HAUSDORFF DIMENSIONS FOR RANDOM WALKS AND THEIR WINDING SECTORS

LPTMS ORSAY CNRS/Université Paris-Sud

with Jean Desbois JSTAT (2008) P08004 arXiv: 0804.1002

about some recent numerical simulations for random walks on a square lattice $\rightarrow$ Brownian curves in the continuous limit (in particular their Hausdorff dimensions)

new definition of a Brownian frontier $\rightarrow$ new Hausdorff dimension ?

as it is well known: the Hausdorff dimension of the external frontier of Brownian curves is $d_H = 4/3$ (numerical conjecture by Mandelbrot, proved by SLE: Lawler, Schramm, Werner ”The dimension of the planar Brownian frontier is $4/3$” Mathematical Research Letters 8 (2001))

Hausdorff dimension d_H

$P =$ perimeter of the frontier and $r =$ typical size

scaling $P = r^{d_H}$

-for example for a circle $P = 2\pi r \Rightarrow d_H = 1$

-for Brownian curves:

the external frontier is defined ”geometrically” as the set of points where one stops when one meets the curve for the first time arriving from infinity

$\longrightarrow$ no excursion inside the curve and problem of orientation for the frontier the question asked:

can we define the frontier in a different way $\longrightarrow$ oriented

Some ideas from:

- site percolation on a square lattice

- recent progress on Brownian winding sectors (SLE)$\longrightarrow$ 0-winding sectors

$\Rightarrow$ excursions inside the curve site percolation on a square lattice:

probability $0 < p < 1$

on each site of a square lattice draw randomly a number $0 < \alpha < 1$:

if $\alpha > p \longrightarrow$ insulator (white) if $\alpha < p \longrightarrow$ conductor (black)

percolation means:

-current flows if at least one edge in common

-current does not flow if one vertex in common

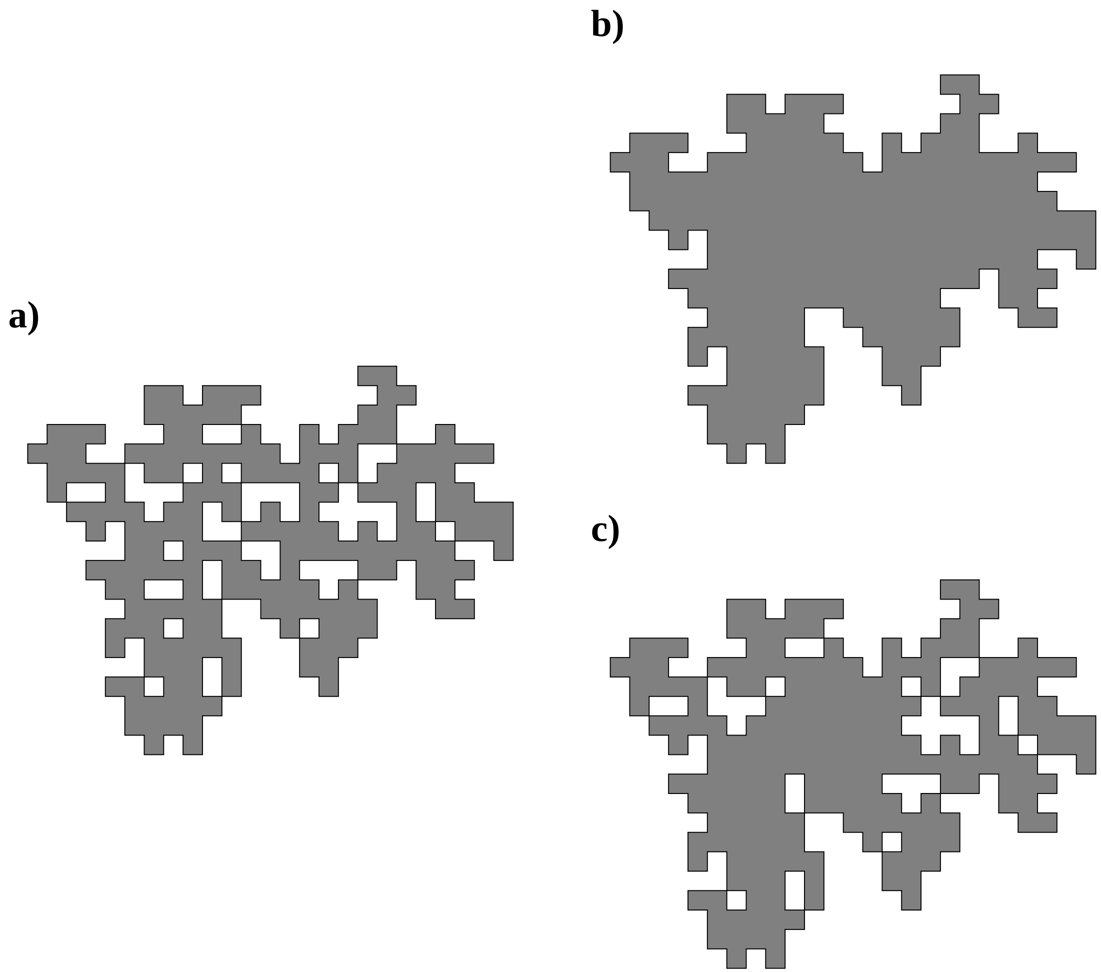
critical percolation: at $p_c = 0.593\ldots$ (numerical) the percolating cluster scales like the overall size of the lattice

still there are non percolating sectors inside the cluster:

-fjords = non percolating sectors which can be connected to the exterior (since the current does not flow)

-lakes = non percolating sectors which cannot be connected to the exterior (deep inside the cluster)

An example:



a) site percolation cluster b) external frontier c) external + fjords frontier

Hausdorff dimension: scaling of the perimeter P with the typical size r of the cluster

at critical percolation $r \simeq$ size of the lattice

-for the perimeter of the external frontier of the cluster: $d_H = 4/3$

-for the perimeter of the external +fjords frontier: $d_H = 7/4$

external $4/3 \rightarrow$ **excursion around the fjords** $7/4$

Can something analogous to fjords and lakes can be defined for Brownian curves ?

2d Brownian curves:

random walk on a square lattice a: probability $1/4 \longrightarrow$ up, down, left, right
number of steps $N \Rightarrow$ continuum limit $N \longrightarrow \infty$, $a \longrightarrow 0$
with $Na^2 = 2t$ fixed t is the time $\Rightarrow$ infinite length $Na \longrightarrow \infty$
$\longrightarrow$ in the continuum: probability for the Brownian particle to reach $\vec{r}$ at time t starting from $\vec{r}_0$ at time 0

$$G(\vec{r},t/\vec{r}_0,0) = \frac{1}{2\pi t} e^{-(\vec{r}-\vec{r}_0)^2/2t}$$

i.e. free 2d quantum propagator

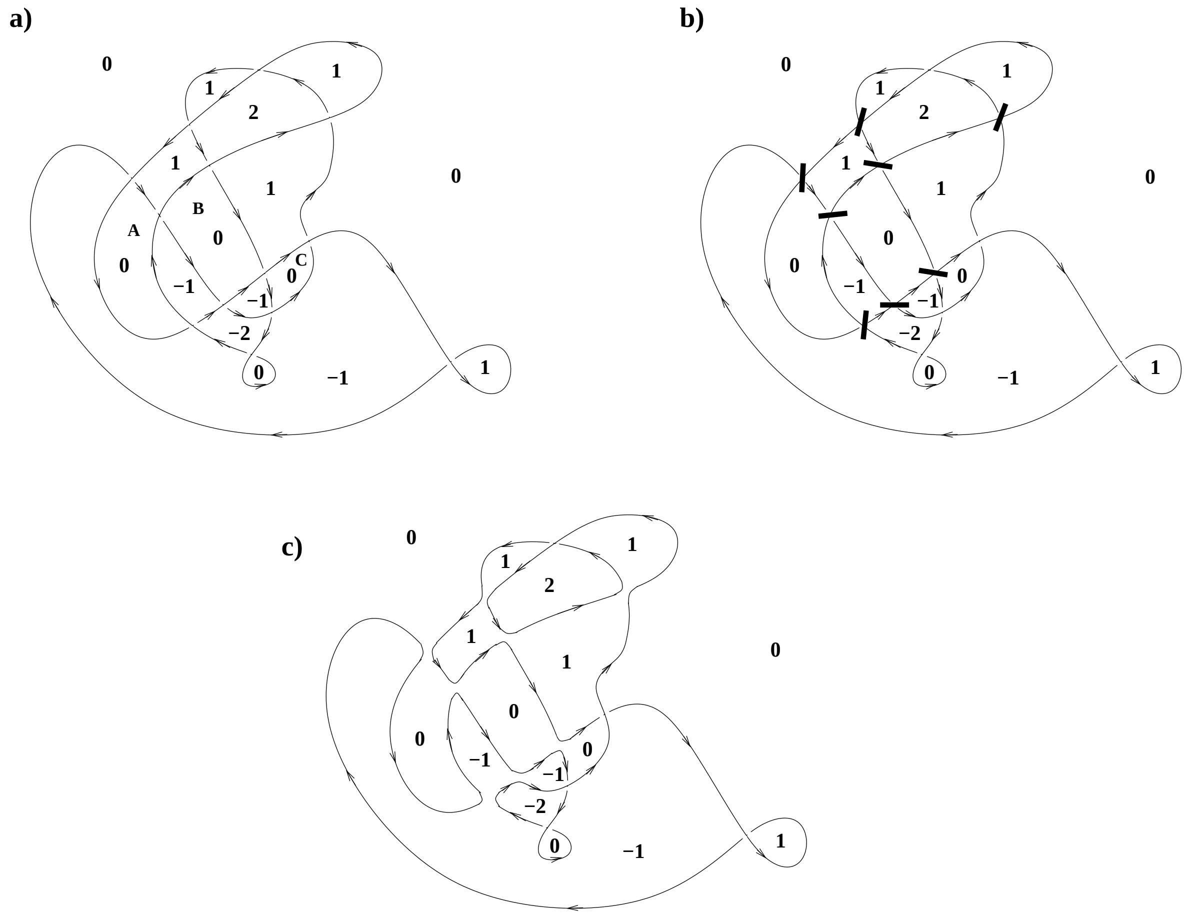
$$\Rightarrow < (\vec{r}-\vec{r}_0)^2 > = \int d^2\vec{r} G(\vec{r},t/\vec{r}_0,0)(\vec{r}-\vec{r}_0)^2 = 2t$$

$\longrightarrow$ typical size of a Brownian curve after time t : $r = \sqrt{2t}$

regularized length : $a(Na) = 2t = r^2 \Rightarrow$ surface filling curve $d_H = 2$ (circle)

perimeter of the external frontier : $P = r^{4/3} \Rightarrow d_H = 4/3$

Look inside the curve: *n*-winding sectors= connected set of points enclosed *n* times by the curve



a) a closed Brownian curve with winding sectors $n = -2, -1, 0, 1, 2$

Comtet, Desbois, S.O. (1990)

random variable S_n = arithmetic area of the n-winding sectors inside a Brownian curve of length t

scaling properties of Brownian curves $\rightarrow S_n$ scales like t

average $< S_n >$ on all closed curves of length t can be computed by path integral technics path integral: $G(\vec{r},t/\vec{r}_0,0) = \frac{1}{2\pi t} e^{-(\vec{r}-\vec{r}_0)^2/2t} = \int_{\vec{r}(0)=\vec{r}_0}^{\vec{r}(t)=\vec{r}} D\vec{r} e^{-\int_0^t \frac{\dot{\vec{r}}^2(\tau)}{2} d\tau}$

closed curve $\vec{r}(t) = \vec{r}_0$

probability $P(\vec{r}_0,n,t)$ to wind n times around the origin after time t:

$\theta = 2\pi n \rightarrow n = \frac{1}{2\pi} \int_0^t \dot{\theta} d\tau$

$\Rightarrow$ constraint $\delta_{n,\frac{1}{2\pi}\int_0^t \dot{\theta} d\tau} = \int_0^1 e^{i2\pi\alpha(n-\frac{1}{2\pi}\int_0^t \dot{\theta} d\tau)} d\alpha$

in the path integral $P(\vec{r}_0,n,t) = \int_0^1 d\alpha e^{i2\pi\alpha n} \int_{\vec{r}(0)=\vec{r}_0}^{\vec{r}(t)=\vec{r}_0} D\vec{r} e^{-\int_0^t (\frac{\dot{\vec{r}}^2(\tau)}{2} + i\alpha\dot{\theta}) d\tau}$

in the action $\alpha\dot{\theta} = \vec{A}.\dot{\vec{r}} \rightarrow \vec{A}$ vector potential $= \alpha\vec{\partial}\theta \Rightarrow$ quantum particle coupled to an Aharanov-Bohm flux $2\pi\alpha$ at the origin

set of all closed curves of length t: integrate over $\vec{r}_0$

$\rightarrow P(n,t) = \int d\vec{r}_0 P(\vec{r}_0,n,t) = \int_0^1 d\alpha e^{i2\pi\alpha n} Z_t(\alpha)$

$Z_t(\alpha) =$ Aharonov-Bohm partition function (with $\beta \rightarrow t$)

S_n = arithmetic area n-winding sectors $\rightarrow B$ field also needed $\Rightarrow$ the result :

$n \neq 0 : < S_n >= \frac{t}{2\pi n^2}$

$n = 0 : < S_0 >= \infty$ since outside 0-winding sea necessarily included (integration over initial point $\vec{r}_0$)

$\Rightarrow < S_0 >$ inside the curve not known

1994: W. Werner thesis : ”Sur l’ensemble des points autour desquels le mouvement Brownien plan tourne beaucoup”

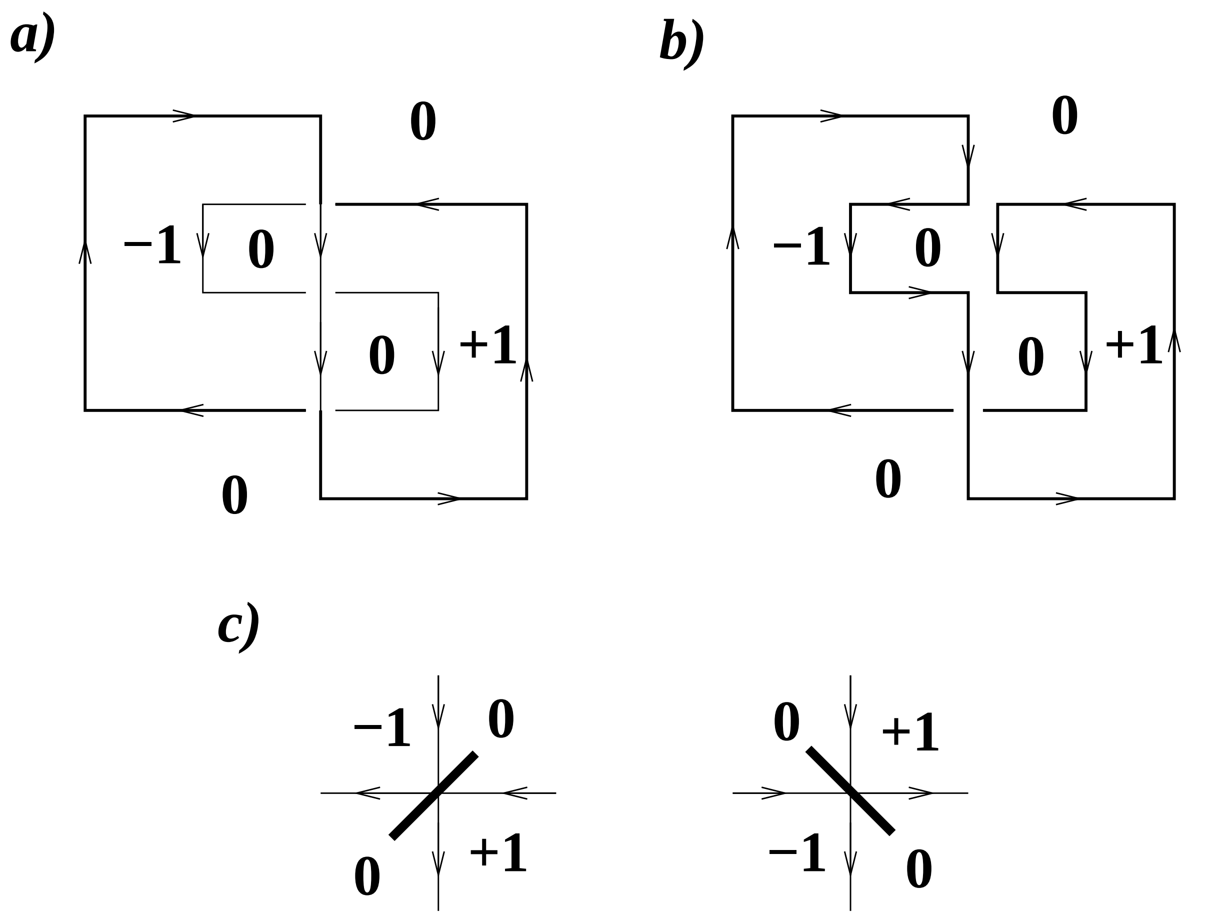
when $n \to \infty$ one has $n^2 S_n \to < n^2 S_n >= \frac{t}{2\pi}$

2005: Garban, Trujillo Ferreras ”The expected area of the filled planar Brownian loop is $\pi/5$” SLE $\to$ total arithmetic area known

$< S >= t\frac{\pi}{5} =< S_0 > +2\sum_{n=1}^{\infty} < S_n >$

$\Rightarrow < S_0 >= t\frac{\pi}{30}$

$\to$ pay more attention to the 0-winding sectors which can be connected to the outside (like percolation fjords) a simple example



a) closed random walk and winding sectors b) same walk with an oriented frontier (the fjord is opened)

a) and b) are two possible time histories of the same random path

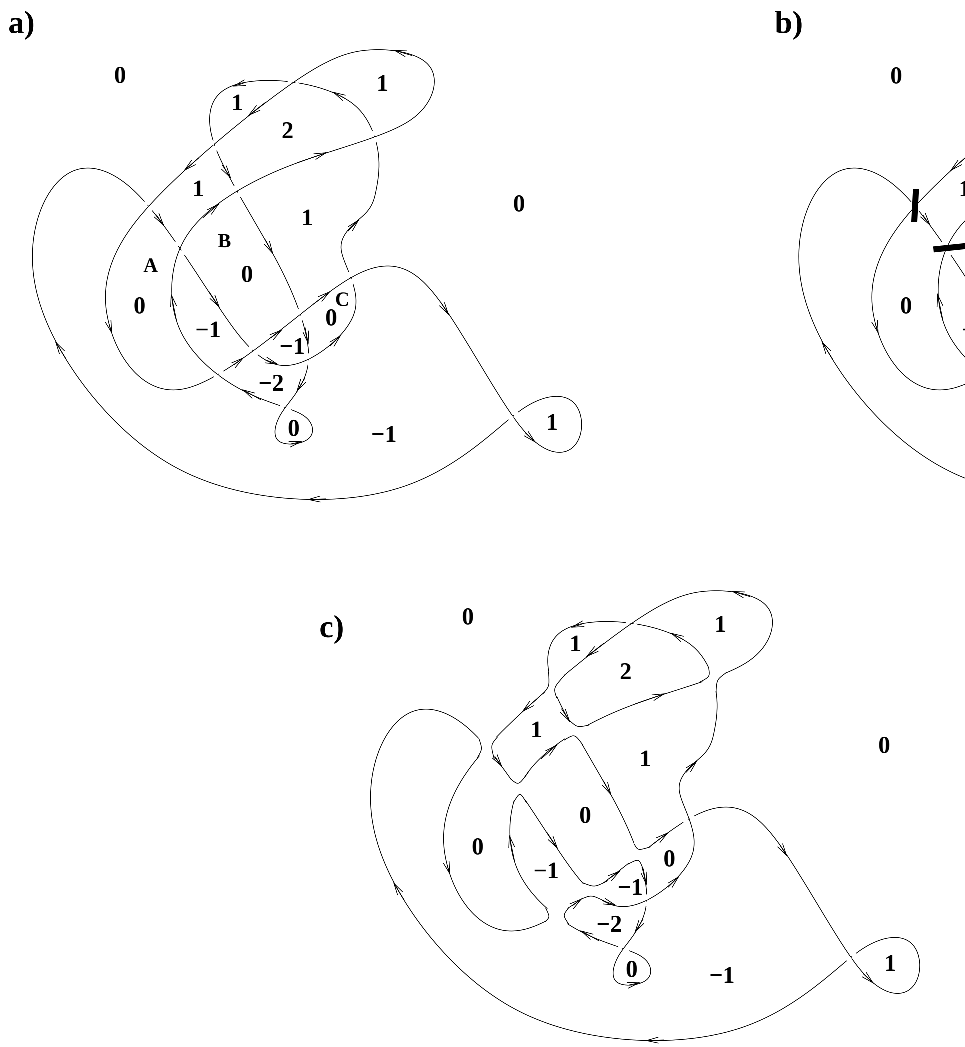
$\Rightarrow$ always possible to follow a time history such that one can define a frontier

- on one side 0-winding sector and on the other side ± 1-winding sector

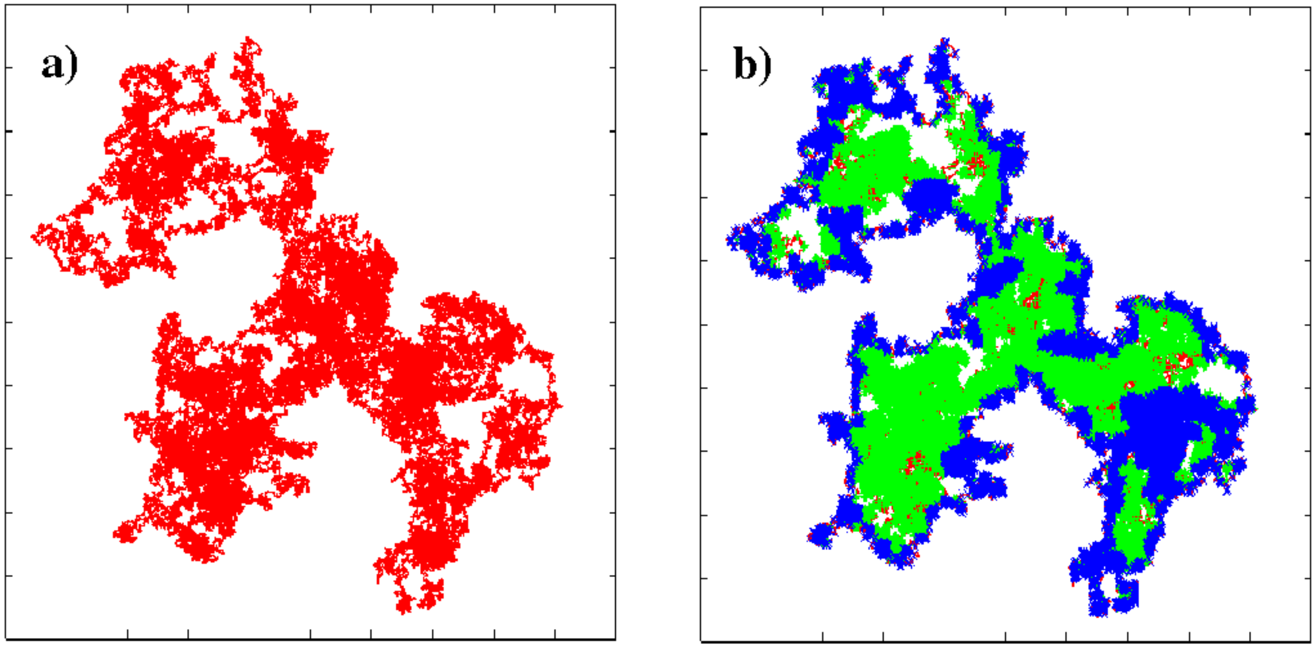
-different from usual exterior (geometric) non oriented frontier

-excursions inside the walk around the fjords

$\longrightarrow$ oriented frontier = external + fjords frontier

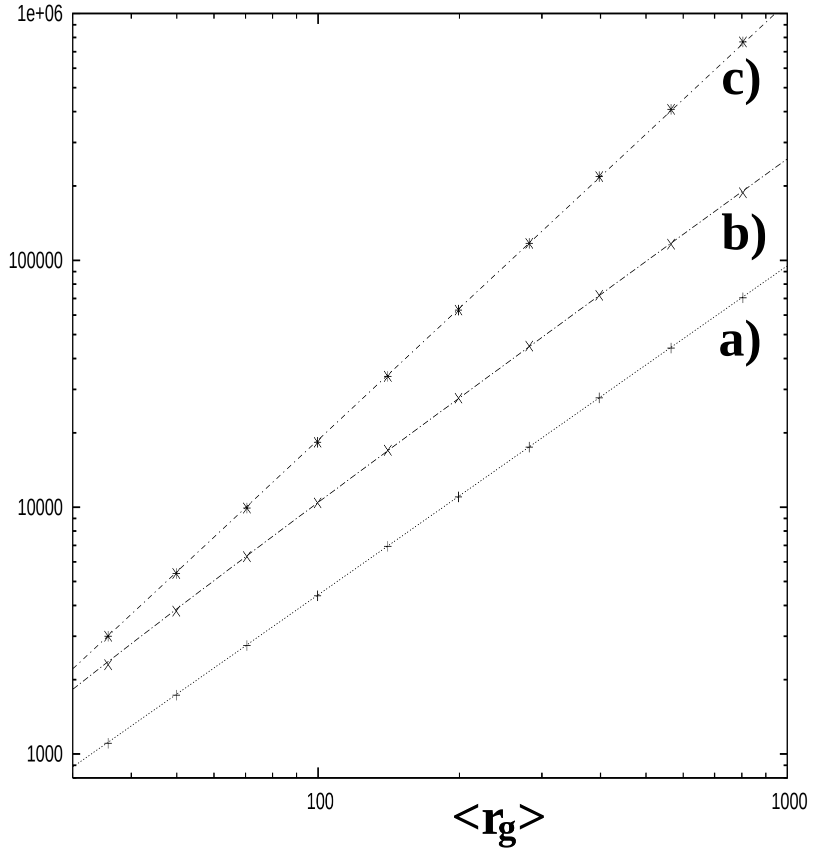


⇒ 0-windings=fjords + lakes: remain inside the curve (as in percolation) in red a closed random walk $N = 1000000$



in blue the fjords, in green the lakes: the fjords remain on the boundary of the walk; the lakes proliferate inside the walk.

numerical simulation for random walks on a square lattice ($N : 10^4 \rightarrow 10^7$)



a)external (geometric) frontier b)external+fjords (oriented) frontier c)external+fjords+lakes(disconnected) : the slopes are a) and b)= 4/3 and c) = $1.77 \approx 7/4$

numerics:

-small $N \simeq 10000 \rightarrow$ big statistics needed

$\rightarrow$ up to 100000 curves

-big $N \simeq 10000000$

$\rightarrow$ up to 30000 curves

for a given N:

-for each curve estimate P and r

-deduce $< P >$ and $< r >$

$\Rightarrow$ a point in the plot

- Hausdorff dimension of the perimeter of the external+fjords oriented frontier

$$d_H = \frac{4}{3}$$

same as the external frontier $\longrightarrow$ the fjords are subleading in the continuum

take also into account the lakes $\Rightarrow$ disconnected frontier:

- Hausdorff dimension of the perimeter of the external+fjords+lakes frontier

$$d_H = \frac{7}{4}$$

$\Rightarrow 4/3 \rightarrow 7/4$ due to the lakes proliferation : leading in the continuum if you take the same point of view in percolation: (numerical simulation with lattice size up to 3200×3200)

- Hausdorff dimension of the perimeter of the external+fjords+lakes disconnected frontier

$$d_H \simeq 1.9 \longrightarrow \frac{91}{48}$$

Summary

BROWNIAN	PERCOLATION
exterior $\frac{4}{3}$	exterior $\frac{4}{3}$
exterior+fjords $\frac{4}{3}$	exterior+fjords $\frac{7}{4}$
exterior+fjords+lakes $\frac{7}{4}$	exterior+fjords+lakes $\frac{91}{48}$

$91/48$: same dimension as the mass (i.e. area) of the percolating cluster percolation: mass of the object (i.e. area) $\simeq$ perimeter : $d_H = \frac{91}{48}$ very porous

Brownian: the same thing happens $< S >= \pi t/5$ and $L = 2t \rightarrow d_H = 2$

percolation and Brownian: analogous and different

with opened questions:

Brownian : oriented frontier (fjords) $= 4/3$ and disconnected frontier $= 7/4$

Percolation : disconnected frontier $= 91/48$

$\rightarrow$ SLE ??