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Noncommutative bosonization and Seiberg-Witten maps

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And yet it moves...

Noncommutative spaces in physics

$$[x^\mu, x^\nu] = i\theta^{\mu\nu} \ , \quad \mu, \nu = 1, \ldots D$$

$\theta^{\mu\nu}$ ordinary (commuting) numbers: 'flat' NC space
Other possibilities (NC sphere, Riemann spaces etc.)

(Connes, Madore, Grosse, Wess,...; Douglas, Schwartz, Seiberg, Witten,...)

- Can arise as specific limits in string theory
- Potential description of Planck scale spacetime physics
- Effective description (e.g., lowest Landau level physics)

Could be viewed as a fundamental effect (cosmological bounds from CMB radiation etc.) or as a tool (e.g., noncommutative Chern-Simons description quantum Hall states)

- No notion of points
- Functions f become operators
- Product of functions $f \cdot g$ associative but noncommutative
- Derivatives and integral defined as

$$\partial_\mu f = [-i\omega_{\mu\nu}x^\nu, f] \qquad \omega_{\alpha\beta} = (\theta^{-1})_{\alpha\beta}$$

$$\int d^D x f = \sqrt{\mathrm{det}(2\pi\theta)}\, \mathrm{Tr} f$$

Most notions of field calculus generalize. E.g.,

- $\int \partial f \sim 0$ translates into $\mathrm{Tr}[f,g] \sim 0$;
- $\partial(f \cdot g) = \partial f \cdot g + f \cdot \partial g$ still true; etc.

Can include ‘internal space’ (spin, flavor,...) as extra copies of the space where the x^μ act

Star products: Weyl ordering of monomials $x^\mu \cdots x^\nu$

$$f = f(\{x^\mu\})_W \leftrightarrow \text{commutative } f(x)$$

In terms of the Fourier transform $\tilde{f}(k)$ of $f(x)$:

$$\tilde{f}(k) = \sqrt{\det(\theta/2\pi)}\, \mathrm{Tr}\, f e^{-ik_\mu x^\mu}$$

$$f = \int dk\, e^{ik_\mu x^\mu} \tilde{f}(k)$$

• Derivatives and integrals map into ordinary commutative expressions

• Product maps into the noncommutative star product

$$(f * g)(k) = \int dq\, \tilde{f}(q)\, \tilde{g}(k-q)\, e^{\frac{i}{2}\theta^{\mu\nu} k_\mu q_\nu}$$

We can view noncommutative field theory as ordinary field theory with a nonlocal, noncommutative product for functions

Noncommutative gauge theory becomes particularly nice in the operator formulation

Covariant coordinates:

$$X^\mu = x^\mu + \theta^{\mu\nu} A_\nu \qquad X^\mu \to U^{-1} X^\mu U$$

$$S_{MYM} \sim \mathrm{Tr}[X^\mu, X^\nu]^2$$

Chern-Simons action:

$$S_{CS} \sim \epsilon_{\mu\nu\rho} \mathrm{Tr} X^\mu X^\nu X^\rho$$

- Unify abelian and nonabelian expressions
- $\theta^{\mu\nu}$, rank of group become superselection parameters
- The above become ordinary-looking expressions in the $*$-product formulation
- Reduce to commutative expressions in $\theta \to 0$ limit

What can we do with it?

What does that have to do with bosonization?

We will use it to describe fuzzy fluids...

...and give an exact description of a many-body fermionic system

Fuzzy fluids: the Lagrangian way

Particle coordinates: $\vec{X}(\vec{x},t)$ ($\vec{x}$ are 'fiducial' particle-fixed coordinates of fixed density)

Velocity: $\vec{v} = \frac{d\vec{X}}{dt}$; density $\frac{1}{\rho} = \left|\det \frac{\partial X^i}{\partial x^j}\right|$

Make particles fuzzy: x^i are noncommutative $\rightarrow$ so are the X^i

• Particle-reparametrization invariance becomes unitary rotations: $X^i \rightarrow U X^i U^{-1}$

• X^i become covariant coordinates

• Noncommutative gauge theory describes the dynamics of a fuzzy fluid (or fuzzy membrane, if dim $X >$ dim x)

(Hoppe, deWitt, Nicolai,...)

• Noncommutative Chern-Simons theory: fuzzy incompressible fluid in 2 dimensions $\rightarrow$ FQH states (Susskind, AP,...)

Fuzzy fluids: the Eulerian way

In Lagrangian fuzzy fluids we can still define commutative currents:

$$\rho(y,t) = \int dx\, \delta(X-y) \qquad \vec{v}(y,t) = \int dx\, \frac{d\vec{X}}{dt}\delta(X-y)$$

They satisfy (commutative) continuity

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{v}) = 0$$

Ordinary Eulerian fluid (avatar of Seiberg-Witten map)

(Jackiw, AP)

Can also start with genuine noncommutative Euler density ρ

Will naturally describe fermions (and parafermions).
So let's jump right there...

Starting point: N non-interacting fermions in D spatial dimensions. (Consider $D = 1$ for notational convenience.)

Single particle hamiltonian $H_{\text{sp}}(x, p)$ and phase space x, p:

$$[x, p]_{\text{sp}} = i\hbar \qquad H_{\text{sp}}|n\rangle = E_n|n\rangle$$

N-body state basis: Fock states; e.g., $|gr\rangle = |1, \ldots 1, 0, \ldots\rangle$

Alternative description: single-particle 'density' operator

$$\rho = \sum_{i=1}^{N} |\psi_i\rangle\langle\psi_i| \ , \qquad \langle\psi_i|\psi_j\rangle = \delta_{ij}$$

with Schrödinger equation of motion

$$i\hbar\dot{\rho} = [H_{\text{sp}}, \rho]_{\text{sp}}$$

(Sakita, Khveshchenko, Nair, Karabali,...)

• ρ must satisfy the algebraic constraints $\rho^2 = \rho \ , \ \mathrm{Tr}\rho = N$

'Solve' the constraints in terms of a unitary field U:

$$\rho = U^{-1}\rho_0 U \ , \qquad \rho_0 = \sum_{n=1}^{N} |n\rangle\langle n| = |gs\rangle$$

An appropriate action for U which leads to EOM is

$$S = \int dt(K - H) = \int dt \mathrm{Tr}\left(i\hbar\rho_0 \dot{U} U^{-1} - U^{-1}\rho_0 U H_{\mathrm{sp}} \right)$$

U encodes both coordinates and momenta. Resulting Poisson brackets for the matrix elements ρ_{mn} of ρ:

$$\{\rho_{mn}, \rho_{rs}\} = \frac{1}{i\hbar}(\rho_{ms}\delta_{rn} - \rho_{rn}\delta_{ms})$$

Drawbacks of the description:

- Can describe only 'factorizable' states
- Violates quantum mechanical superposition principle

Still it reproduces the full Hilbert space of the N fermions upon quantization!

- S is the Kirillov-Kostant-Souriau (KKS) form for the group of unitary transformations on the Hilbert space
- Truncate to K first energy levels ($K \gg N$): S becomes the KKS action for the group $U(K)$
- $\rho = U^{-1}\rho_0 U$ and S have the gauge invariance

$$U(t) \to V(t)U(t) \ , \quad [\rho_0, V(t)] = 0$$

which reduces the left degrees of freedom

• Gauge invariance introduces Gauss law and a 'global gauge anomaly'

• Quantization condition: eigenvalues of ρ_0 must be integers

PBs for ρ become upon quantization

$$[\rho_{mn}, \rho_{rs}]_{QM} = \rho_{ms}\delta_{rn} - \rho_{rn}\delta_{ms}$$

This is the $U(K)$ algebra in Cartesian basis

• Quantum states: irreps of $U(K)$ with Young tableau $= \rho_0$

In our case ρ_0 gives the N-fold fully antisymmetric irrep of $U(K)$
$\rightarrow$ Hilbert space of N fermions on K single-particle states.

Realize ρ_{mn} à la Jordan-Wigner with K fermionic oscillators Ψ_n:

$$\rho_{mn} = \Psi_n^\dagger \Psi_m \ , \qquad \sum_{n=1}^{K} \Psi_n^\dagger \Psi_n = N$$

Ψ: second-quantized Fermi field

$$H = \mathrm{Tr}(\rho H_{\mathrm{sp}}) = \sum_{m,n} \Psi_m^\dagger (H_{\mathrm{sp}})_{mn} \Psi_n$$

H becomes the second-quantized Fermi hamiltonian.

• $\rho_0^2 = q\rho_0$ describes parafermions of order q
• In the limit $K \to \infty$, ρ_{mn} reproduces the W_∞ algebra.
Conditions $\rho^2 = \rho$ and $\mathrm{Tr}\rho = N$ fix highest weight state
$\to$ pick fermionic vacuum

We can think of (classical) ρ as a fuzzy (noncommutative) fluid:

• $\rho^2 = \rho$ is the characteristic function of a domain in the non-commutative plain x, p
• ρ represents a 'droplet' filling the domain
• The density inside the droplet becomes $1/2\pi\hbar$
• Semiclassical picture of a Liouville fluid with evolving boundary

The ground state ρ_0 corresponds to a droplet filling states up to Fermi energy E_F:

$$H_{\text{sp}}(x, p) \leq E_F$$

• U can become singular in the limit $\hbar \to 0$
• U generates a canonical transformation in that case
• A nonsingular U becomes a phase and generates infinitesimal boundary waves

Recast the model as a noncommutative field theory with $\theta = \hbar$:
$H_{\mathrm{sp}}(x,p)$, $\rho(x,p)$, $U(x,p) * U(x,p)^{\dagger} = 1$

$$H = \frac{1}{2\pi\hbar}\int dxdp\ H_{\mathrm{sp}}(x,p)\rho(x,p)$$

$$[\rho(x,p), \rho(x',p')]_{QM} = [\rho(x,p), \delta(x-x',p-p')]_{*}$$

$$S = \frac{1}{2\pi}\mathrm{Tr}\int dtdxdp\ \vartheta(p-K_F) * \partial_t U(x,p) * U(x,p)^{*} - \int dtH$$

We obtained an exact bosonization of the fermion system in any dimension. Still we paid some price:
- Noncommutative, nonlocal action
- U must satisfy 'star-unitarity' condition
- 2 (in general $2D$) spatial dimensions

Where is the conventional bosonization?

Can be recovered in the $\hbar \to 0$ limit:

- $U = e^{i\phi} + O(\hbar^2)$
- $\rho = \rho_0 + \hbar \partial_p \rho_0 \partial_x \phi + O(\hbar^2)$

The action in $D = 1$ becomes

$$S = \frac{\hbar}{2} \int dt dx \; \partial_x \phi (\partial_t \phi - v_F \partial_x \phi)$$

Linear abelian bosonization ($v_F = \left. \frac{\partial H_{\text{sp}}}{\partial p} \right|_F$)

Assuming there are also n internal degrees of freedon on which x, p do not act, fields become $n \times n$ matrices. A similar (subtler) limit yields the Wess-Zumino-Witten model on the Fermi surface with an additional potential (from H)

$\longrightarrow$ Nonabelian bosonization

Still how can we recover the familiar exact (not $\hbar \to 0$) nonlinear bosonization in $D = 1$?

• S is the noncommutative version of the WZW action (yields it in commutative limit)
• Need an exact transformation that maps it to its commutative counterpart
$\to$ one coordinate becomes auxiliary $\to$ reduction to $2D - 1$ dimensions

Such transformations are called Seiberg-Witten maps
• First arose in noncommutative gauge theory
• Map noncommutative to commutative gauge fields respecting gauge transformations

In general the form of the action changes under a SW map

However, Chern-Simons and Wess-Zumino actions are special:
Exist Seiberg-Witten maps that leave them invariant

(Moreno, Schaposhnik; Grandi, Silva; Lopez-Sarrión, AP)

The infinitesimal transformation (in operator notation)

$$\delta U = \frac{i\delta\theta}{2\theta^2}(xpU + Upx - xUp - UpU^{-1}xU)$$

leaves noncommutative WZW action invariant and has a smooth commutative limit when driven to $\theta = 0$

$\longrightarrow$ standard (abelian or nonabelian) bosonization

This can be exported to higher dimensions!

- Seiberg-Witten map only works in $D = 2$
- Pick a 2-dim submanifold of phase space and perform transformation there
- End up with $2D-2$ noncommutative and 1 commutative variable

Leads to higher dimensional noncommutative bosonization

Calling the noncommutative coordinates ϕ and the commutative one σ, the boundary field is defined as

$$R(\sigma, \phi) = iU^{-1} * \partial_\sigma U$$

It satisfies the fundamental commutator

$$[R_1, R_2]_{QM} = \frac{1}{(2\pi\hbar)^{D-2}} \quad \Big(\delta'(\sigma_1 - \sigma_2)\,\delta(\phi_1 - \phi_2)$$
$$\delta(\sigma_1 - \sigma_2)\,[R_1, \delta(\phi_1 - \phi_2)]_*\Big)$$

- Partly density, partly current
- Its quantization reproduces the full N-body fermionic set of states.
- The hamiltonian may become complicated
- Quadratic potentials remain simple

Overview, Outlook

• A 'fuzzy fluid' description in the Euler picture achieves exact bosonization in any dimension

• (Fuzzy fluid in Lagrange picture $\rightarrow$ noncommutative Chern-Simons description of FQH states)

• Seiberg-Witten map makes contact with standard bosonization (in $D = 1$)...

• ...and gives 'minimal' bosonization in $D > 1$

$\star$ Is this really the minimal?

$\star$ Expression of ρ in terms of R? (Known in $D = 1$)

$\star$ Fermi operator?

$\star\star$ *Any interesting applications...?*