Topological Quantum Computation with non-Abelian Anyons

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  Not a big help on the scale of 1000 qubits
  (huge overhead)

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Fourier transform of n-qubits, $N = 2^n$ can be executed by the Hadamard gate H and Controlled-R_k operations

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- execute controlled-U^{2^j} (modular exponentiation), for integer j

$$CU^{2^j} = |0\rangle\langle 0|\otimes \mathbb{I} + |1\rangle\langle 1|\otimes U^{2^j}$$

creates entanglement crucial for the quantum speed-up

- ▷ **Order finding:** for x and N positive integers find the smallest positive integer r such that

$$x^r = 1 \pmod{N}, \quad \text{where} \quad x < N \quad \text{and} \quad \gcd(x, N) = 1$$

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- **hidden even from us!** no local measurement could determine fusion channel - need non-local measurement of

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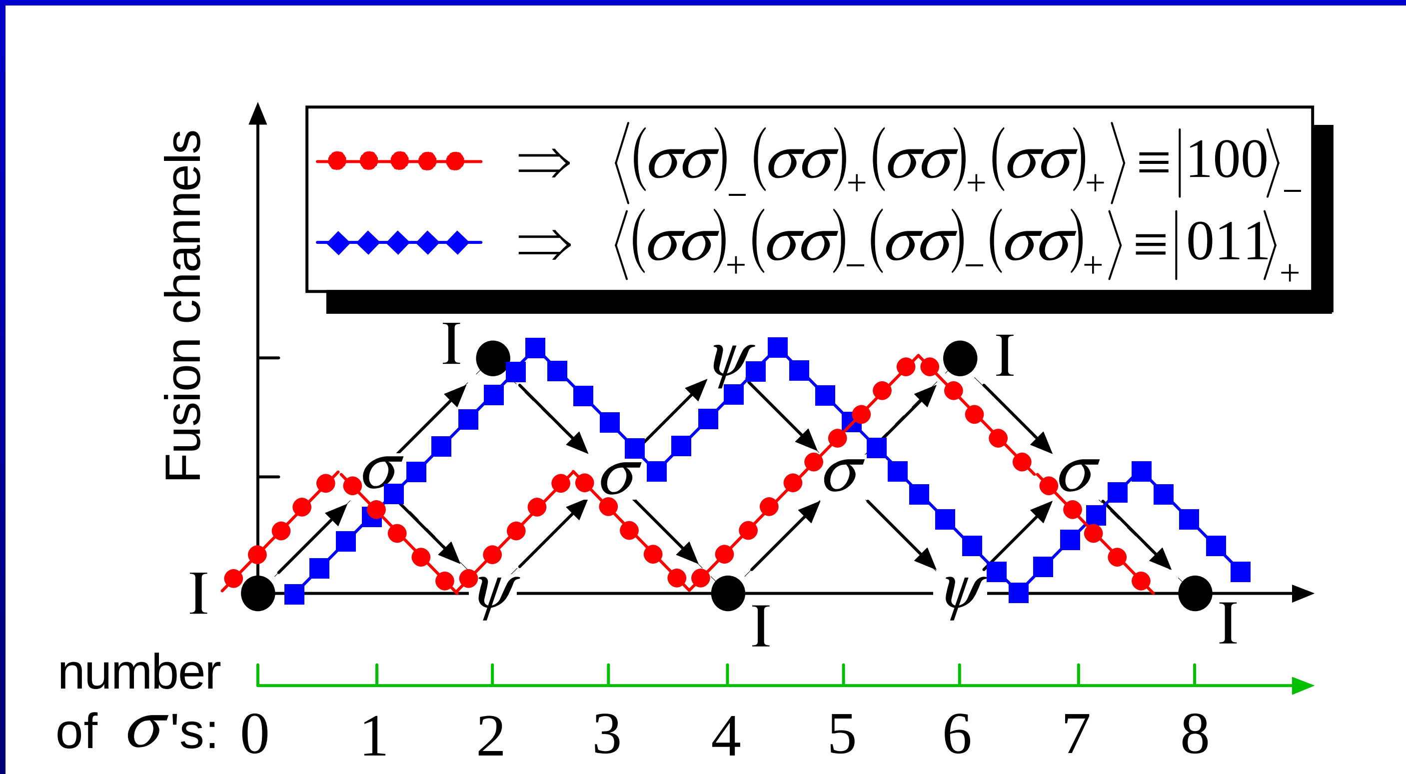
i.e., for an array of anyonic fields we must fix the fusion channels of each neighbors

▷ **Remember:** Multi-anyon states are labeled by paths in Bratteli diagrams



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Alexander von Humboldt Stiftung/Foundation
ИЯИЯЕ INRNE
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- non-Abelian statistics might be easier to be observed than the Abelian one [Stern-Halperin, Kitaev et al.]

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- **Incompressibility:** entire dynamics is on the edge $\Longrightarrow$ the low-E effective QFT on the edge must be **unitary**
- **Luscher–Mack Theorem:** $\Longrightarrow$ full CFT symmetry, i.e., the Laurent modes of the stress-energy tensor satisfy the Virasoro algebra

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- **Witten:** (2+1)D Chern–Simons TQFT $\Longleftrightarrow$ (1+1)D RCFT

▷ FQH wave functions by CFT correlators:

- $\nu = 1/m$ **Laughlin state:** standard second-quantized relation between wave functions and state in filling number representation

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- **GS ansatz [N. Read] :** Bose condensate or homogeneous uniform charged two-dimensional plasma

$$|\Psi_N\rangle = \left(\int d^2z\,\psi^\dagger(z)\right)^N |0\rangle \propto \exp\left(-i\int d^2z\sqrt{m}\,\rho_0\phi(z)\right)|0\rangle$$

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where $\rho_0 = \nu/2\pi$ and U_1 is the charge-shift automorphism of $\widehat{u(1)}$ used in bosonization, e.g., of the 1D Dirac fermion

$$\psi_D(z) =: \mathrm{e}^{i\phi(z)} :\equiv U_1 \mathrm{e}^{i\phi_+(z)} z^{J_0} \mathrm{e}^{i\phi_-(z)}$$

Or, put differently, the second-quantized ground state is just a screening charge in the center of the disk

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- **CFT correlators – wave function relation:**

$$\Psi_{\mathrm{L}}(z_1,\ldots,z_N) = \langle N\,\sqrt{m}| : \mathrm{e}^{-i\,\sqrt{m}\phi(z_1)} : \cdots : \mathrm{e}^{-i\,\sqrt{m}\phi(z_N)} : |0\rangle$$

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- **Quasiholes:** are represented by the operators $\psi_{\rm qh}(z) =: \mathrm{e}^{i\frac{1}{\sqrt{m}}\phi(z)}:$ and the quasihole wave function is

$$
\begin{aligned}
\Psi_{\rm qh}(\eta, z_1, \ldots, z_N) &= \langle N\sqrt{m} - 1/\sqrt{m}| : \mathrm{e}^{i\frac{1}{\sqrt{m}}\phi(\eta)} : \times \\
&\times : \mathrm{e}^{-i\sqrt{m}\phi(z_1)} : \cdots : \mathrm{e}^{-i\sqrt{m}\phi(z_N)} : |0\rangle = \\
&= \prod_{i=1}^{N}(\eta - z_i) \prod_{j<k}(z_j - z_k)^m
\end{aligned}
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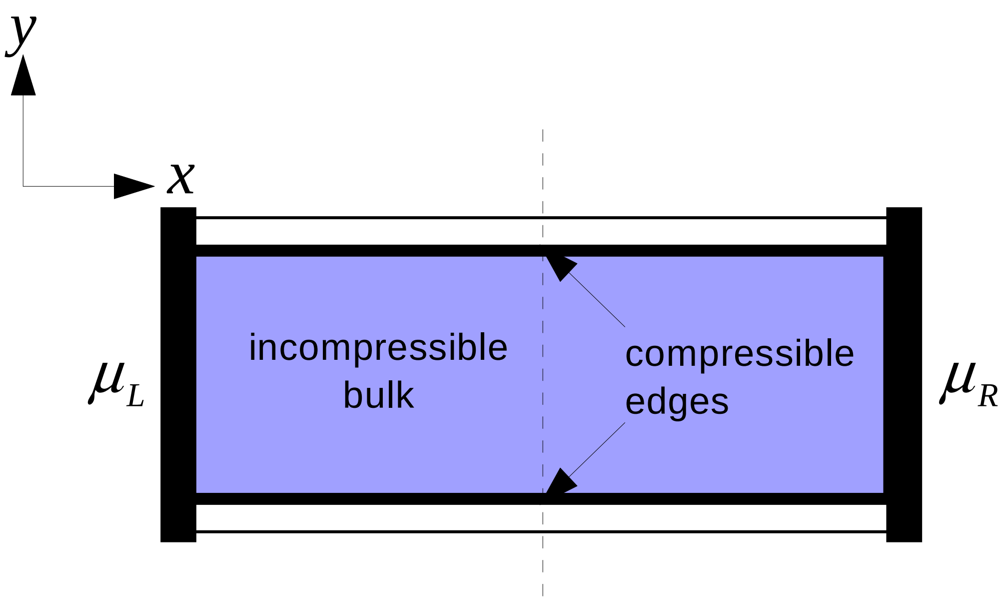
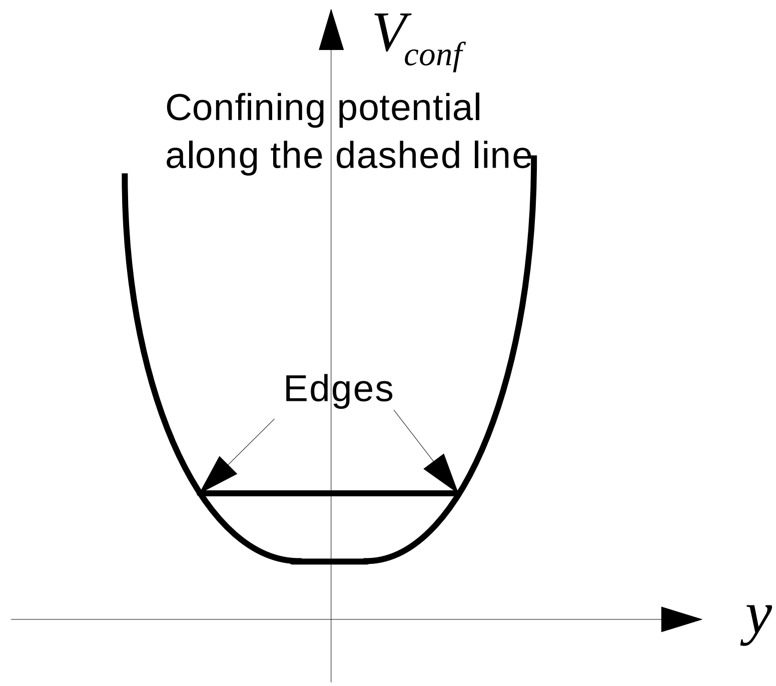
Quantum Hall bar: edge currents





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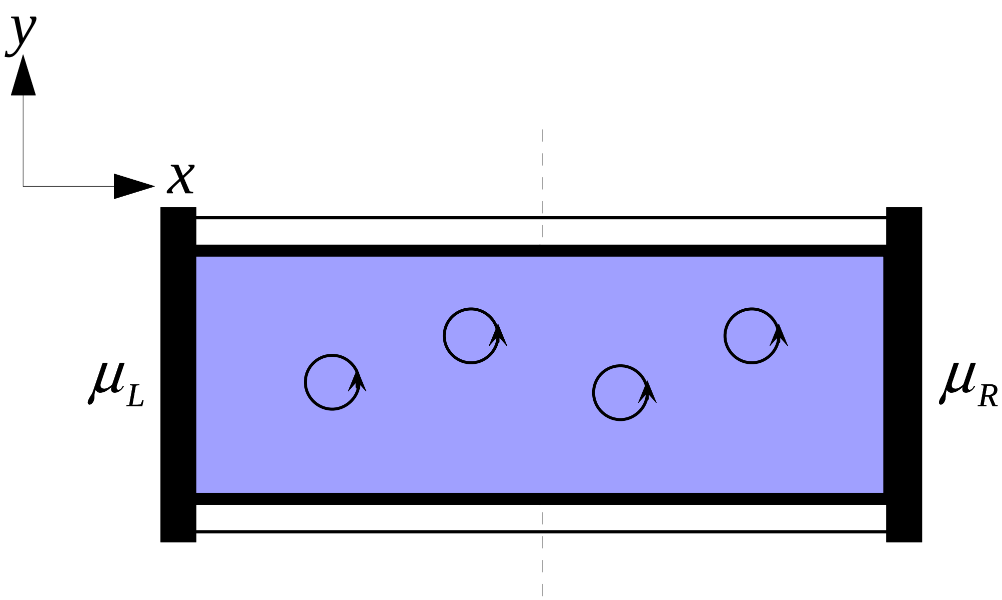

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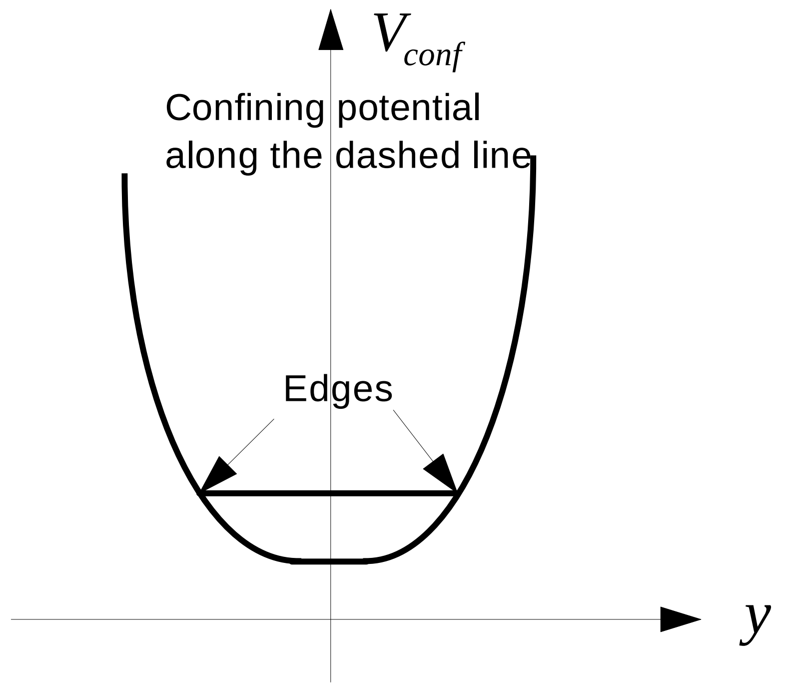

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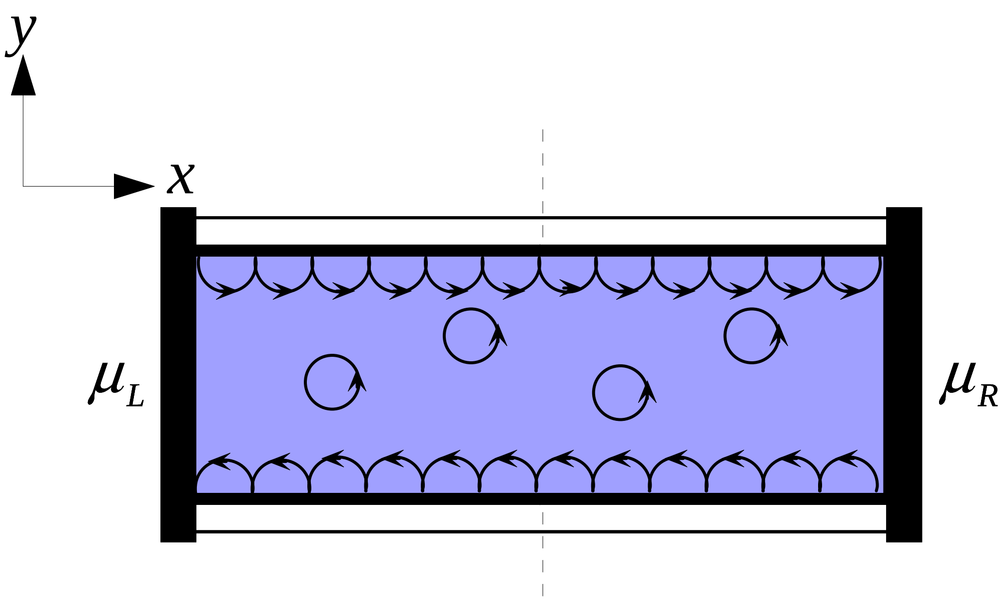

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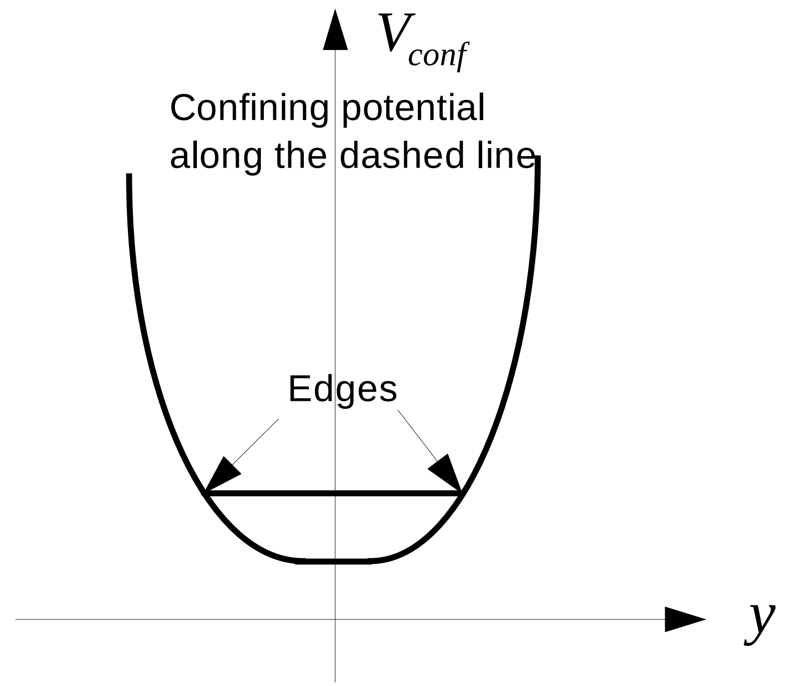

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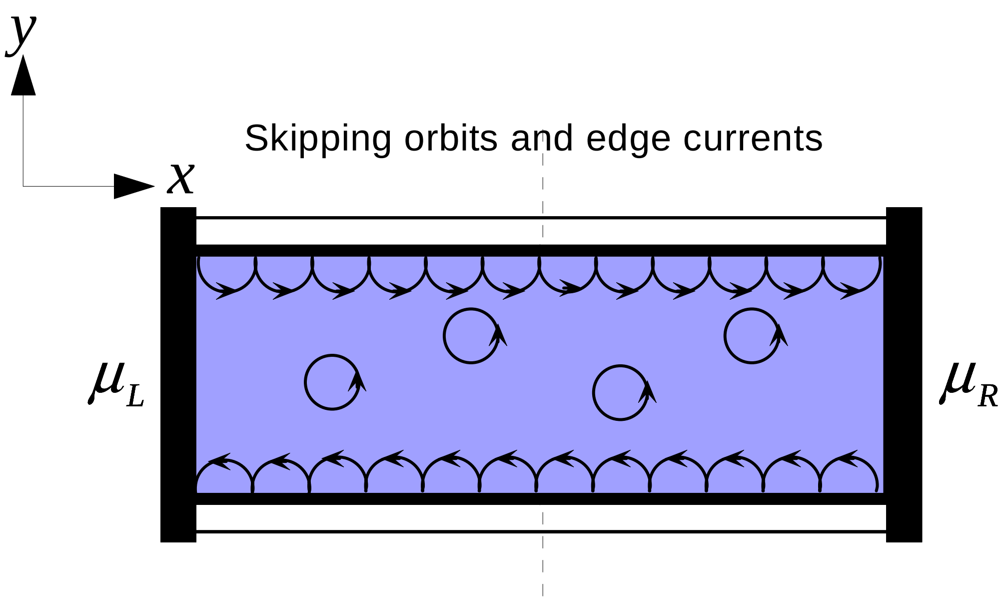

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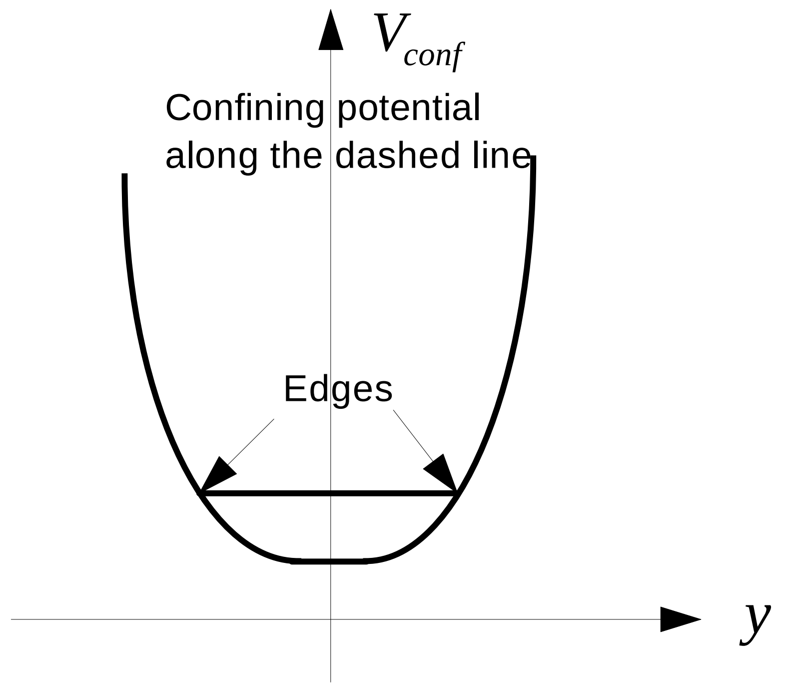

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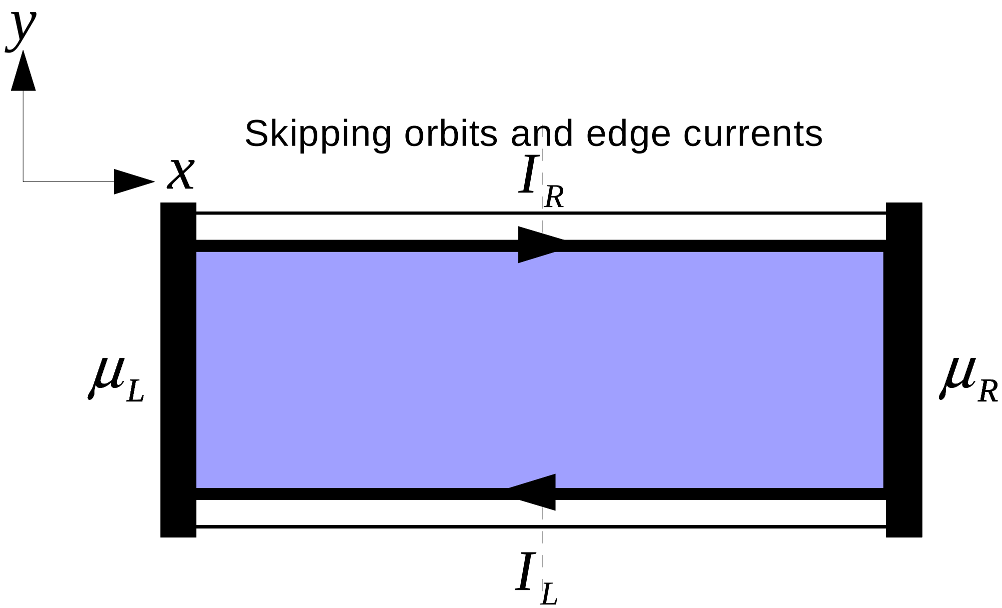

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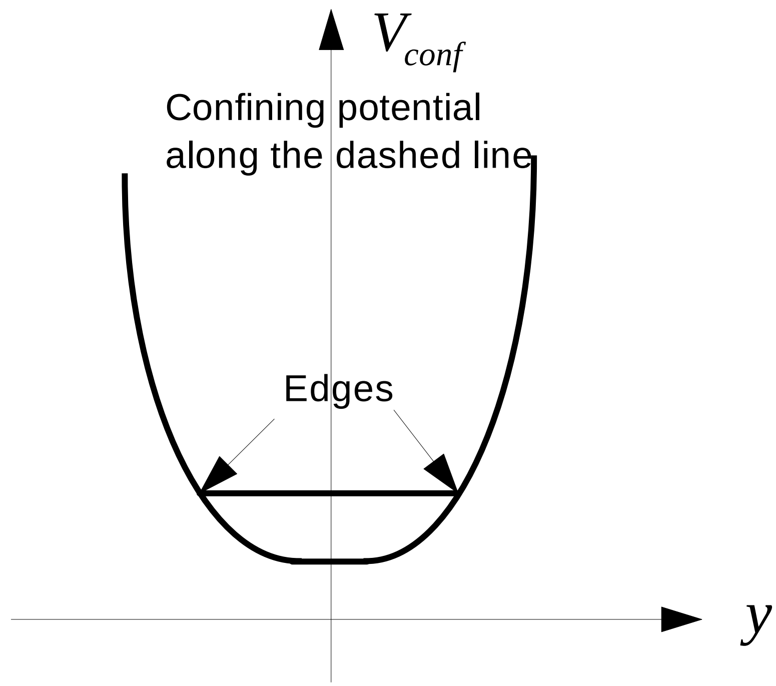
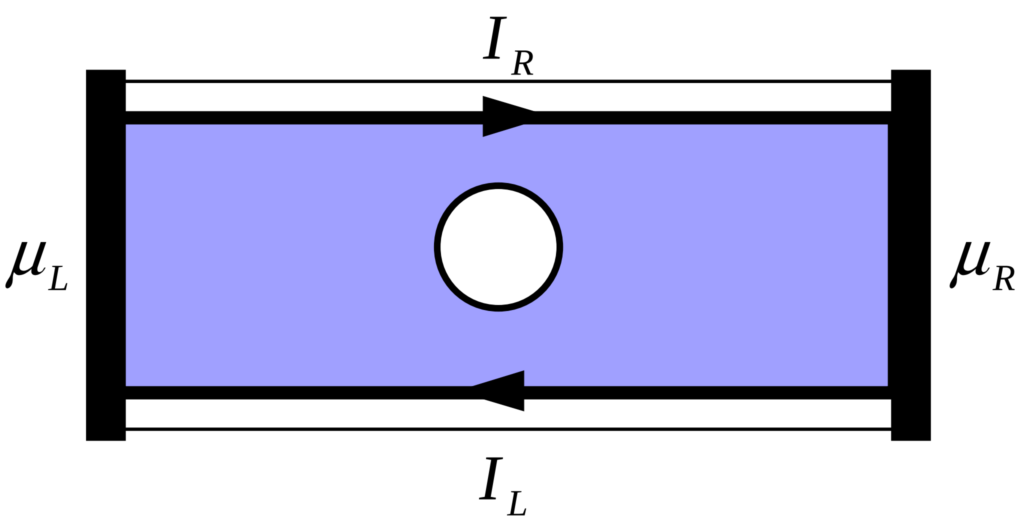
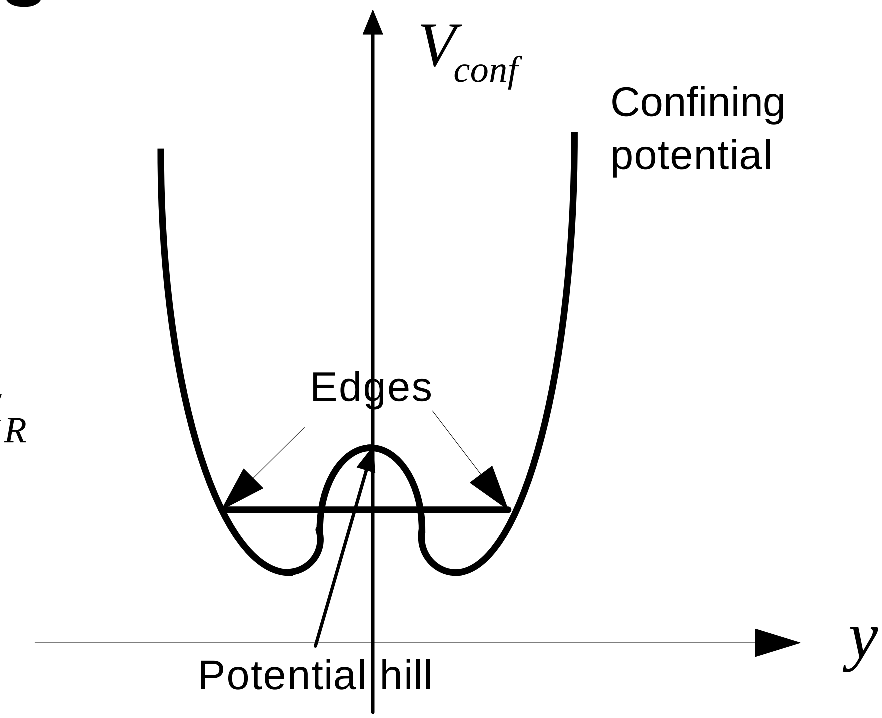


Antidots: localizing quasiholes in FQH droplets





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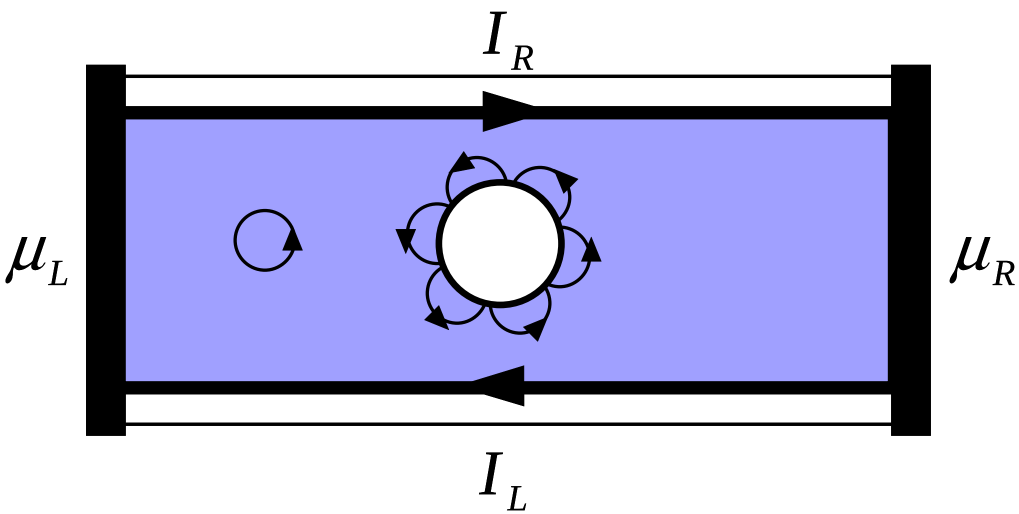

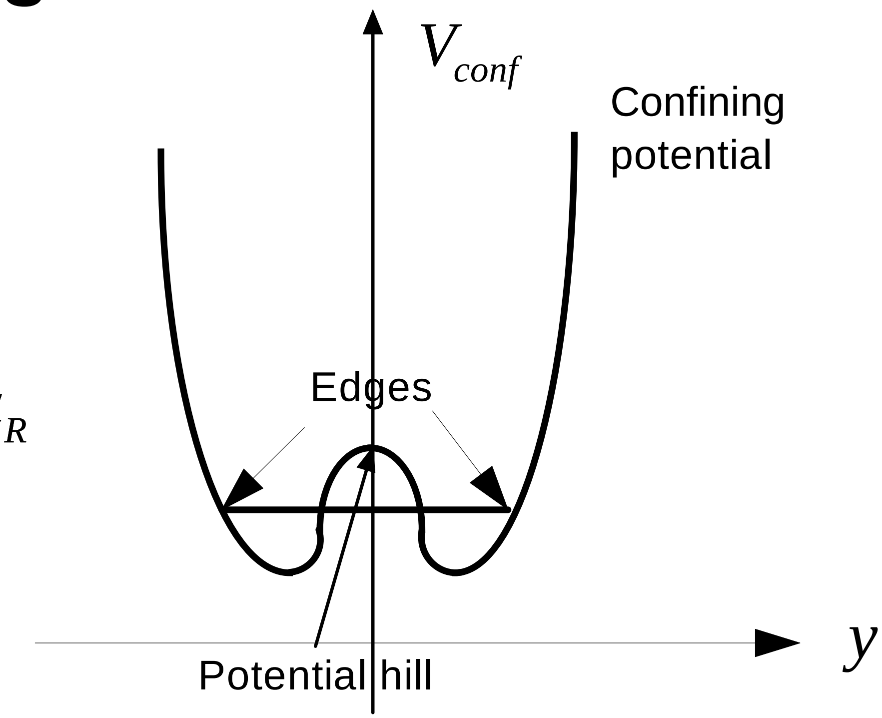

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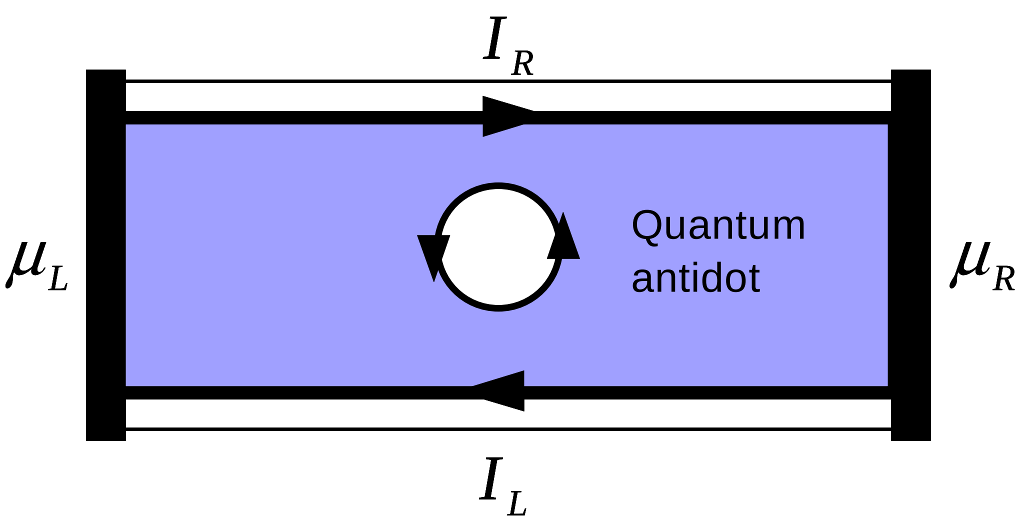

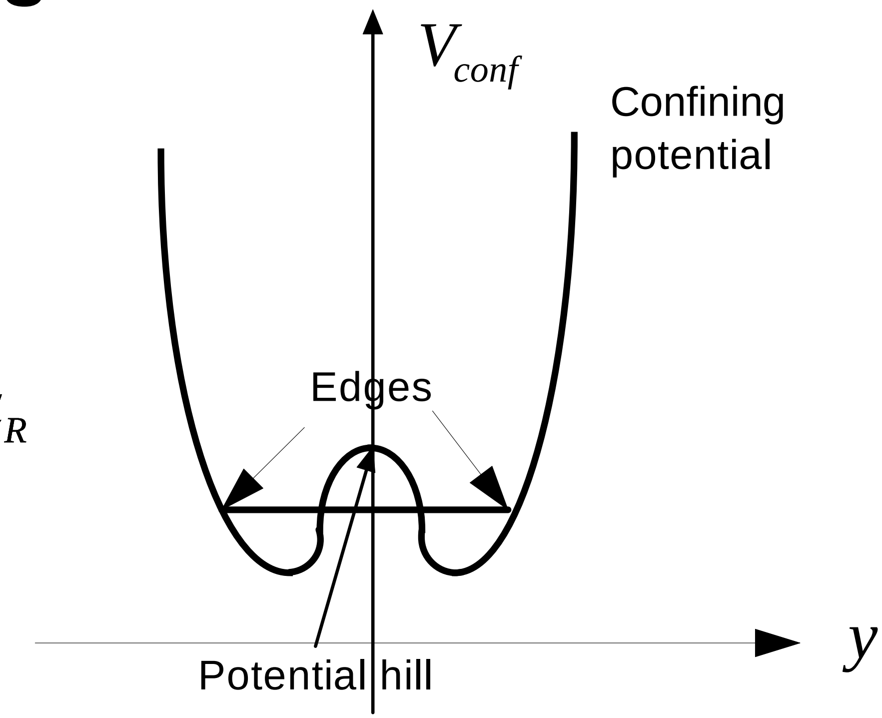

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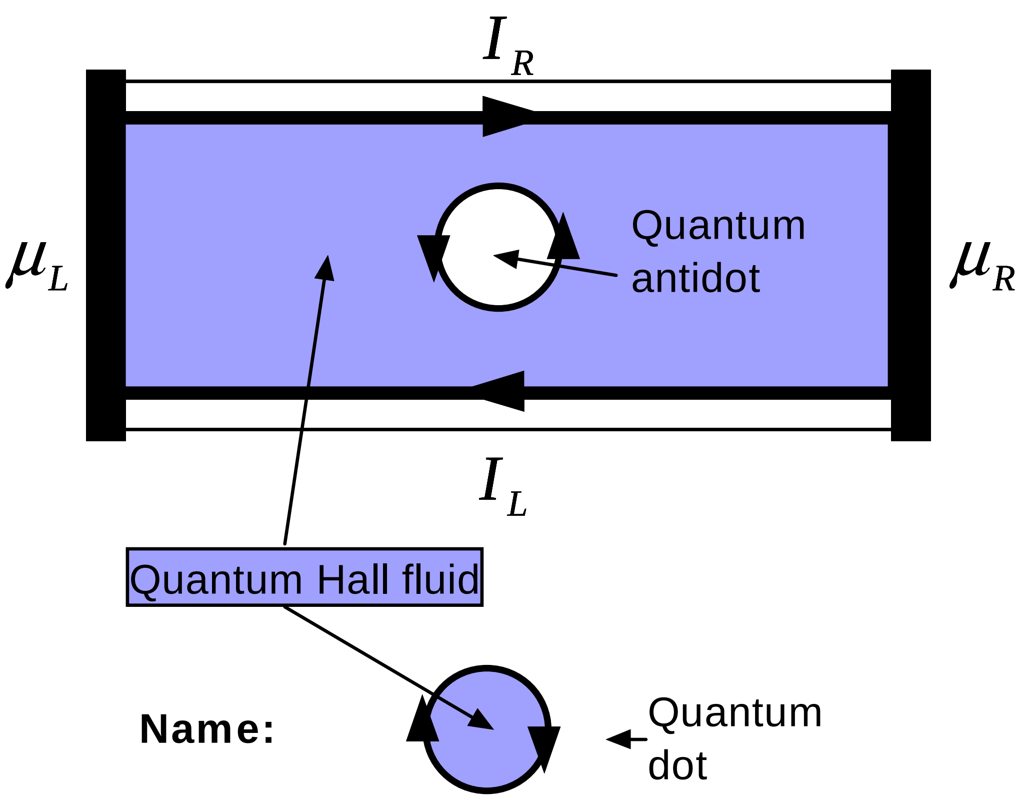

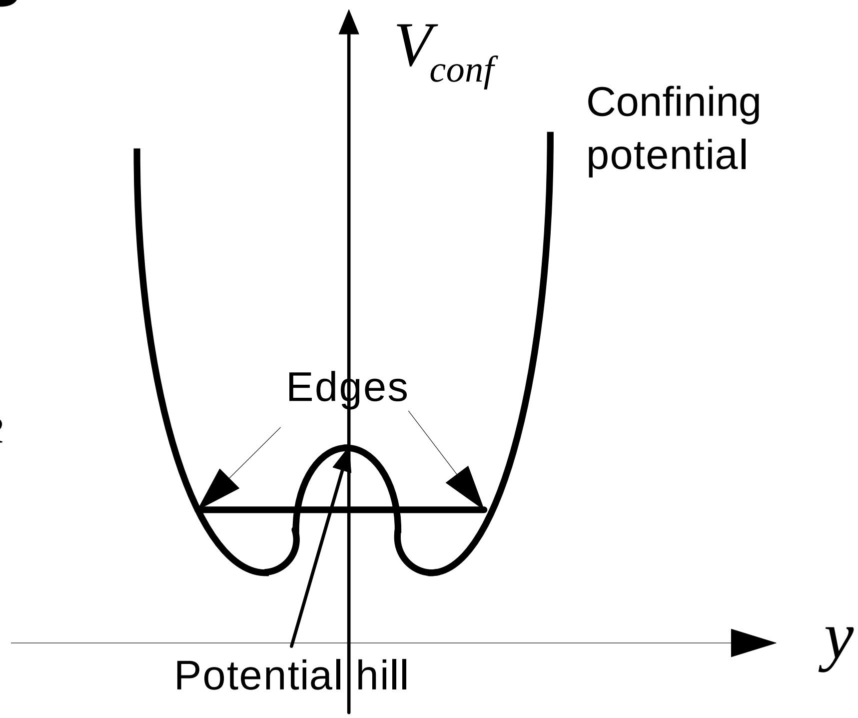

TQC scheme of Das Sarma et al.: SUMMARY

- **The Pfaffian FQH state:** $\text{CFT} = \widehat{u(1)} \times \text{Ising}$

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- **Single qubit construction:**

$\nu = 5/2$ FQHE sample with 4 non-Abelian quasiparticles at fixed positions η_a (e.g., localized on 4 quantum antidots)

- N **-electron wave function:** CFT correlation function

$$\Psi_{4\mathrm{qh}}(\eta_1, \eta_2, \eta_3, \eta_4, \{z_i\}) =$$

$$\langle \psi_{\mathrm{qh}}(\eta_1)\psi_{\mathrm{qh}}(\eta_2)\psi_{\mathrm{qh}}(\eta_3)\psi_{\mathrm{qh}}(\eta_4) \prod_{i=1}^{N} \psi_{\mathrm{el}}(z_i)\rangle$$

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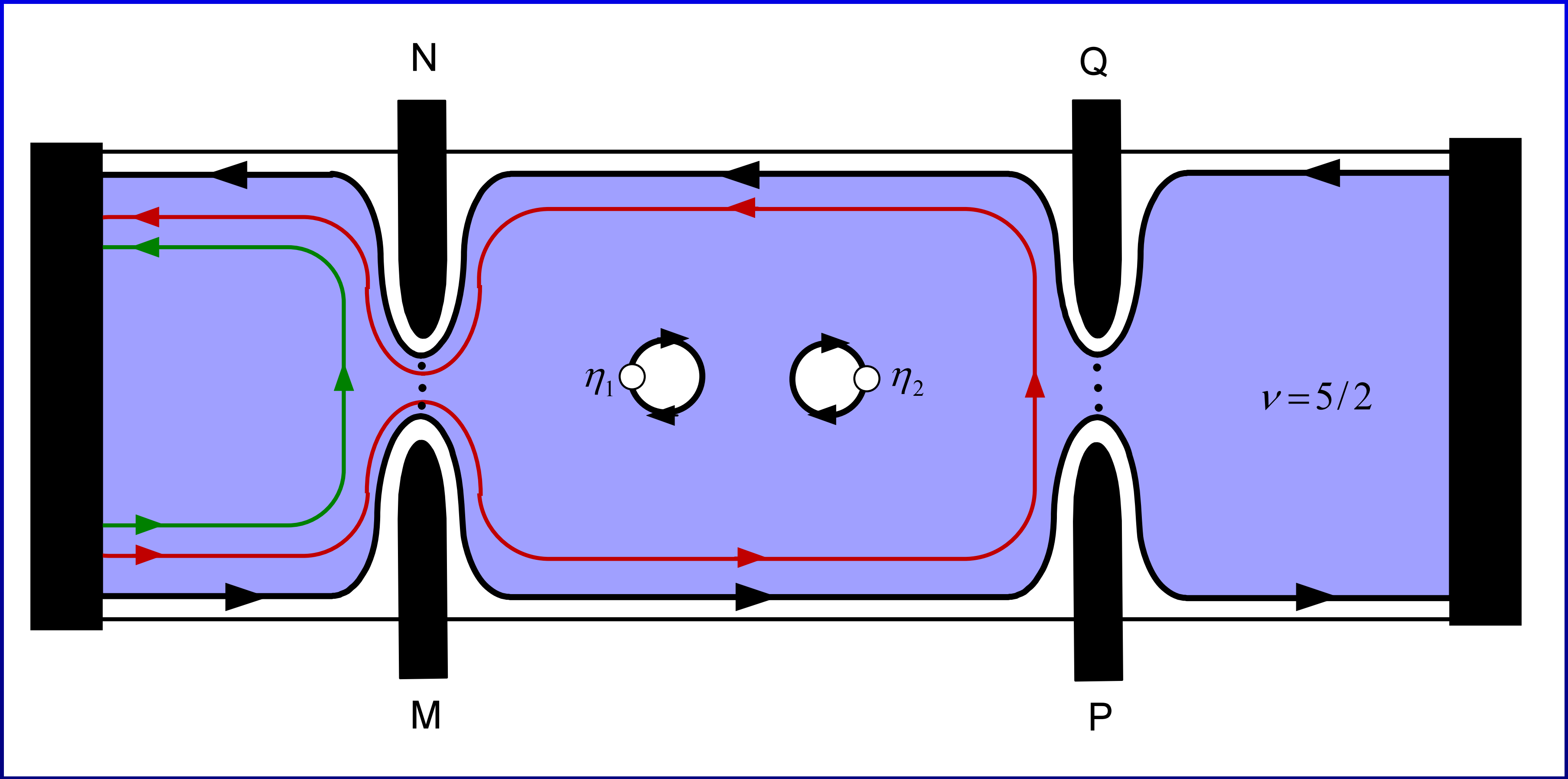
N
Q
η_1
η_2
$\nu = 5/2$
M
P

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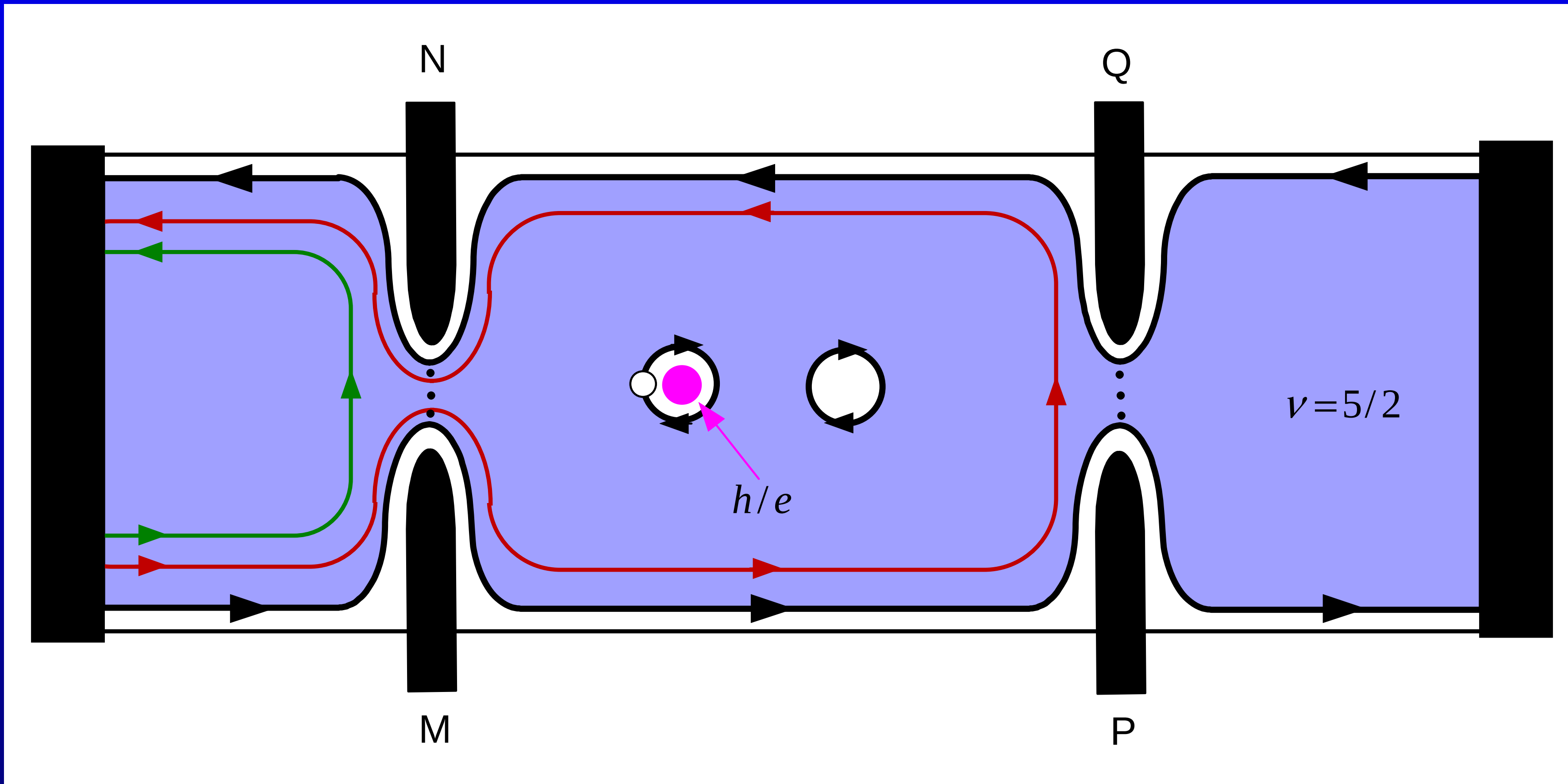
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$$\sigma \equiv \frac{\sigma_+ + \sigma_-}{\sqrt{2}} \quad \Longrightarrow \quad \sigma \times \sigma = \mathbb{I} + \psi$$

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- **Positive parity** (N =even)

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corresponds to the fusion path $\sigma\times\sigma\to\mathbb{I}\times\sigma\to\sigma\times\sigma\to\mathbb{I}$

• **Negative parity** (N =even again)
However put one extra fermion at $z_\infty = \infty$ or at $z_0 = 0$)

$$|0\rangle_- \equiv \langle \psi(z_\infty)\sigma_+(\eta_1)\sigma_+(\eta_2)\sigma_+(\eta_3)\sigma_-(\eta_4) \prod_{j=1}^{N} \psi(z_j)\rangle,$$

$$|1\rangle_- \equiv \langle \psi(z_\infty)\sigma_+(\eta_1)\sigma_-(\eta_2)\sigma_+(\eta_3)\sigma_+(\eta_4) \prod_{j=1}^{N} \psi(z_j)\rangle$$

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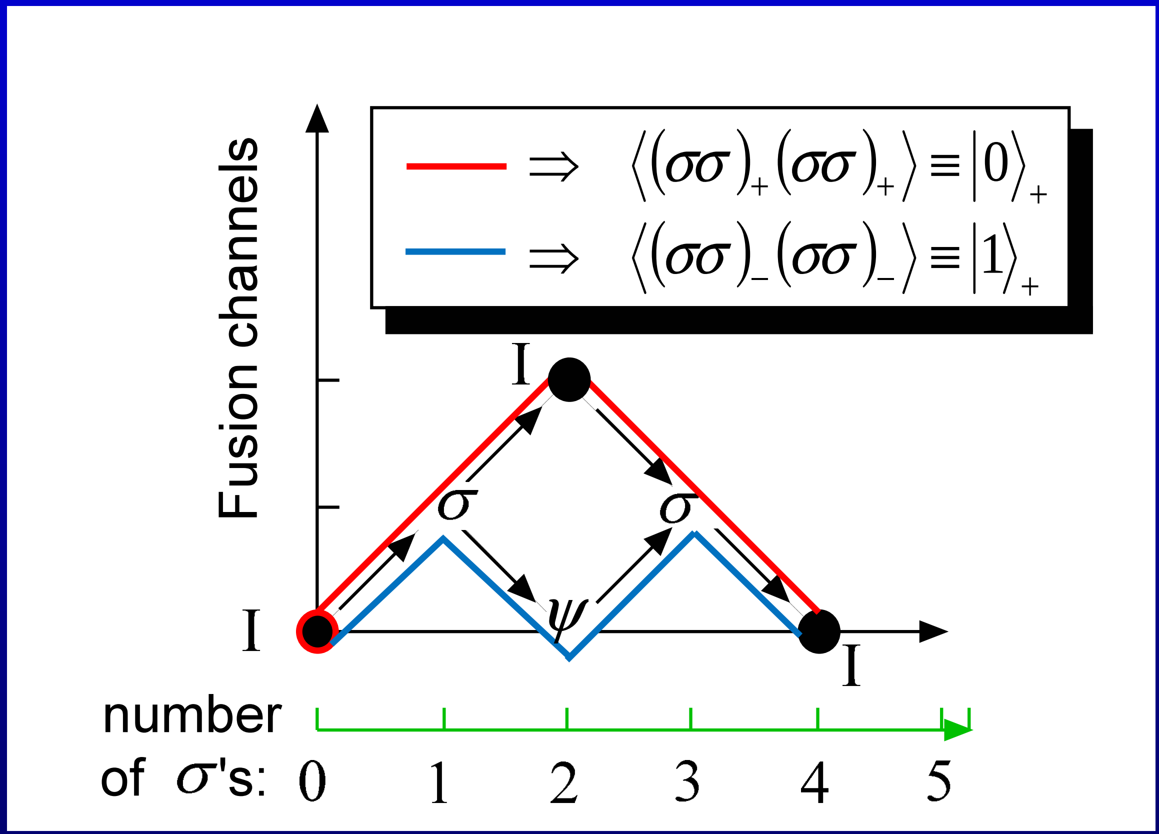
• Fusion paths for the computational basis:

(4 anyons, positive parity)



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☞ L.G., Nucl. Phys. B 789 (2008) 552

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- **Using fusion rules:** Neveu–Schwarz sector

$$\sigma_+(\eta_1)\sigma_+(\eta_2) = \mathbb{I}, \quad \sigma_+(\eta_1)\sigma_-(\eta_2) = \sqrt{\frac{\eta_{12}}{2}}\psi(\eta_2)$$

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$\Longrightarrow$ Compute in this way $R_{12}^{(4,\pm)}$ and $R_{34}^{(4,\pm)}$

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Bottom line: $\Longrightarrow$ it is possible to compute $R_{23}^{(4,\pm)}$ like this

- **Result: finite 2-dimensional representations of $\mathcal{B}_4$:**
  2 inequivalent IRs: positive vs. negative fermion parity

$$R_{12}^{(4,+)} = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}, \quad R_{23}^{(4,+)} = \frac{e^{i\frac{\pi}{4}}}{\sqrt{2}} \begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix}, \quad R_{34}^{(4,+)} = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$$

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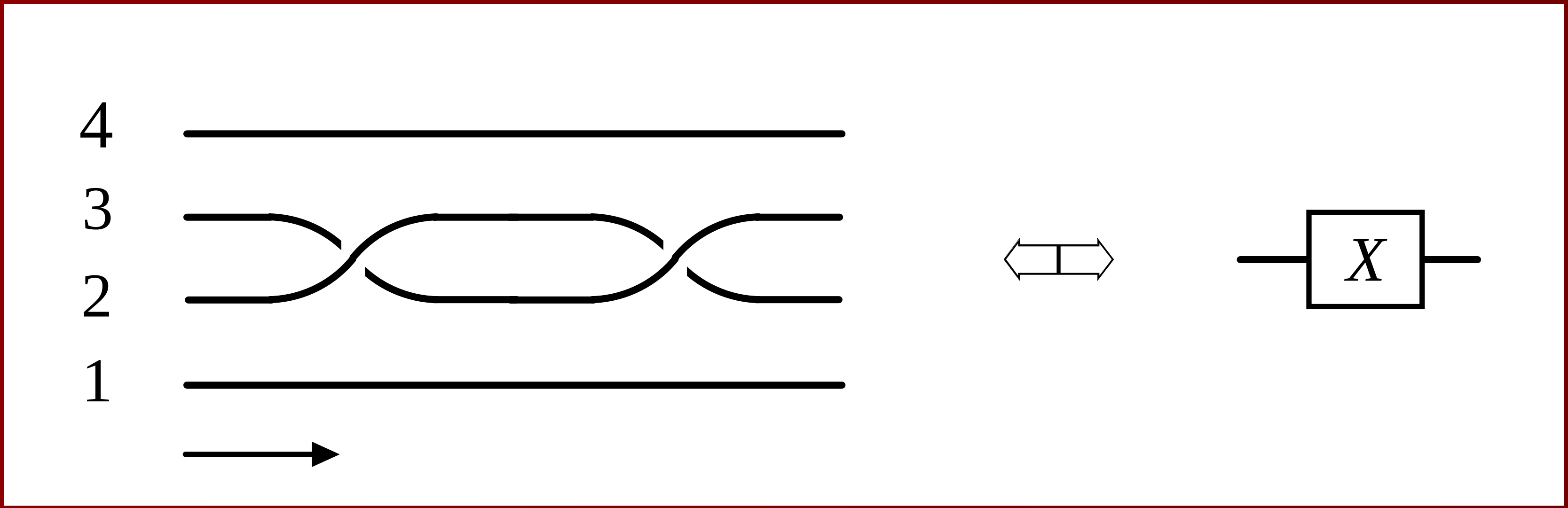
- **Dimino's algorithm:** (Maple)

$$|\mathrm{Image}\, (\mathcal{B}_4)\,| = 96.$$

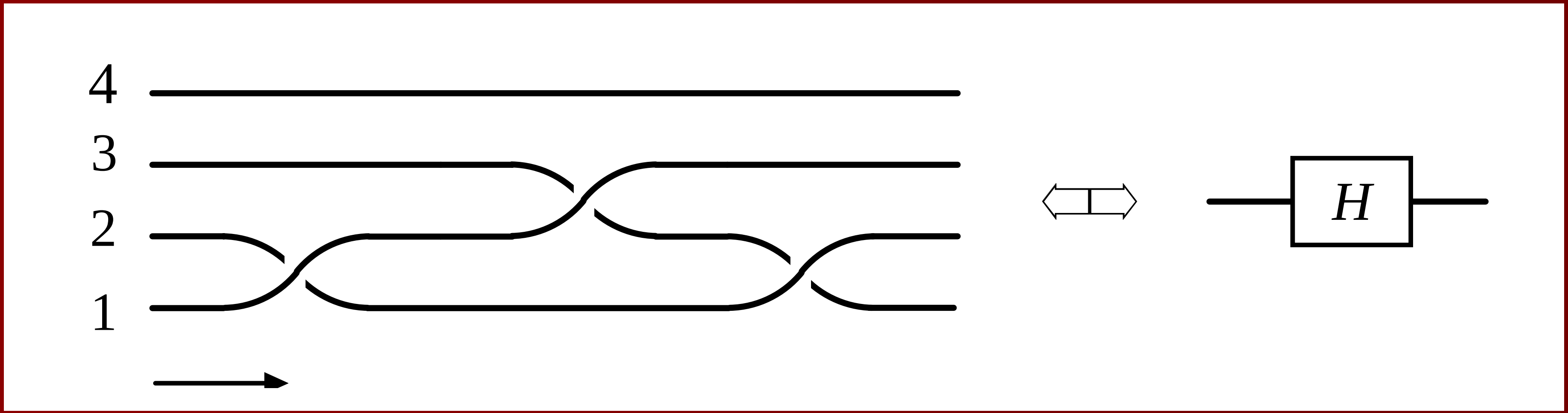
Single-qubit gates: The Pauli X gate

The NOT gate [Das Sarma et al.] $X \equiv R_{23}^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$



Single-qubit gates: the Hadamard gate

$$H \simeq R_{12}^2 R_{13} = R_{12} R_{23} R_{12} = \frac{e^{i\frac{\pi}{4}}}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$



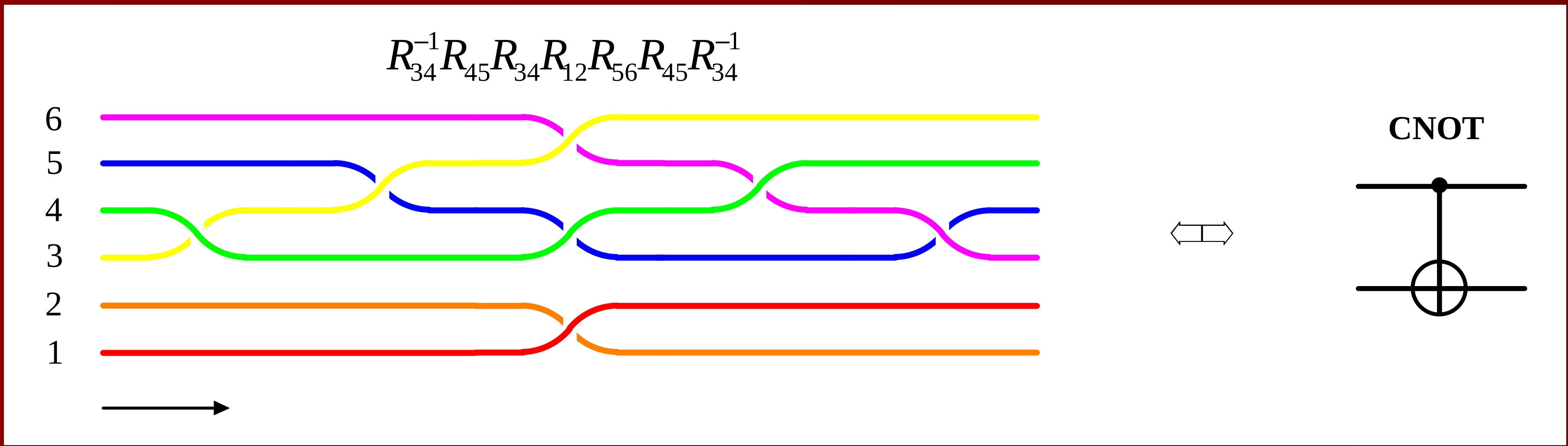
Two-qubit gates: The Controlled-NOT gate

- **CNOT in terms of 6-anyon braidings**

$$\begin{aligned} \mathrm{CNOT} = H_2\, \mathrm{CZ}\, H_2 &= R_{56} R_{45} R_{56}^{-1} R_{34}^{-1} R_{12} R_{45} R_{56} \\ &\simeq \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{aligned}$$

- **Alternatively**

$$\mathrm{CNOT} = R_{34}^{-1} R_{45} R_{34} R_{12} R_{56} R_{45} R_{34}^{-1}.$$



- **N.B.: 100 % topological protection for CNOT**
- **Just 7 elementary braids!**

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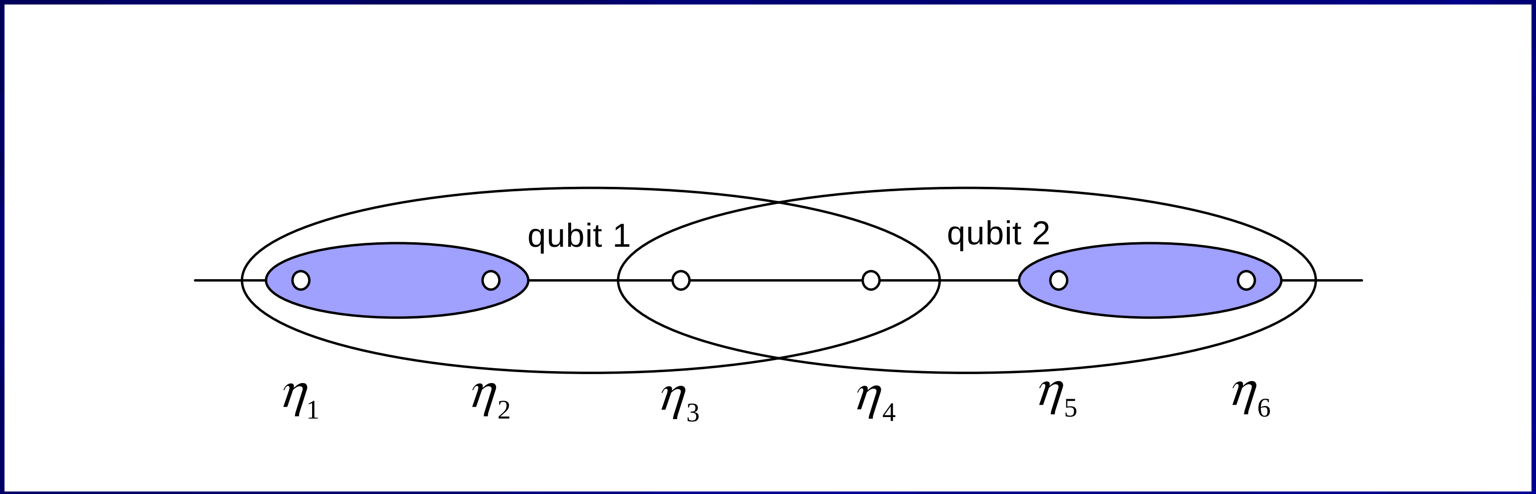

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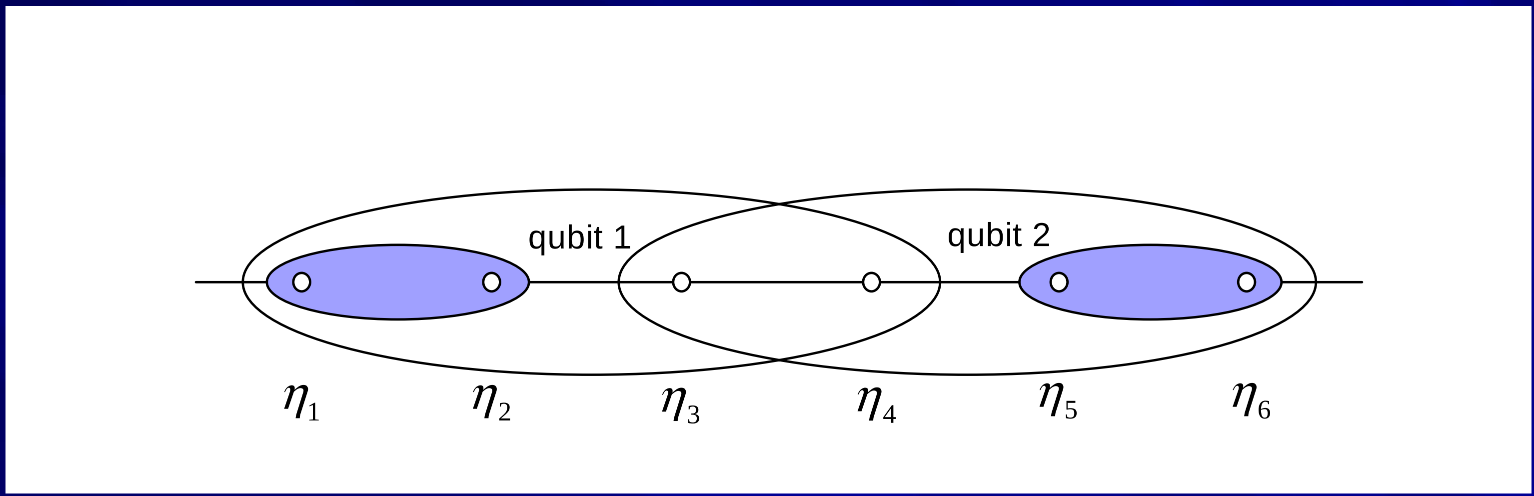
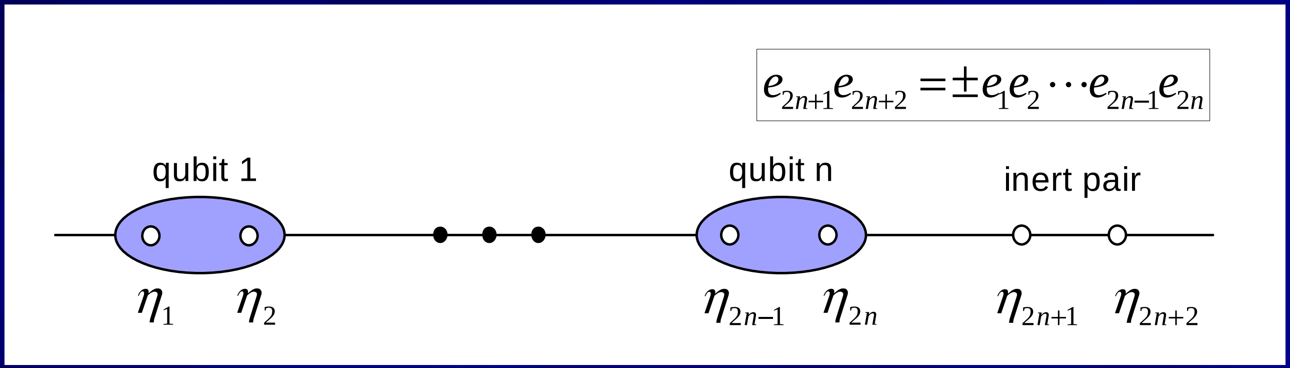


$$|00\rangle \equiv \langle\sigma_+\sigma_+\sigma_+\sigma_+\sigma_+\sigma_+\rangle, \quad |01\rangle \equiv \langle\sigma_+\sigma_+\sigma_+\sigma_-\sigma_+\sigma_-\rangle$$

$$|10\rangle \equiv \langle\sigma_+\sigma_-\sigma_+\sigma_-\sigma_+\sigma_+\rangle, \quad |11\rangle \equiv \langle\sigma_+\sigma_-\sigma_+\sigma_+\sigma_+\sigma_-\rangle$$

n-qubits: encoding



$(n-1)$-qubits: $\mathrm{SO}(2n)$ spinor IRs

Clifford algebra

$$\left\{\gamma_i^{(n)}, \gamma_j^{(n)}\right\} = 2\delta_{ij}, \quad 1 \le i, j \le 2n.$$

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$$\gamma_1^{(n)} \quad = \quad \sigma_1 \otimes \sigma_3 \otimes \cdots \otimes \sigma_3$$

$$
\begin{aligned}
\gamma_2^{(n)} &= \sigma_2 \otimes \sigma_3 \otimes \cdots \otimes \sigma_3 \\
&\vdots \\
\gamma_{2i-1}^{(n)} &= \underbrace{\mathbb{I}_2 \otimes \cdots \otimes \mathbb{I}_2}_{i-1} \otimes \sigma_1 \otimes \underbrace{\sigma_3 \otimes \cdots \otimes \sigma_3}_{n-i} \\
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&\vdots \\
\gamma_{2n-1}^{(n)} &= \mathbb{I}_{2^{n-1}} \otimes \sigma_1 \\
\gamma_{2n}^{(n)} &= \mathbb{I}_{2^{n-1}} \otimes \sigma_2 .
\end{aligned}
$$

The “gamma-five” matrix

$$\gamma_F^{(n)} = (-i)^n \gamma_1^{(n)} \cdots \gamma_{2n}^{(n)}$$

and can be used to define two projectors

$$P_\pm^{(n)} = \frac{\mathbb{I}_{2^n} \pm \gamma_F^{(n)}}{2}, \text{ i.e., } \left(P_\pm^{(n)}\right)^2 = P_\pm^{(n)} = \left(P_\pm^{(n)}\right)^\dagger .$$

Generators of the IRs $\mathcal{B}_{2n}$

$$R_{i,i+1}^{(\pm)} \quad = \quad \mathrm{e}^{i\frac{\pi}{4}} P_\pm^{(n)} \exp\left(-\frac{\pi}{4}\gamma_i\gamma_{i+1}\right) P_\pm^{(n)}$$

$$= \frac{e^{i\frac{\pi}{4}}}{\sqrt{2}} \left(\mathbb{I} - \gamma_i^{(n)} \gamma_{i+1}^{(n)} \right) P_{\pm}^{(n)} .$$

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