Effective Hamiltonians of anyon lattice models

Alexander Protogenov

Institute of Applied Physics of the RAS, Nizhny Novgorod, Russia

with **Luigi Martina**
Dipartimento di Fisica, Università di Lecce, Lecce, Italy
and **Valery Verbus**
Institute for Physics of Microstructures of the RAS, Nizhny Novgorod, Russia
(***E.I.N.S.T.E.IN - RFBR collaboration***)

Low-dimensional Field Theories and Applications

GGI Florence, 24 oct 2008

Some exact soluvable Hamiltonians of spin lattice models

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Outline

- Introduction: main tools
- Temperley-Lieb algebra projectors
- Mappings to transverse field Ising models
- Universal form of the effective Hamiltonians
- Conclusions

Algebraic data: $(d_s, F^{ijk}{}_{lmn}, \delta_{abc}, N)$

Quantum dimension $\mathbf{d_s}$ in the case of $SU(2)_k$ theory is equal to:

$$d_s = \frac{Sin[\pi(2s+1)/(k+2)]}{Sin[\pi/(k+2)]}$$

The quantum dimension d_a is an irrational number which illustrates that Hilbert space has no natural decomposition as a tensor product of subsystems because the topologically encoded information is a collective property of anyon systems.

$$d_s = \frac{Sin\,[\pi(2s+1)/(k+2)]}{Sin\,[\pi/(k+2)]} \qquad (\ \Rightarrow\ \ 2s+1 \quad if \quad k >> 1\ )$$

$$d_{1/2} = \sqrt{2} = 1.4142... \qquad for\ \ k = 2, \qquad instead \quad of\ \ 2\cdot 1/2+1 = 2 \qquad for\ s = 1/2$$

It controls the rate of growth of the n-particle Hilbert space $V^{b}_{aaa...a} = (d_a)^n d_b/D^2$ for anyons of type a.

Here $D^2 = \Sigma_{s=0}^{k/2}\,[2s+1]_q^{\,2}$.

Additional meaning of the d_a's

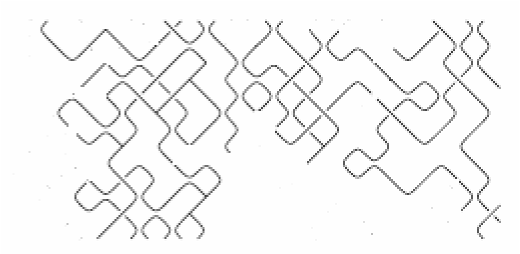
Operator product expansion (OPE) fusion rules for primary fields:

$$\varphi_a \circ \varphi_b = \sum_c N_{ab}^c \varphi_c$$

$$N_a \vec{d} = d_a \vec{d}$$

- p($a \bar{a} \rightarrow 1$) = $1/d_a^2$
- p(a b $\rightarrow$ c) = $N^c_{ab} d_c/(d_a d_b)$

String-net condensates



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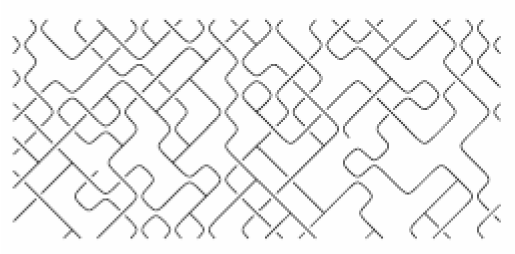

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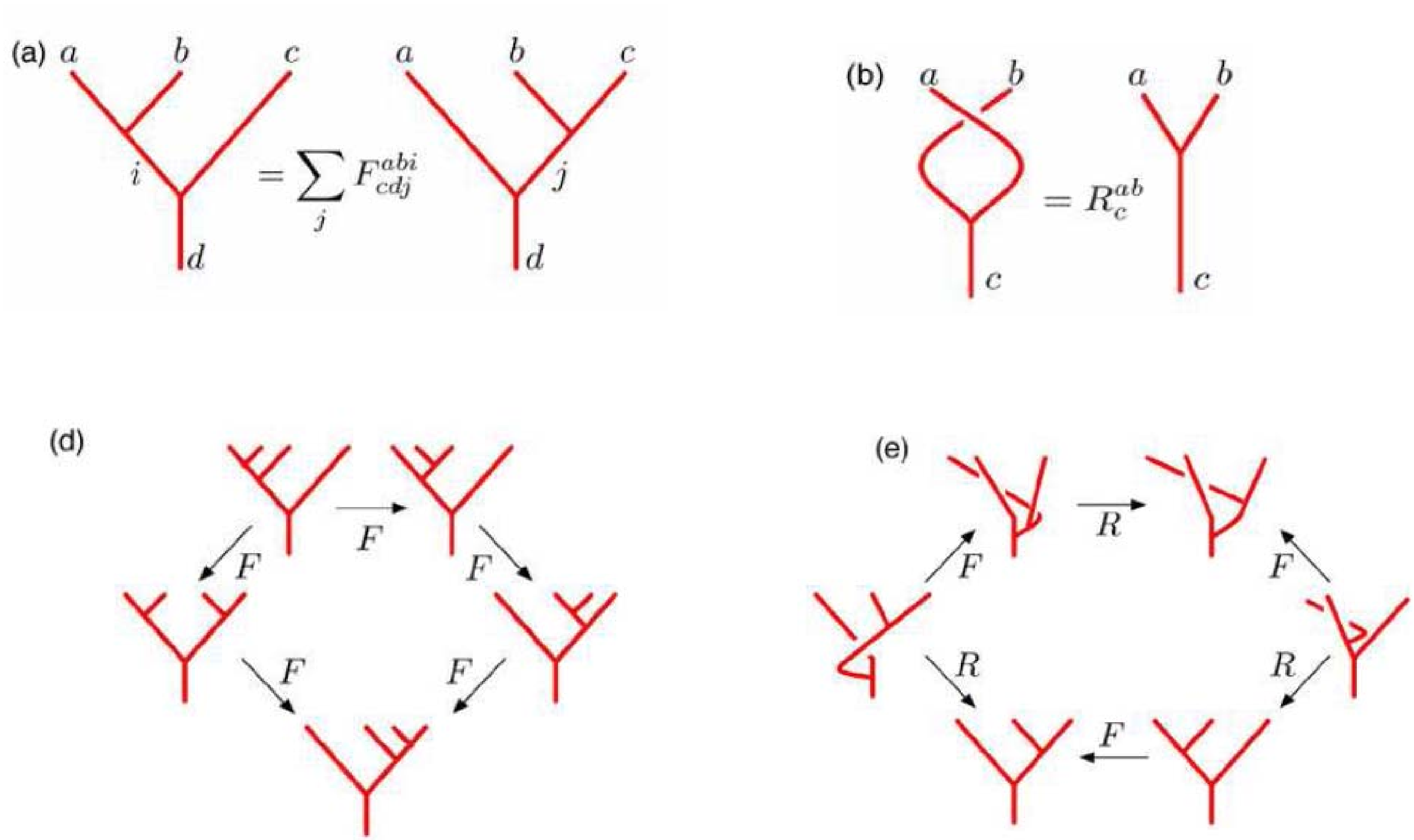
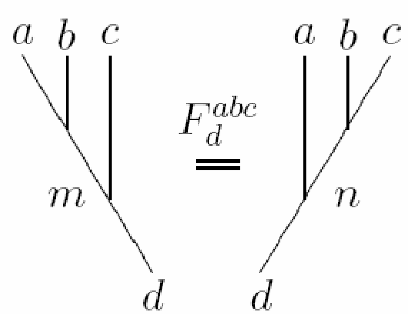
Equations of the tensor modular categories are nonlinear pentagon and hexagon ones



5.1. **F-matrices.** Given an MTC $\mathcal{C}$. A 4-punctured sphere $S^2_{a,b,c,d}$, where the 4 punctures are labelled by a, b, c, d, can be divided into two pairs of pants(=3-punctured spheres) in two different ways. In the following figure, the 4-punctured sphere is the boundary of a thickened neighborhood of the graph in either side, and the two graphs encode the two different pants-decompositions of the 4-punctured sphere. The F-move is just the change of the two pants-decompositions.

When bases of all pair of pants spaces $\mathrm{Hom}(a \otimes b, c)$ are chosen, then the two pants decompositions of $S^2_{a,b,c,d}$ determine bases of the vector spaces $\mathrm{Hom}((a \otimes b) \otimes c, d)$, and $\mathrm{Hom}(a \otimes (b \otimes c), d)$, respectively. Therefore the F-move induces a matrix $F_d^{a,b,c} : \mathrm{Hom}((a \otimes b) \otimes c, d) \to \mathrm{Hom}(a \otimes (b \otimes c), d)$, which are called the F-matrices. Consistency of the F matrices are given by the pentagon equations.



R. Rowell, R.Stong, Z. Wang, **math-QA/07121377**

These equations have a form

$$\sum_n F(m\ell kp)_n^q F(jimn)_s^p F(js\ell k)_r^n = F(jiqk)_r^p F(rim\ell)_s^q.$$

$$R_r^{mk} F(\ell mkj)_r^q R_q^{m\ell} = \sum_p F(\ell kmj)_r^p R_j^{mp} F(m\ell kj)_p^q$$

$$F_{cdj}^{abi} = \left\{ \begin{matrix} a & b & j \\ c & d & i \end{matrix} \right\}_q$$

V. Turaev, N. Reshetikhin, O. Viro, L. Кauffman 1992

The braid group

- $B_i\, B_{i+1}\, B_i \;=\; B_{i+1}\, B_i\, B_{i+1}$
- $B_i\, B_k \;=\; B_k\, B_i \qquad | i - k | \geq 2$

- For bosons the B_i matrices are all the identity.
- If the matrices B_i are diagonal, then the particles have Abelian statistics.
- For anyons their entries are phases.
- The wave function changes form depending on the order in which the particle are braided in the case of non-Abelian representations of the braid group when particles obey non-Abelian statistics.

Temperley-Lieb algebra

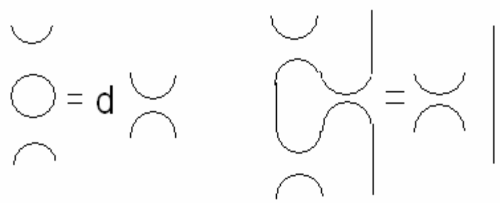
The generators e_i of the **TL algebra** are defined as follows

$e_i^2 = d\, e_i$,

$e_i\, e_{i+1}\, e_i = e_i$,

$e_i\, e_k = e_k\, e_i$ ($|k-i| \geq 2$).

e_i acts non-trivially on the *i*th and (*i+1*)th particles:



where $d=q+q^{-1}$ is theBeraha number (a weight of the Wilson loop)

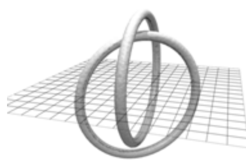
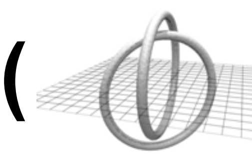
$$d = 2\cos[\pi/(k+2)]\,.$$

1. The values of the parameter d are nontrivial restriction, which leads to the finite-dimensional Hilbert spaces.

2. Besides, it turns out, that for the mentioned values of d the theory is unitary.

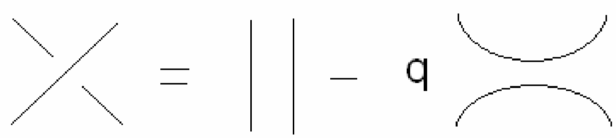
Meaning of the Beraha number d

The wave function **ψ(α)**, defined on the one-dimensional manifold **α**,
which is a joining up of the arbitrary tangle β and the Wilson loop, equals **dψ(β)**:

Ψ ( **+ a loop) = d Ψ (**  **)**

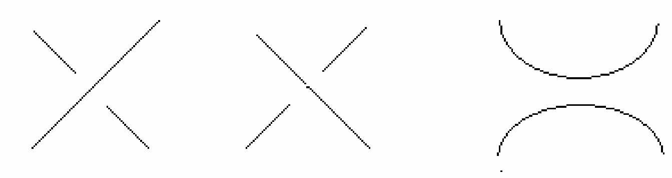
Braid group and TL algebra

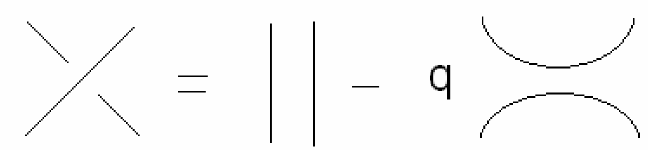
$B_i = I - qe_i, \qquad B_i^{-1} = I - q^{-1}e_i$



It obeys the braid group if $d = q + q^{-1} = 2\cos(\pi/(k+2))$.

But how does the Hamiltonian look like?





Temperley-Lieb algebra projectors

Due to $e_i^2 = d\, e_i$, $(e_i/d)^2 = e_i/d$.

Therefore, effective Hamiltonians of the Klein type (J. Phys. A 15, 661, 1982) could have a form of the sum of the Temperley-Lieb algebra projectors:

$$H = -\Sigma_i\, e_i/d$$

Irreps of e_i's

$$\mathbf{e}[i]|j_{i-1}j_ij_{i+1}\rangle \;=\; \sum_{j'_i} \left(\mathbf{e}[i]^{j_{i+1}}_{j_{i-1}}\right)^{j'_i}_{j_i} |j_{i-1}j'_ij_{i+1}\rangle$$

$$\left(\mathbf{e}[i]^{j_{i+1}}_{j_{i-1}}\right)^{j'_i}_{j_i} \;=\; \delta_{j_{i-1},j_{i+1}} \sqrt{\frac{S^0_{j_i}S^0_{j'_i}}{S^0_{j_{i-1}}S^0_{j_{i+1}}}}$$

$$S^{j'}_{j} := \sqrt{\tfrac{2}{(k+2)}}\;\sin[\pi\tfrac{(2j+1)(2j'+1)}{k+2}]$$

V. Jones, V. Pasquier, H. Wenzl; A. Kuniba, Y. Akutzu, M. Wadati; P. Fendley, 1984 - 2006

Mapping to the transverse field Ising model

In the case k=2 (when d=$(2)^{1/2}$), we have the transverse field Ising model:

$$H = -h\sum_j [\sigma_j^z \sigma_{j+1}^z + (g/h)\sigma_j^x]$$

Correlation functions:

A.R. Its, A.G. Izergin, V.E. Korepin, V. Ju. Novokshenov,
Nucl. Phys. B 340, 752 (1990).

J. Yu, S.-P. Kou, X.-G. Wen, quant-ph/07092276

Some steps of the proof

N.E. Bonesteel, K. Yang, '06

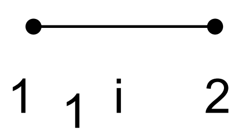
$$P_i = 1 - c^{+}_{i,i+1} c_{i,i+1}$$

$$n_{i,i+1} = c^{+}_{i,i+1} c_{i,i+1} = \tfrac{1}{2}(1+\sigma^3)_{i,i+1}$$

$$c_{i,i+1} = (\gamma_{1,i} - i\gamma_{2,i+1})/\sqrt{2},\ c^{+}_{i,i+1} = (\gamma_{1,i} + i\gamma_{2,i+1})/\sqrt{2}$$

$$P_i = 1 - c^{+}_{i,i+1} c_{i,i+1} = 1 - n_{i,i+1}$$

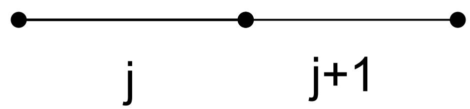
$$\{\gamma_k, \gamma_l\} = 2\delta_{kl} \qquad \gamma_i^{+} = \gamma_i$$



$$\gamma_1 = (c^{+} + c)/\sqrt{2}, \gamma_2 = (c^{+} - c)/i\sqrt{2} \qquad 2i\gamma_2\gamma_1 = 2n - 1 = \sigma^3$$

$$P = 1 - n = \frac{1}{2}(1 - \sigma^3) = \frac{1}{2} + i\gamma_1\gamma_2$$

$$H = -\sum_j H_j = -\sum_j i\gamma_{1,j}\gamma_{2,j} - \sum_j i\gamma_{2,j}\gamma_{1,j+1}$$



$$\gamma_{1,j} \;=\; \sigma_j^1 \prod_{k=1}^{j-1} \sigma_k^3 \qquad\qquad \gamma_{2,j} = \sigma_j^2 \prod_{k=1}^{j-1} \sigma_k^3$$

$$H = -J \sum_j (g\sigma_j^1 + \sigma_j^3 \sigma_{j+1}^3)$$

Jg = h

$$H \;=\; \sum_{\mathbf{k}} \epsilon_{\mathbf{k}} (\gamma_{\mathbf{k}}^{+} \gamma_{\mathbf{k}} - 1/2)$$

$$\epsilon_{\mathbf{k}} \;=\; \sqrt{k^2 + \Delta^2}$$

$$\Delta = 2J\,|1 - g\,|$$

Wen's model

$$H = g\sum_{i,j} F_{ij} + h\sum_{i,j}\sigma^2_{i,j} = g\sum_{i,j}\sigma^2_{i,j}\sigma^1_{i+1,j}\sigma^2_{i+1,j+1}\sigma^1_{i,j+1} + h\sum_{i,j}\sigma^2_{i,j}$$

$$\sigma^1_{i,j} + i\sigma^2_{i,j} = 2\left[\prod_{j'<j}\prod_{i'}\sigma^3_{i',j'}\right]\left[\prod_{i'<i}\sigma^3_{i',j}\right]c^+_{i,j}$$

$$\sigma^3_{i,j} = 2c^+_{i,j}c_{i,j} - 1$$

$$A_{i,j} = c^+_{i,j} + c_{i,j},\ B_{i,j} = i(c^+_{i,j} - c_{i,j})$$

$$d_{i,j} = (A_{i,j} + iB_{i,j+1})/2$$

$$d^+_{i,j} = (A_{i,j} - iB_{i,j+1})/2$$

$$iA_{i,j}B_{i,j+1} = 2d^+_{i,j}d_{i,j} - 1 = \mu^3_{i,j}$$

$$h = 0$$

$$H = g\sum_{i,j}\mu^3_{i,j}\mu^3_{i+1,j}$$

$$A_i = \sigma^1_i\sigma^2_{i+e_1}\sigma^1_{i+e_1+e_2}\sigma^2_{i+e_2} = \tau^1_{i+1/2}$$

$$B_i = \sigma^1_i = \tau^3_{i-1/2}\tau^3_{i+1/2}$$

$$H = -\sum_a\sum_i(g\tau^1_{a,i+1/2} + h\tau^3_{a,i-1/2}\tau^3_{a,i+1/2})$$

Universal form of effective Hamiltonians

$$\tau^1_{a,j+1/2} = 2c^+_{a,j}c_{a,j} - 1,\ \tau^3_{a,j+1/2} = (-1)^{j-1}\exp\left(\pm i\pi \sum_{n=1}^{j-1} c^+_{a,n}c_{a,n}\right)(c^+_{a,j} + c_{a,j})$$

$$H = h\sum_j \left[(c_j - c_j^+)(c_{j+1} + c^+_{j+1}) + (g/h)(c_j^+ - c_j)(c_j^+ + c_j)\right]$$

$$H = \sum_{\alpha,\mathbf{k}} \overline{\Psi}_{\mathbf{k}} h_\alpha(\mathbf{k})\sigma^\alpha\, \Psi_{\mathbf{k}}, \qquad \sigma_\alpha = (\mathbb{I}, \boldsymbol{\sigma})$$

$$E_{\mathbf{k}} = \sqrt{h_3^2(\mathbf{k}) + |\Delta(\mathbf{k})|^2},\ \Delta(\mathbf{k}) \equiv h_1(\mathbf{k}) + ih_2(\mathbf{k})$$

In the case of the Kitaev model

$$H = -J_x \sum_{x-links} \sigma_i^x \sigma_j^x - J_y \sum_{y-links} \sigma_i^y \sigma_j^y - J_z \sum_{z-links} \sigma_i^z \sigma_j^z$$

$$h_1(\mathbf{k}) = -J_y \sin\alpha(\mathbf{k}) + J_x \sin\beta(\mathbf{k})\,,$$

$$h_2(\mathbf{k}) = J_3 + J_y \cos\alpha(\mathbf{k}) + J_x \cos\beta(\mathbf{k})\,,$$

$$h_3(\mathbf{k}) = 2J^{'} \sin(\sqrt{3}k_x)\,,$$

$$\alpha(\mathbf{k}) = (\sqrt{3}k_x - 3k_y)/2, \quad \beta(\mathbf{k}) = (\sqrt{3}k_x + 3k_y)/2$$

$$|\mathbf{h}(\mathbf{k})| = 0 \qquad\qquad |J_x - J_y| < J_z < J_x + J_y$$

$$(k_x, k_y) \rightarrow (h_1, h_2, h_3)$$

$$\widetilde{P}(\mathbf{q}) = \frac{1}{2}\left(1 + m_x(\mathbf{q})\sigma^x + m_y(\mathbf{q})\sigma^y + m_z(\mathbf{q})\sigma^z\right)$$

h≡m

$$\frac{1}{2\pi i}\int \mathrm{Tr}\left(\widetilde{P}\, d\widetilde{P} \wedge d\widetilde{P}\right) = \frac{1}{2\pi i}\int \mathrm{Tr}\left(\widetilde{P}\left(\frac{\partial \widetilde{P}}{\partial q_x}\frac{\partial \widetilde{P}}{\partial q_y} - \frac{\partial \widetilde{P}}{\partial q_y}\frac{\partial \widetilde{P}}{\partial q_x}\right)\right) dq_x\, dq_y$$

$$\mathcal{P} = \frac{1}{8\pi}\int d^2k\ \epsilon^{\mu\nu}\hat{\mathbf{h}} \cdot (\partial_{k_\mu}\hat{\mathbf{h}} \times \partial_{k_\nu}\hat{\mathbf{h}})$$

ĥ = **h**/h

1. **Effective Hamiltonians in systems with topologically ordered states in the case k=2 have a form of the Bloch matrix**

$$H = \begin{pmatrix} h_3(\mathbf{k}) & \Delta(\mathbf{k}) \\ \Delta^*(\mathbf{k}) & -h_3(\mathbf{k}) \end{pmatrix} \qquad \Delta(\mathbf{k}) \equiv h_1(\mathbf{k}) + i h_2(\mathbf{k})$$

2. **Z_2 invariants are significant for the classification of the classes of universality in 2D systems. In particular,**

$$\mathcal{P} = \frac{1}{8\pi} \int d^2k \, \epsilon^{\mu\nu} \hat{\mathbf{h}} \cdot (\partial_{k_\mu} \hat{\mathbf{h}} \times \partial_{k_\nu} \hat{\mathbf{h}}) \in$$ **Z_2**, i.e. equals **0** or **1**

3. In the case k=3, irreps of the e_i's lead to the Hamiltonian

of the Fibonnacci anyons, where φ=(1+ $\sqrt{5}$)/2 is the golden ratio. This is the k=3 RSOS model which is a lattice version of the tricritical Ising model at its critical point (A. Feiguin, S. Trebst, A.W.W. Ludwig, M. Troyer, A. Kitaev, Z. Wang, M. Freedman, PRL, 2007.)

$$H = \sum_i \left[(n_{i-1} + n_{i-1} - 1) - n_{i-1} n_{i+1} (\varphi^{-3/2} \sigma_i^x + \varphi^{-3} n_i + 1 + \varphi^{-2}) \right]$$

$$(\mathbf{H}_i)_{x_i}^{x'_i} := -(F^{x_{i+1}}_{x_{i-1}\tau\tau})^1_{x_i} (F^{x_{i+1}}_{x_{i-1}\tau\tau})^1_{x'_i}$$

$$\mathbf{F}^{\tau}_{\tau\tau\tau} = \begin{pmatrix} \varphi^{-1} & \varphi^{-1/2} \\ \varphi^{-1/2} & -\varphi^{-1} \end{pmatrix}, \quad \mathbf{H}_i = -\begin{pmatrix} \varphi^{-2} & \varphi^{-3/2} \\ \varphi^{-3/2} & \varphi^{-1} \end{pmatrix}$$

4. What about larger values of the linking number ?
For example, k=4.