Pohlmeyer reduction revisited

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Outlook

- Introduction and motivation
  - Pohlmheyer reduction in string theory
- Non-linear sigma models on symmetric spaces
- SSSG equations for F/G of definite signature
  - Connection with the non-abelian affine Toda equations
  - Lagrangian formulation
  - Example: Pohlmheyer reduction of the principal chiral models
- SSSG equations for AdS_n (indefinite signature)
- Conclusions and open problems

Pohlmeyer reduction: The basics

▶▶ **Non-local** transformation of variables that maps the equations of motion of two-dimensional **non-linear sigma-models** to multi-component generalizations of the sine-Gordon equation.

▶▶ Relies on **classical conformal invariance**

$$\Rightarrow \boxed{T_{++} = T_{--} = \mu^2 \quad \text{and} \quad T_{+-} = 0}$$

▶▶ Preserves **integrabiliy** and two-dimensional **Lorentz invariance**.

■ Examples

[Pohlmeyer'76]

$$S^2 = SO(3)/SO(2) \longrightarrow \text{sine-Gordon}$$

$$S^3 = SO(4)/SO(3) \longrightarrow \text{complex sine-Gordon/Lund-Regge model}$$

....

- Group theoretical generalization to sigma-models associated to generic symmetric spaces $\mathcal{M} = F/G$

[Pohlmeyer,Eichenherr,Forger,D'Auria,Regge,....'79-81]

$\longrightarrow$ **Symmetric Space sine-Gordon** (SSSG) equations

★ **Classical integrability:**

Zero-curvature equations of **non-abelian affine Toda type**

[Bakas-Park-Shin'95]

★ **Lagrangian formulation:**

Gauged Wess-Zumino-Witten for G/H with a potential term

►► **Perturbed CFT**

Pohlmeyer reduction in string theory

[Tseytlin'03]

$$\boldsymbol{T_{\pm\pm} = \mu^2} \quad \longrightarrow \quad \text{Virasoro constraints on } \boldsymbol{R_t \times \mathcal{M}}$$

Pohlmeyer reduction in string theory

[Tseytlin'03]

$$T_{\pm\pm} = \mu^2 \quad \longrightarrow \quad \text{Virasoro constraints on } R_t \times \mathcal{M}$$

- Construction of **bosonic string** configurations on **curved space-times**

$$R_t \times S^n \subset AdS_5 \times S^5$$

[Hofman-Maldacena'06]

- $n = 2 \;\rightarrow$ **Giant magnons** $\rightarrow$ sine-Gordon solitons

[Chen-Dorey-Okamura'06]

- $n = 3 \;\rightarrow$ **Dyonic giant magnons** $\rightarrow$ complex sine-Gordon solitons

.... $R_t \times \mathcal{M} \subset AdS_4 \times CP^3$ ....

★ $\mathcal{M} = S^n, CP^n, \ldots$ are of **definite signature** $\Rightarrow$ $\boldsymbol{\mu^2 > 0}$ $\rightarrow$ $\boxed{\boldsymbol{R_t \times \mathcal{M}}}$

But. . . . . .

★ $\mathcal{M} = S^n, CP^n, \ldots$ are of **definite signature** $\Rightarrow$ $\boldsymbol{\mu^2 > 0}$ $\rightarrow$ $\boldsymbol{R_t \times \mathcal{M}}$

But. . . . . .

★ $\mathcal{M} = AdS_n$ is of **indefinite signature** (Lorentzian)

- $\mu^2 > 0$ $\rightarrow$ bosonic strings on $\boxed{\boldsymbol{R_t \times AdS_n}}$

[Grigoriev-Tseytlin'08]

- $\mu^2 < 0$ $\rightarrow$ bosonic strings on $\boxed{\boldsymbol{AdS_n \times S^1 \subset AdS_5 \times S^5}}$

[de Vega et al.'93-96]
[Jevicki-Jin-Kalousios-Volovich'08]

- $\mu^2 = 0$ $\rightarrow$ bosonic strings on $\boxed{\boldsymbol{AdS_n}}$

[Grigoriev-Tseytlin'08]
[Mikhailov-SchaferNakemi'08]

- Novel **Lorentz invariant** formulation of $AdS_5 \times S^5$ superstring theory

  - Generalized Pohlmeyer reduction of the $\frac{PSU(2,2|4)}{Sp(2,2)\times Sp(4)}$ supercoset model

  - Virasoro constraints $T_{\pm\pm} = T^{AdS_5}_{\pm\pm} + T^{S^5}_{\pm\pm} = 0 \rightsquigarrow$ $\boxed{\begin{array}{l} T^{S^5}_{\pm\pm} = +\mu^2 \\ T^{AdS_5}_{\pm\pm} = -\mu^2 \ \leftarrow \end{array}}$

►► Lorentz invariant, integrable Lagrangian action

$$\boxed{S = S_{gWZW}\left[\frac{Sp(2,2)}{SU(2)\times SU(2)} \times \frac{Sp(4)}{SU(2)\times SU(2)}\right] + \text{potential} + \textbf{fermions}}$$

▶▶▶ QUESTIONS :

- General structure of the SSSG equations that can be associated to a given symmetric space.
- Do all of them correspond to bosonic string configurations in curved space-times?

- Precise relation between the degrees of freedom of the original sigma model and those of gauged WZW actions with a potential term.
- Is a Lagrangian formulation of this type always possible??

Non-linear sigma models on symmetric spaces

- F/G symmetric space

  $\Rightarrow\ \mathfrak{f} = \mathfrak{g} \oplus \mathfrak{p}$ such that $[\mathfrak{g},\mathfrak{g}] \subset \mathfrak{g}, \quad [\mathfrak{g},\mathfrak{p}] \subset \mathfrak{p}, \quad [\mathfrak{p},\mathfrak{p}] \subset \mathfrak{g}$

$$\boxed{F/G = \left\{f \in F,\ B_\mu \in \mathfrak{g}\right\} / \left[\, f \to fg^{-1},\ B_\mu \to g\left(B_\mu + \partial_\mu\right)g^{-1} : g \in G \,\right]}$$

$$\Rightarrow\ J_\mu = f^{-1}D_\mu f = f^{-1}\partial_\mu f - B_\mu \ \to\ gJ_\mu g^{-1}$$

- $$\boxed{\mathcal{L} = -\frac{1}{2\kappa}\mathrm{Tr}\left(J_\mu J^\mu\right)}$$ ►►► $$\boxed{\begin{array}{l} J_\mu \in \mathfrak{p} \\ \partial_\pm J_\mp + \left[B_\pm, J_\mp\right] = 0 \leftarrow \\ \left[J_+, J_-\right] + F_{+-} = 0 \end{array}}$$

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$$\boxed{\begin{array}{l} J_\mu \in \mathfrak{p} \\ \partial_\pm J_\mp + \left[B_\pm, J_\mp\right] = 0 \ \leftarrow \\ [J_+, J_-] + F_{+-} = 0 \end{array}}$$

$\Rightarrow$ $\boxed{\boldsymbol{\partial_\pm \mathrm{Tr}\left(J_\mp^n\right) = 0}}$ local chiral densities, and $\boldsymbol{\mathrm{Tr}\left(J_\pm^2\right) = -2\kappa\, T_{\pm\pm}}$

★★ **Pohlmeyer reduction constraints**

$$\boxed{\mathrm{Tr}\left(J_{\mp}^{n}\right) = \text{constant} \qquad \forall\, n \geq 2}$$

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▶ Different number of SSSG equations depending on

- **Signature** of F/G

  $\boldsymbol{S^n} = SO(n+1)/SO(n)$, $\boldsymbol{CP^n} = SU(n+2)/U(n), \ldots \rightarrow$ **Definite**

  $\boldsymbol{AdS_n} = SO(2, n-1)/SO(1, n-1), \ldots \rightarrow$ **Indefinite (Lorentzian)**

- **Rank** of F/G = Dimension of the maximal abelian subspaces of $\mathfrak{p}$

  $$\boxed{\text{rank}(F) - \text{rank}(G) \leq \text{rank}(F/G) \leq \text{rank}(F)}$$

  $\boldsymbol{S^n}, \boldsymbol{CP^n}, \boldsymbol{AdS_n}, \ldots \rightarrow$ **rank=1**

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    $\boldsymbol{S^n}, \boldsymbol{CP^n}, \boldsymbol{AdS_n}, \ldots \rightarrow$ **rank=1**

- Not all of them admit a Lagrangian formulation

  - The Lagrangian is not unique $\rightarrow$ **Target-space duality**

SSSG equations for F/G of definite signature

$\boxed{G \text{ is compact}}$ $\quad (\mathfrak{f} = \mathfrak{g} \oplus \mathfrak{p})$

[$\cdots$Eichenherr-Forger'81$\cdots$]

$\Rightarrow$ **Polar coordinate decomposition:**

Let $\mathfrak{a}$ be a **maximal abelian subspace** of $\mathfrak{p}$: $\boxed{\dim(\mathfrak{a}) = \mathrm{rank}(F/G)}$

$\forall\, k \in \mathfrak{p}$ there is $\overline{g} \in G$ such that $\overline{g}^{-1} k \overline{g} \in \mathfrak{a}$

▶ $J_\pm \in \mathfrak{p} \;\Rightarrow\; \boxed{J_\pm = \overline{g}_\pm\, c_\pm\, \overline{g}_\pm^{-1}, \quad \overline{g}_\pm \in G}$

★ $\mathrm{Tr}\big(J_\pm^n\big) = \mathrm{Tr}\big(c_\pm^n\big) = \text{constant} \;\Rightarrow\; \boxed{\boldsymbol{c}_\pm \in \boldsymbol{\mathfrak{a}} \text{ constant}}$

[Evans-Mountain'00]

●● Only $\mathrm{rank}(F/G)$ **primitive** chiral densities for each chirality

●● $T_{\pm\pm} = \mathrm{Tr}\big(J_\pm^2\big) < 0 \;\Rightarrow\;$ bosonic strings on $R_t \times F/G$

$$\boldsymbol{J}_+ = \boldsymbol{\mu}_+ \boldsymbol{\Lambda}_+ \,, \qquad \boldsymbol{J}_- = \boldsymbol{\mu}_- \boldsymbol{\gamma}^{-1} \boldsymbol{\Lambda}_- \boldsymbol{\gamma} \,; \qquad \gamma \in G, \;\; \Lambda_\pm \in \mathfrak{a}$$

$$B_- = A_-^{(R)} \in \mathrm{Ker}\big(\mathrm{Ad}_{\Lambda_+}\big) \cap \mathfrak{g}, \qquad \gamma B_+ \gamma^{-1} - \partial_+ \gamma\, \gamma^{-1} = A_+^{(L)} \in \mathrm{Ker}\big(\mathrm{Ad}_{\Lambda_-}\big) \cap \mathfrak{g}$$

►► $\Big(\boldsymbol{J}_{\boldsymbol{\mu}} = \mathsf{P}_{\mathfrak{p}}\big(f^{-1}\partial_\mu f\big),\; \boldsymbol{B}_{\boldsymbol{\mu}} = \mathsf{P}_{\mathfrak{g}}\big(f^{-1}\partial_\mu f\big) \Big) \longrightarrow \big(\boldsymbol{\gamma}, \boldsymbol{A}_+^{(\boldsymbol{L})}, \boldsymbol{A}_-^{(\boldsymbol{R})} \big)$

► $\mathrm{Tr}\big(J_\pm^n\big) = \mu_\pm^n \mathrm{Tr}\big(\Lambda_\pm^n\big) = \text{constant}$

$$J_+ = \boldsymbol{\mu}_+ \boldsymbol{\Lambda}_+ \,, \qquad J_- = \boldsymbol{\mu}_- \boldsymbol{\gamma}^{-1} \boldsymbol{\Lambda}_- \boldsymbol{\gamma} \,; \qquad \gamma \in G, \quad \Lambda_\pm \in \mathfrak{a}$$

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▸ $\mathrm{Tr}\big(J_\pm^n\big) = \mu_\pm^n \mathrm{Tr}\big(\Lambda_\pm^n\big) = \text{constant}$

$$\Big[\, \boldsymbol{\partial}_+ + \boldsymbol{\gamma}^{-1} \boldsymbol{\partial}_+ \boldsymbol{\gamma} + \boldsymbol{\gamma}^{-1} \boldsymbol{A}_+^{(L)} \boldsymbol{\gamma} + \boldsymbol{z} \, \boldsymbol{\mu}_+ \boldsymbol{\Lambda}_+, \; \boldsymbol{\partial}_- + \boldsymbol{A}_-^{(R)} + \boldsymbol{z}^{-1} \, \boldsymbol{\mu}_- \boldsymbol{\gamma}^{-1} \boldsymbol{\Lambda}_- \boldsymbol{\gamma} \, \Big] = \boldsymbol{0}$$

$\Rightarrow$

$$\gamma \rightarrow h_- \, \gamma \, h_+^{-1}, \qquad A_-^{(R)} \rightarrow h_+ \big(A_-^{(R)} + \partial_- \big) h_+^{-1}, \qquad A_+^{(L)} \rightarrow h_- \big(A_+^{(L)} + \partial_+ \big) h_-^{-1}$$

$$h_\pm \in \boldsymbol{H}^{(\pm)} = \big\{ g \in G \; : \; g^{-1} \Lambda_\pm g = \Lambda_\pm \big\},$$

Zero-curvature condition on $G/H_L^{(-)} \times H_R^{(+)}$

Connection with the NAAT equations

Particular **gauge-fixing conditions**

$$A_-^{(R)} = A_+^{(L)} = \mathsf{P}_{\mathfrak{h}_+}(\gamma^{-1}\partial_+\gamma) = \mathsf{P}_{\mathfrak{h}_-}(\partial_-\gamma\gamma^{-1}) = 0$$

▸▸

$$\partial_-(\gamma^{-1}\partial_+\gamma) = \mu_+\mu_-\left[\,\Lambda_+,\, \gamma^{-1}\Lambda_-\gamma\,\right]$$
$$\mathsf{P}_{\mathfrak{h}_+}(\gamma^{-1}\partial_+\gamma) = \mathsf{P}_{\mathfrak{h}_-}(\partial_-\gamma\gamma^{-1}) = 0$$

[Leznov-Saveliev'83]
[Ferreira-JLM-SanchezGuillen'97]
[Nirov-Razumov'07]

Non-abelian affine Toda equations associated to the **affine Lie algebra**

$$\bigoplus_{n\in Z}\left(\lambda^{2n}\otimes\mathfrak{g} + \lambda^{2n+1}\otimes\mathfrak{p}\right)$$

Lagrangian formulation

[Bakas-Park-Shin'96]
[JLM'08]

The SSSG equations are the equations of motion of a gauged WZW action with a potential term provided that $\boldsymbol{H^{(+)}}$ **is isomorphic to** $\boldsymbol{H^{(-)}}$

$$H^{(+)} = \epsilon_R(H), \quad H^{(-)} = \epsilon_L(H)$$

$$\boxed{S\big[\gamma, \mathcal{A}_\pm\big] = S_{\mathsf{gWZW}}\big[\gamma, \mathcal{A}_\pm\big] - \mu_+\mu_- \int d^2x \ \mathsf{Tr}\big(\Lambda_+\gamma^{-1}\Lambda_-\gamma\big)\,, \quad \gamma \in G,\ \mathcal{A}_\pm \in \mathfrak{h}}$$

- $\boldsymbol{S_{\rm gWZW}}$ is the Lagrangian action associated to the **(asymmetric)** coset

$$\boxed{G/H = G/\big[\,\gamma \sim \epsilon_L(h)\,\gamma\,\epsilon_R(h^{-1});\ \gamma \in G,\ h \in H\,\big]}$$

Equations of motion

- $\left[\partial_+ + \gamma^{-1}\partial_+\gamma + \gamma^{-1}\epsilon_L(\mathcal{A}_+)\gamma + z\mu_+\,\mu_+\Lambda_+,\, \partial_- + \epsilon_R(\mathcal{A}_-) + z^{-1}\mu_-\,\gamma^{-1}\Lambda_-\gamma\right] = 0$

  Zero-curvature condition $\Rightarrow$ $\boxed{A_+^{(L)} = \epsilon_L(\mathcal{A}_+),\;\; A_-^{(R)} = \epsilon_R(\mathcal{A}_-)}$ ◀◀

- $\mathsf{P}_{\mathfrak{h}_-}\Big(-\partial_-\gamma\gamma^{-1} + \gamma\epsilon_R(\mathcal{A}_-)\gamma^{-1}\Big) = \epsilon_L(\mathcal{A}_-)$

  $\mathsf{P}_{\mathfrak{h}_+}\Big(\gamma^{-1}\partial_+\gamma + \gamma^{-1}\epsilon_L(\mathcal{A}_+)\gamma\Big) = \epsilon_R(\mathcal{A}_+)$

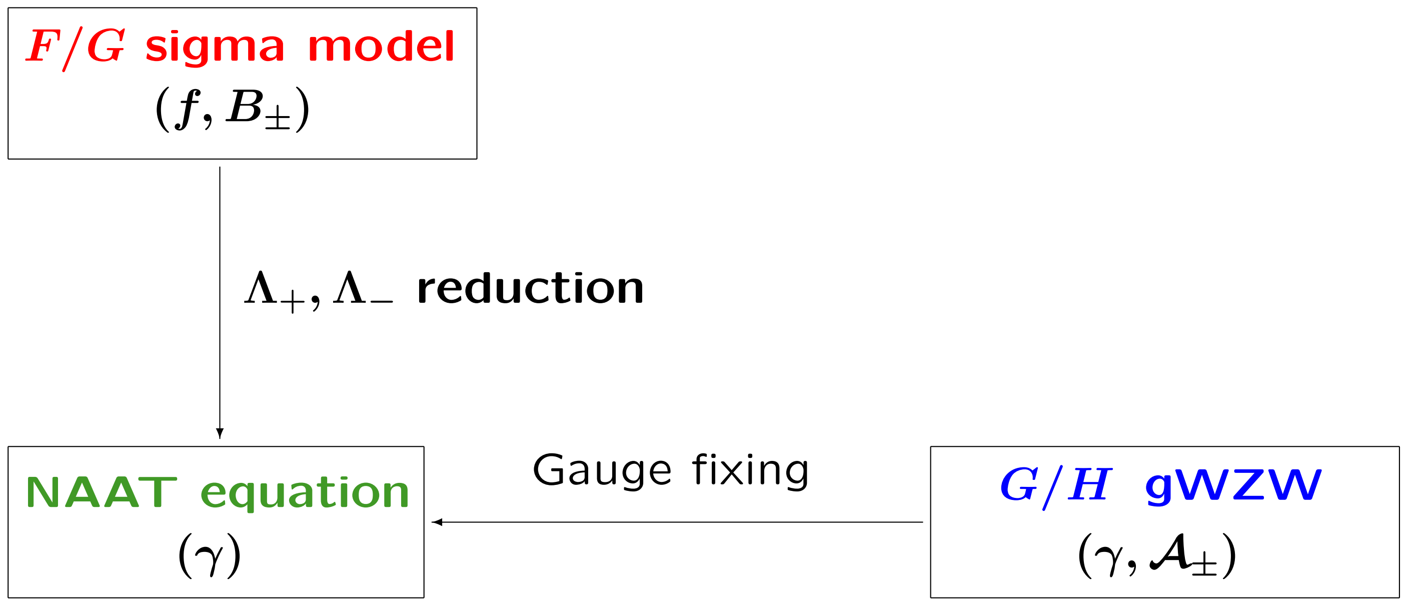
  [Grigoriev-Tseytlin'08]

  Gauge-fixing conditions $\boldsymbol{H_L^{(-)}} \times \boldsymbol{H_R^{(+)}} \;\longrightarrow\; \boldsymbol{H}$

★ **Different** choices of ϵ_L and ϵ_R are related by **gauge transformations**

[JLM'04]

$\Rightarrow$ **Different** Lagrangian actions related by **T-duality transformations**


F/G sigma model
$(f, B_\pm)$
Λ_+, Λ_- reduction
NAAT equation
(γ)
Gauge fixing
G/H gWZW
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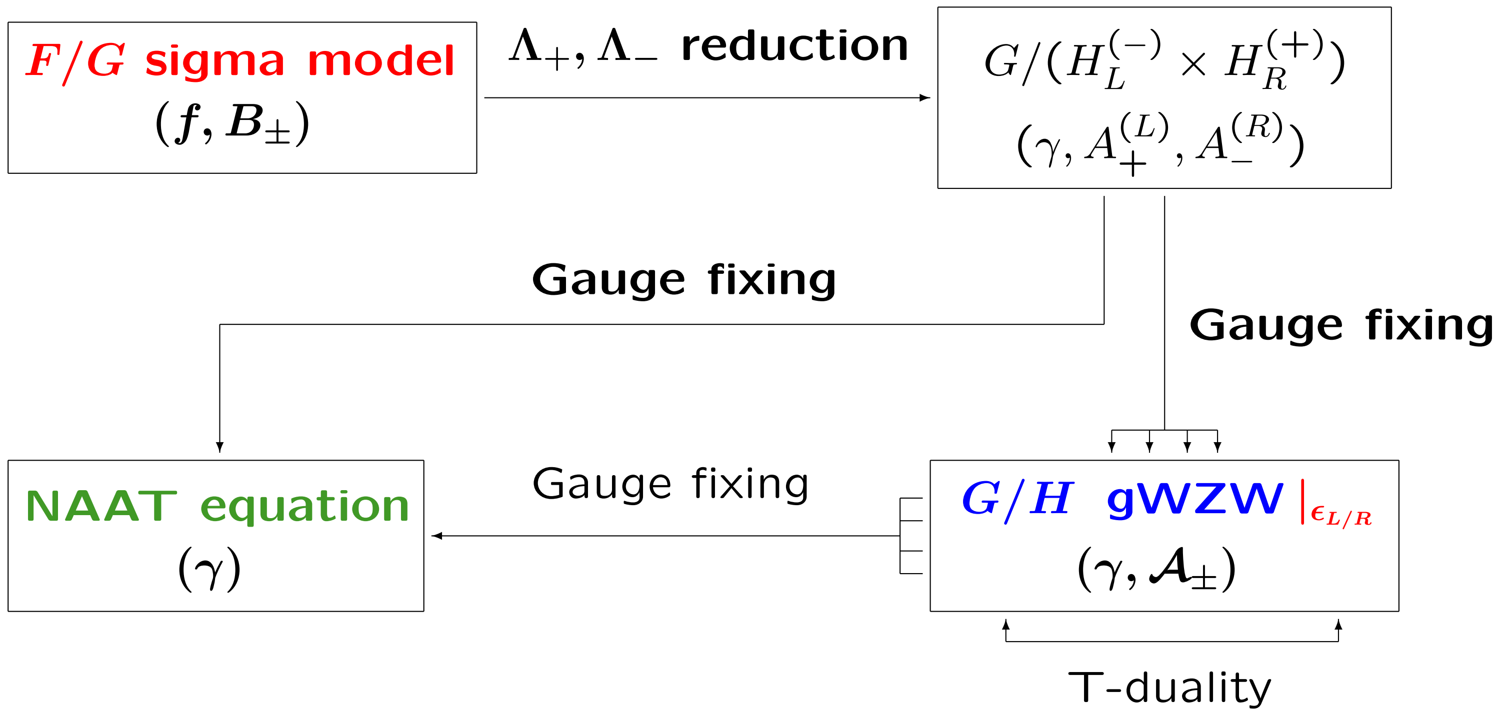
F/G sigma model
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T-duality

Example: reductions of principal chiral models

$$\boxed{F/G = G \times G/G_D \simeq G} \quad \longrightarrow \quad \boxed{G\text{-principal chiral model}}$$

- rank$\big(G \times G/G_D\big) = $ rank(G)

Example: reductions of principal chiral models

$$\boxed{F/G = G \times G/G_D \simeq G} \;\longrightarrow\; \boxed{G\text{-principal chiral model}}$$

- $\text{rank}\big(G \times G/G_D\big) = \text{rank}(G)$

- $\boldsymbol{G = SU(3)} \;\rightarrow\; \Lambda_\pm = \begin{pmatrix} i\mu_1^\pm & 0 & 0 \\ 0 & i\mu_2^\pm & 0 \\ 0 & 0 & i\mu_3^\pm \end{pmatrix}, \qquad \mu_1 + \mu_2 + \mu_3 = 0$

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  - $\boxed{\mu_i^\pm \neq \mu_j^\pm \;\text{ for }\; i \neq j} \;\Rightarrow\; H^{(+)} = H^{(-)} = U(1)^2$

[F.Pousa-Gallas-Hollowood-JLM'97]
[Dorey-JLM'04]

$SU(3)$ **Homogeneous sine-Gordon model**

- $\boxed{S = S_{gWZW}\big[SU(3)/U(1)^2\big] + \text{potential}}$
- Three **massive** particles with $U(1)^2$ charge
- •• Integrable theories with **adjustable** parameters

- $\boxed{\mu_1^{\pm} = \mu_2^{\pm}}$ $\Rightarrow$ $H^{(+)} = H^{(-)} = SU(2) \times U(1)$

  - $\boxed{S = S_{gWZW}\left[SU(3)/SU(2) \times U(1)\right] + \text{potential}}$

  - $SU(2) \times U(1)$ multiplet of two **massive** particles

- $\boxed{\mu_1^{\pm} = \mu_2^{\pm}}$ $\Rightarrow$ $H^{(+)} = H^{(-)} = SU(2) \times U(1)$

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  - $SU(2) \times U(1)$ multiplet of two **massive** particles

- $\boxed{\mu_1^{+} = \mu_2^{+} \text{ and } \mu_2^{-} = \mu_3^{-}}$ $\Rightarrow$ $H^{(+)} \neq H^{(-)} \simeq SU(2) \times U(1)$ ★★

  - $\boxed{S = \widetilde{S}_{gWZW}\big[SU(3)/SU(2) \times U(1)\big] + \text{potential}}$

  - One **massless** and one **massive** particle

  - •• Integrable perturbations of **asymmetric** coset models

SSSG equations for AdS_n (indefinite signature)

$$\boldsymbol{AdS_n} = \{(x_1, \ldots, x_{n+1}) \in R^{n+1} \; : \; -x_1^2 - x_2^2 + x_3^2 + \cdots + x_{n+1}^2 = -1\}$$
$$= \boldsymbol{SO(2, n-1)/SO(1, n-1)} \equiv F/G$$

- $\mathbf{rank}(\boldsymbol{AdS_n}) = \mathbf{1}$

- $G = SO(1, n-1)$ is **non-compact**

  - The sign of $T_{\pm\pm} = \mathrm{Tr}\big(J_\pm^2\big)$ is indefinite

    $\Rightarrow$ bosonic strings on $\boxed{R_t \times AdS_n}$, $\boxed{AdS_n \times S^1}$ or $\boxed{AdS_n}$

  - The polar coordinate decomposition is not satisfied

$\boxed{\mathfrak{f} = \mathfrak{g} \oplus \mathfrak{p}}$

- $\mathfrak{g} = so(1, n-1)$

- $k \in \mathfrak{p}$ is of the form $k = \begin{pmatrix} 0 & -v_0 & v_1 & \cdots & v_{n-1} \\ v_0 & 0 & 0 & \cdots & 0 \\ v_1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ v_{n-1} & 0 & 0 & \cdots & 0 \end{pmatrix} \equiv \widehat{k}[\boldsymbol{v}]$

[JLM'08]

★ **Generalized** polar coordinate decomposition

$\forall\, k \in \mathfrak{p}$ there is $\overline{g} \in G$ such that

$$\boxed{\overline{g}^{\,-1} k \overline{g} = \begin{cases} \mu\, \widehat{k}[(0,\ldots,0,1)] \equiv \mu\, T^{(s)}, & \text{if } \operatorname{Tr}(k^2) > 0, \\ \mu\, \widehat{k}[(1,0,\ldots,0)] \equiv \mu\, T^{(t)}, & \text{if } \operatorname{Tr}(k^2) < 0, \\ \mu\, \widehat{k}[(1,0,\ldots,0,1)] \equiv \mu\, T^{(l)}, & \text{if } \operatorname{Tr}(k^2) = 0. \end{cases}}$$

●● $\mathbf{rank}(\boldsymbol{AdS_n}) = \mathbf{1}$

Spacelike reduction: Strings on $\mathbf{R_t} \times \mathbf{AdS_n}$

- $\Lambda_+ = \Lambda_- = T^{(s)} \Rightarrow \boxed{\boldsymbol{T_{\pm\pm} = +\mu_\pm^2 > 0}}$

$$\boxed{S = \widetilde{S}_{gWZW}\big[SO(1,n-1)/SO(1,n-2)\big] + \text{potential}}$$

- $n = 2 \;\longrightarrow\;$ sinh-Gordon
- $n = 3 \;\longrightarrow\; \mathcal{L} = \frac{1}{4}\partial_\mu\theta\partial^\mu\theta - \tan^2(\theta/2)\partial_\mu\rho\partial^\mu\rho - \mu_+\mu_- \sin^2(\theta/2)$

- **Spacelike** reduction: Strings on $\mathbf{R_t} \times \mathbf{AdS_n}$

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- **Timelike** reduction: Strings on $\mathbf{AdS_n} \times \mathbf{S^1}$

  - $\Lambda_+ = \Lambda_- = T^{(t)} \;\Rightarrow\; \boxed{\boldsymbol{T_{\pm\pm} = -\mu_\pm^2 < 0}}$

    $$\boxed{S = \widetilde{S}_{gWZW}\big[SO(1,n-1)/SO(n-1)\big] + \text{potential}}$$

    - $n = 2 \;\longrightarrow\;$ sinh-Gordon
    - $n = 3 \;\longrightarrow\; \mathcal{L} = \frac{1}{4}\partial_\mu\chi\partial^\mu\chi + \tanh^2(\chi/2)\partial_\mu\phi\partial^\mu\phi - \mu_+\mu_-\sinh^2(\chi/2)$

■ **Lightlike** reduction: Strings on $\mathbf{AdS_n}$

▶ $\Lambda_+ = \Lambda_- = T^{(l)} \;\Rightarrow\; \boxed{\boldsymbol{T_{\pm\pm} = 0}}$

Conformal invariance is not broken

Zero-curvature conditions on $\boxed{SO(1,n-1)/E_L(n-2) \times E_R(n-2)}$

Lagrangian formulation is **not known**

- $n = 2 \;\rightarrow\;$ Liouville
- $n = 3 \;\rightarrow\; \partial_+\partial_-\chi + 2\mathrm{e}^{\chi}\big(b_+b_- + \mu_+\mu_-\big) = \partial_\pm(\mathrm{e}^{\chi}b_\mp) = 0$

Invariant under $x_\pm \rightarrow \mathrm{e}^{-\eta_\pm}x_\pm, \quad \chi \rightarrow \chi + \eta_+ + \eta_-, \quad b_\pm \rightarrow \mathrm{e}^{\pm(\eta_+-\eta_-)}b_\pm$

[de Vega et al.'93-96]
[Jevicki-Jin-Kalousios-Volovich'08]

•• Can be mapped into the sinh-Gordon equation

Conclusions and open problems

- Systematic group theoretical formulation of the Pohlmeyer reduction of two-dimensional non-linear sigma models with target-space a symmetric space F/G.
- The conditions required to ensure a Lagrangian formulation of the resulting SSSG equations in terms of gauged WZW actions with a potential term are clarified.
- The relation between the degrees of freedom of the original sigma model and those of the relevant Lagrangian actions become explicit.

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- The relation between the degrees of freedom of the original sigma model and those of the relevant Lagrangian actions become explicit.
- The (stringy) interpretation of the constraints for rank$(F/G) > 1$ should be investigated.
- The Lagrangian formulation of the "lightlike" SSSG equations corresponding to AdS_n remains an open problem.
- Soliton solutions `[in progress]`
- ....