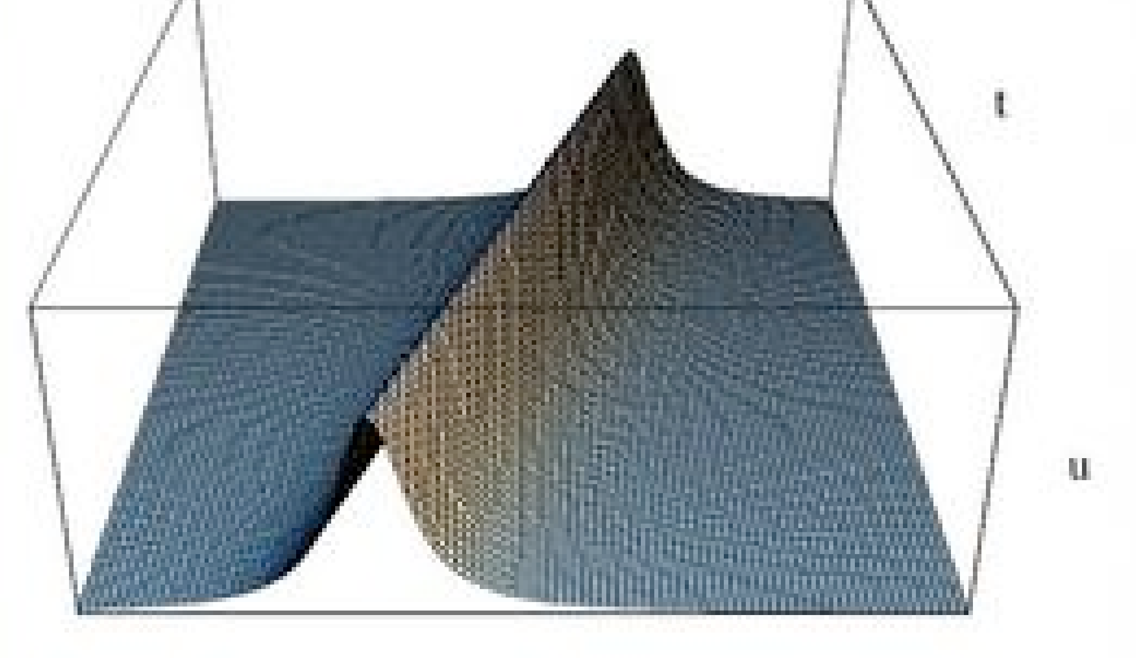
$$H_0 = c^\dagger (-\frac{\nabla^2}{2m} - E_F) c$$

$$H = H_0 + \int c^\dagger(x) c(x) V(x-y) c^\dagger(y) c(y) dx dy$$

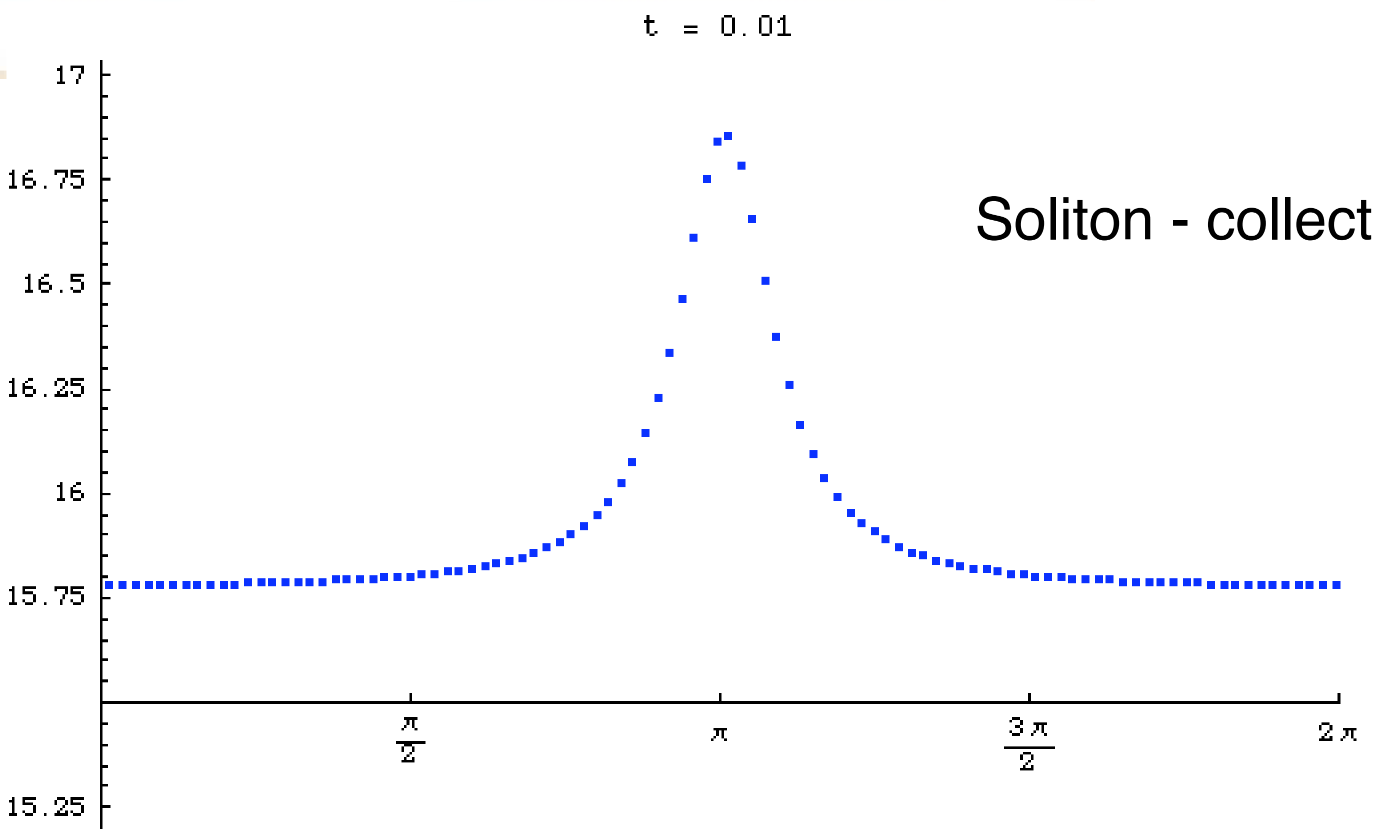
Model Hamiltonian: Calogero model

$$V(x) = \wp(x) \to \frac{1}{x^2}, \quad \frac{1}{\sinh^2 x},$$

Interpolates between Lattinger liquid and Calogero model
- quantum wires, edge states of FQHE

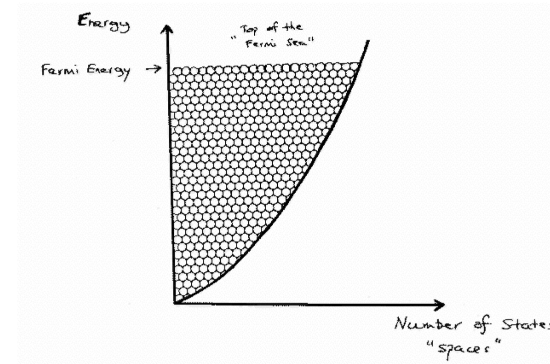


$$u(x,t) = \frac{2V}{V^2(x - x_0 - Vt)^2 + 1}$$



Soliton - collective excitation of particles

Free Fermions: $H = \sum_p \frac{p^2}{2m} \psi_p^\dagger \psi_p$



$$H = \int \left(\frac{\rho v^2}{2} + \frac{\pi^2}{6} \rho^3 \right) dx.$$

Kinetic energy Potential (Fermi Energy)

$$E = \int_{|p|<p_F} \frac{p^2}{2m} \frac{dp}{2\pi} = \frac{\pi^2 \ {}^2}{6m} \rho_0^3;$$

$$\rho_0 = \int_{|p|<p_F} \frac{dp}{2\pi}$$

Quantum Riemann Equation

$$H = \int \left(\frac{\rho v^2}{2} + \frac{\pi^2}{6}\rho^3 \right) dx.$$

$$\dot{\rho} + \nabla(\rho v) = 0 \qquad \text{Continuity equation}$$

$$\dot{v} + \frac{1}{2}\nabla(v^2 + \pi^2\rho^2) = 0 \qquad \text{Euler's equation}$$

$$\nabla\varphi_{R,L} \equiv \pm J^{R,L}(x,t) = v \pm \pi\rho$$

$$\dot{\varphi} + \frac{1}{2}(\nabla\varphi)^2 = 0 \qquad \dot{J} + J\nabla J = 0$$

Current Algebra

$$J^{R,L}(x) = \sum_k e^{ikx} J_k^{R,L}, \quad J_k^{R,L} = \sum_{\pm p > 0} \psi_p^\dagger \psi_{p+k}.$$

$$\rho = \frac{1}{2\pi}\left(J^R + J^L\right), \quad v = \frac{1}{2}\left(J^L - J^R\right)$$

$$[J_k^R, J_l^R] = [J_k^L, J_l^L] = \frac{k}{2\pi}\delta_{k+l,0}, \quad [J_k^R, J_l^L] = 0$$

$$[\rho(x),\, v(y)] = -i\nabla\delta(x-y)$$

F. Bloch 1934, Tomonaga 1950

Equation of motion

$$\dot{J} = i[H,\, J] = -J\nabla J$$

$$\dot{\varphi} + \frac{1}{2}(\nabla\varphi)^2 = 0$$

$$H = \sum_p \frac{p^2}{2m}\psi_p^\dagger\psi_p$$

$$H = \int \left(\frac{\rho v^2}{2} + \frac{\pi^2}{6}\rho^3\right) dx.$$

Schick 1968, Haldane 1978, Sakita 1974, Jevicki 1991

Non-linear hydrodynamics

Free fermions (exact):

Non-linear bosons

$$\boxed{\dot{\varphi} + (\partial_x \varphi)^2 = 0}$$

Quantum Riemann equation

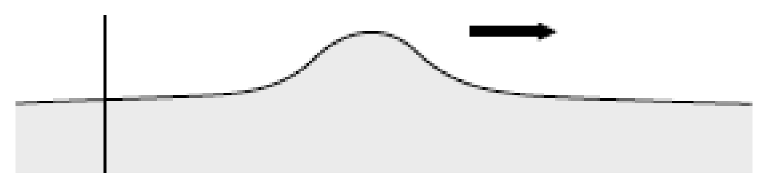
Non-linear effects reflect Fermi-statistics and dispersion

Linearized equation - sound modulation of density

$$\dot{\varphi} + v_F \partial_x \varphi = 0$$

$$\frac{p^2}{2m} - E_F \sim \pm v_F (p \pm k_F)$$

$$\varphi(x,t) = \varphi(x - v_F t, 0)$$



Shape does not change!?

Free fermions: Shock waves are stabilized by quantum corrections

$$\boxed{\dot{\varphi} + (\partial_x \varphi)^2 = 0} \qquad \langle \varphi^2 \rangle = \langle \varphi \rangle^2 + \langle\langle \varphi^2 \rangle\rangle$$

Non-linear hydrodynamics for Free Fermions are essentially quantum;

Different matrix elements have different behavior;

After-shock oscillations with a period $\sim 1/k_F$

Vertex Operators

$$V_a(x) =: e^{ia\varphi} : \qquad a = 1 - \text{fermion}$$

$$(i\partial_t - \frac{1}{2}\nabla^2)e^{ia\varphi} = \frac{1}{2}a(a+1)e^{ia\varphi}T,$$
$$(i\partial_t + \frac{1}{2}\nabla^2)e^{i(a+1)\varphi} = -\frac{1}{2}a(a+1)e^{i(a+1)\varphi}\bar{T}$$

$$T = (\nabla\varphi)^2 - i\nabla^2\varphi$$

holomorphic components of the stress-energy tensor of a chiral Bose field (with the central charge 1/2)

Hirota's bilinear form

$$\left(iD_t - \frac{\hbar}{2m}D_x^2\right) e^{ia\phi} \cdot e^{i(a+1)\phi} = 0$$

Hirota derivatives

$$Df \cdot g = \partial f g - f \partial g,$$

$$D^2 f \cdot g = \partial^2 f g - 2\partial f \partial g + \partial^2 g f$$

$$\varrho = \exp\left(\sum_{p,q} A_{pq}\psi_p^\dagger \psi_q\right).$$

Density matrix - coherent state

$$gl(\infty)$$

$$\tau_a =< 0|e^{ia\varphi(x,t)}\varrho|0 >$$

$$\left(iD_t - \frac{\hbar}{2m}D_x^2\right)\tau_a \cdot \tau_{a+1} = 0$$

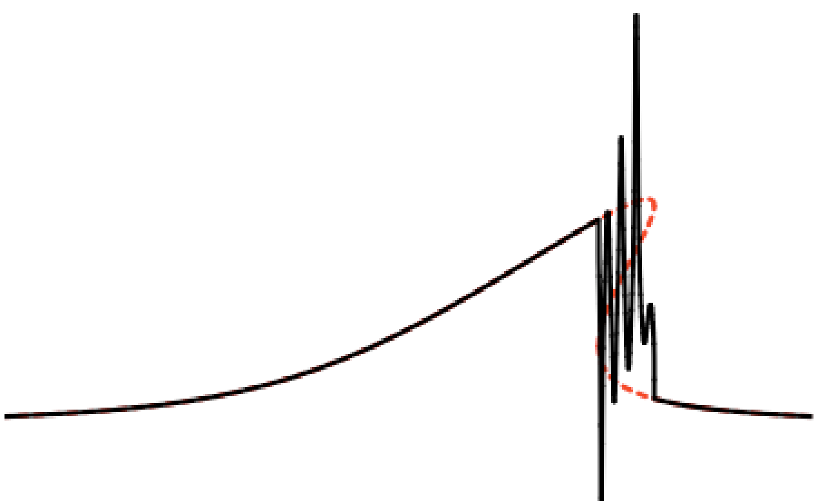
$$\left(iD_t - \frac{\hbar}{2m}D_x^2\right)V_a \cdot V_{a+1} = 0, \quad V_a = e^{ia\varphi}$$

$$\left(iD_t - \frac{\hbar}{2m}D_x^2\right)\tau_a \cdot \tau_{a+1} = 0$$

modified KP equation

$$\Phi = i\frac{\hbar}{m}\log\left(\frac{\tau_a}{\tau_{a+1}}\right), \qquad \tilde{\Phi} = \frac{\hbar}{m}\log(\tau_a\tau_{a+1})$$

$$\partial_t\Phi + \frac{1}{2}(\nabla\Phi)^2 - \frac{\hbar}{2m}\nabla^2\tilde{\Phi} = 0$$



Higher derivative stabilizes the overhang

General Periodic Solution

$$\tau_a = e^{\frac{i}{\hbar} a\theta_F} \det_{i,j}(\delta_{ij} + K_a(p_i, q_j))$$

$$K_a(p_i, q_j) = \frac{\sin(\pi a)}{\pi} A_{p_i q_i} \left(\frac{p_i - p_F}{p_F - q_i} \right)^a \frac{e^{\frac{i}{\hbar}\theta_i(x,t)}}{p_i - q_j}$$

$$\theta(p_i, q_i) = (p_i - q_i)x - \frac{1}{2m}(p_i^2 - q_i^2)t$$

$$\theta_F = p_F x - E_F t$$

Moduli $\quad p_i > p_F > q_i, \quad E_F$

Whitham Theory:

1) Before and after the shock use dispersionless equation

$$x < X_-(t), \quad x > X_+(t) \qquad\qquad \dot{\varphi} + \frac{1}{2}(\nabla\varphi)^2 = 0$$

It is solvable explicitly

$$\varphi(x,t) = f(x - \varphi(x,t) \cdot t), \quad \varphi(x, t=0) = f(x).$$

2) Insert a simple periodic solution at $X_-(t) < x < X_+(t)$

3) Give slow space time dependence to moduli to glue the solutio

1) Simplest periodic solution (genus 1)

$$\tau_a = e^{\frac{i}{\hbar} a\Theta_F} \left[1 + A_{pq} \frac{\sin(\pi a)}{\pi} \left(\frac{P - P_F}{P_F - Q}\right)^a \frac{e^{\frac{i}{\hbar}\Theta(P,Q)}}{P - Q}\right]$$

2) Moduli obey Whitham modulation equations

$$\dot{P} + \partial_x E(P) = 0.$$

$$\dot{P}_F + \partial_x E(P_F) = 0$$

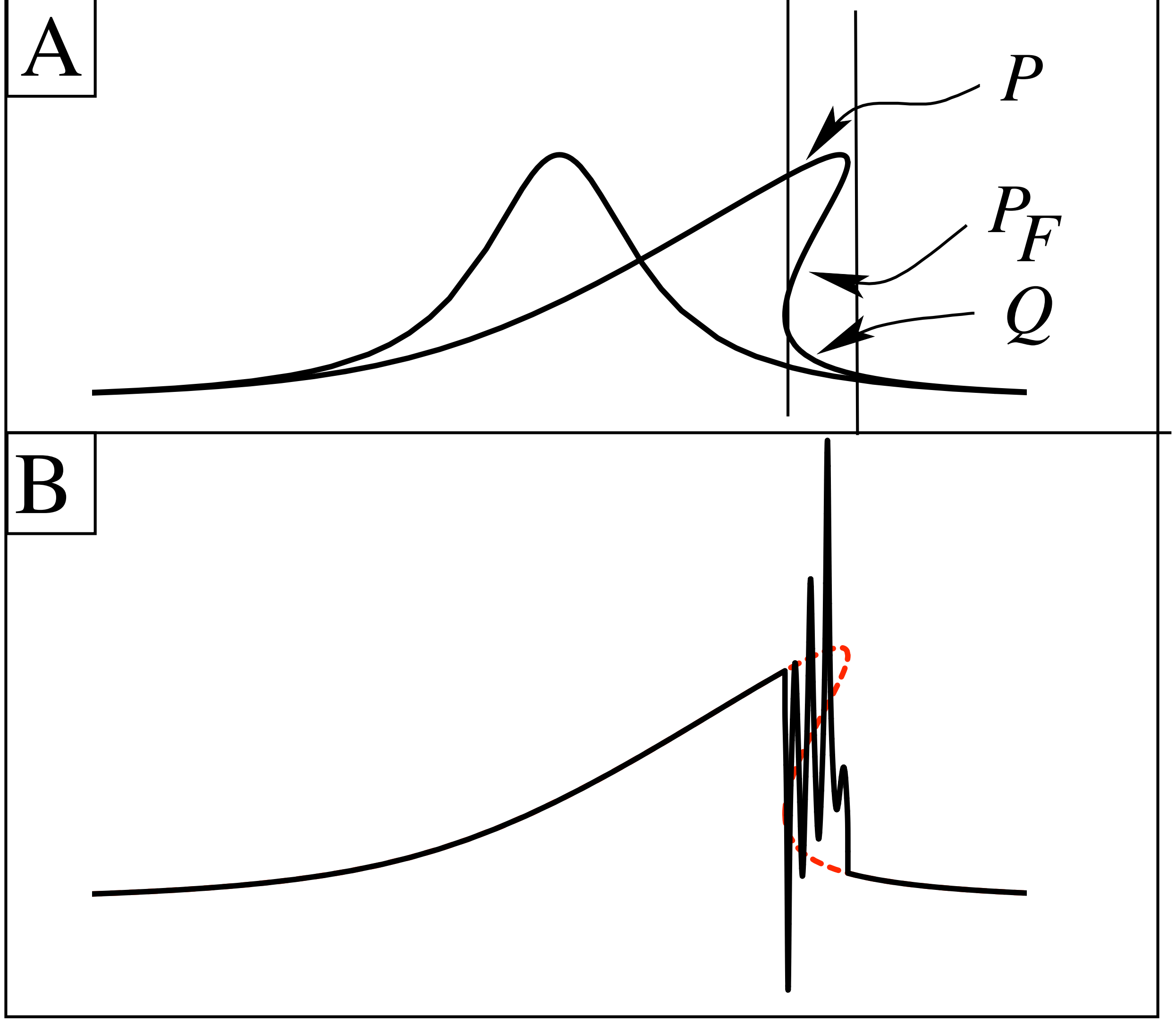
$$\dot{Q} + \partial_x E(Q) = 0$$

$$E(P) = \frac{P^2}{2}$$

Hamilton Jacobi Equations

$$\dot{\Theta} = E(P) - E(Q), \qquad \partial_x \Theta = P - Q,$$

$$\dot{\Theta}_F = E(P_F) \qquad \partial_x \Theta_F = P_F$$



$$\tau_a = e^{\frac{i}{\hbar} a \Theta_F} \left[1 + A_{pq} \frac{\sin(\pi a)}{\pi} \left(\frac{P - P_F}{P_F - Q} \right)^a \frac{e^{\frac{i}{\hbar}\Theta(P,Q)}}{P - Q} \right]$$