qKZ equations and critical percolation

Low-dimensional quantum field theory and applications

GGI

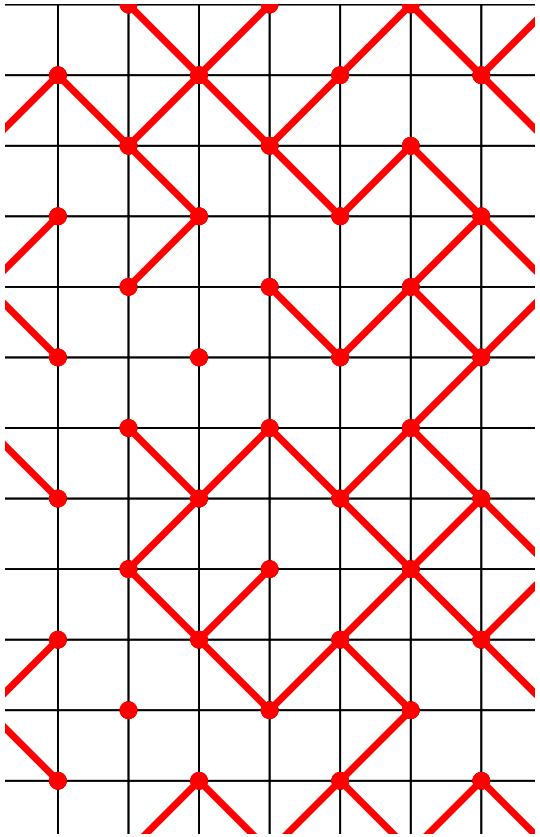
October 27, 2008

Bernard Nienhuis

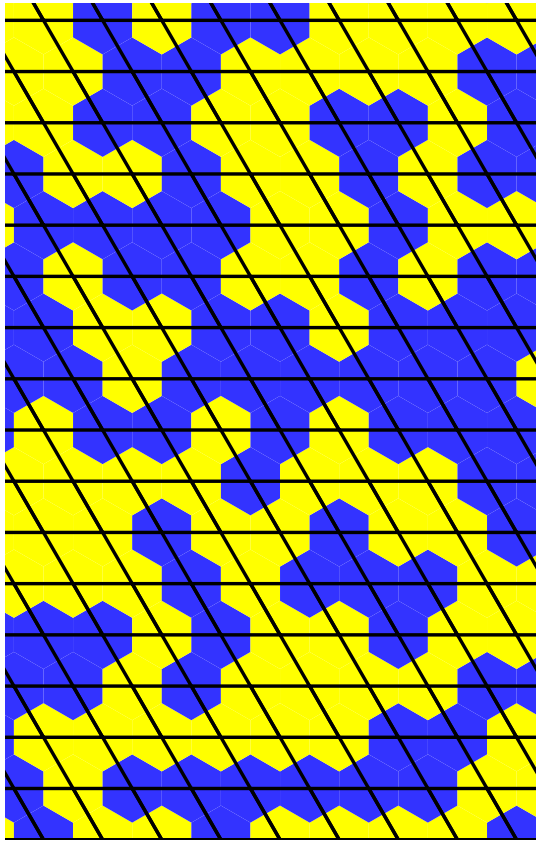
in collaboration with

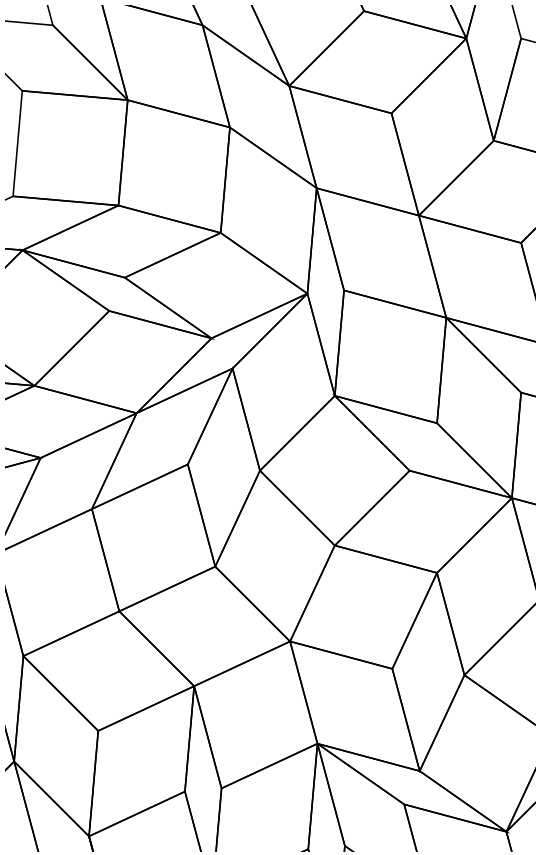
Jesper Jacobsen, Philippe di Francesco

Percolation



rhombus tilings





What is the occasion?

- For (square) bond percolation many correlation functions have been found exactly from the eigenvector of the transfer matrix. Not just in the scaling limit

- Similar approach for (triangular) site percolation has been largely unsuccessful.

- Yet, there is an example of a correlation function where the reverse is true.

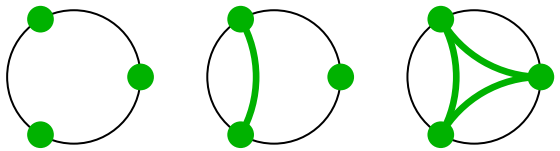
- Generalizing the lattice opens new avenues,
(also to answer questions for regular lattices.)

Percolation on the half cylinder
the transfer matrix

Consider transfer matrix which keeps 'in mind' which of the boundary sites are connected with each other:
the index of the matrix labels the various ways the boundary sites can be mutually connected.

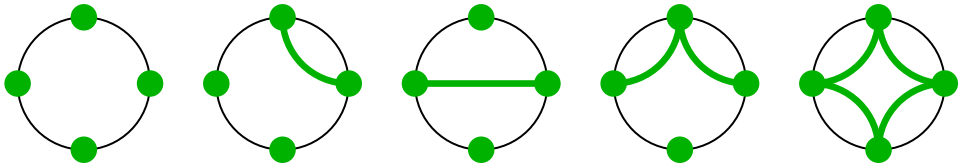
Examples of the eigenvector for bond percolation:

$L = 3$



2 1 2

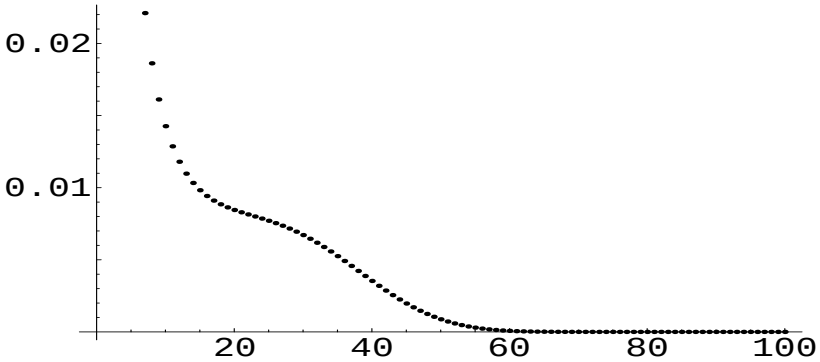
$L = 4$



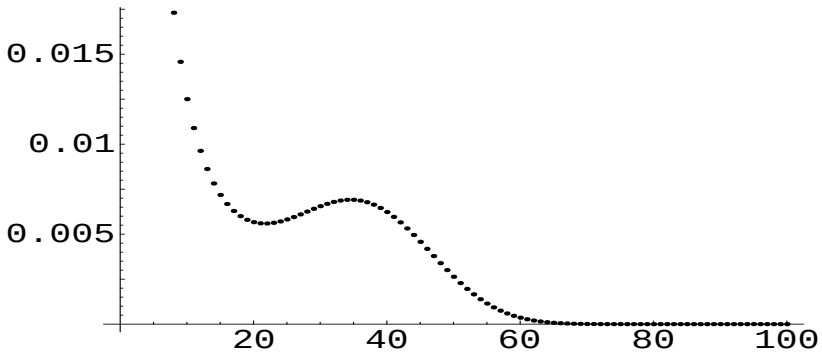
7 3 1 3 7

From eigenvectors like this that the following correlation functions were inferred.

An example of an explicit correlation function



All clusters PBC



All clusters anti − PBC

Probability that a site on the rim of a half cylinder is in a cluster with n sites on the rim.

$$P(L,n) = \frac{3n\left[L^2 + 2n(3n^2-1)\right](2n-3)!!(L-n-1)!(L+2n-2)!!}{(2n+2)!!(L+n)!(L-2n)!!}$$

Another example

Probability of being surrounded by $n/2$ nested clusters (n nested cluster boundaries) is generated by:

$$F(L, 2\cos 2\theta) \;=\; C \det_{0\le j,k\le L} \left[\binom{j+k}{j} \mathrm{e}^{\mathrm{i}\theta} \;+\; \delta_{j,k} \mathrm{e}^{-\mathrm{i}\theta} \right]$$

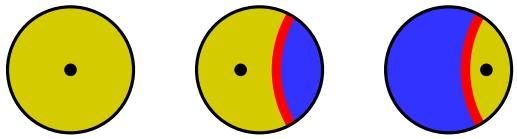
These are just examples of many correlation functions that have been conjectured, and are in agreement with all data with $L < 30$.

Site percolation on the half cylinder
the transfer matrix

Here sites come in two colors, adjacent sites of the same color are connected. Non-adjacent sites may or may not be connected.

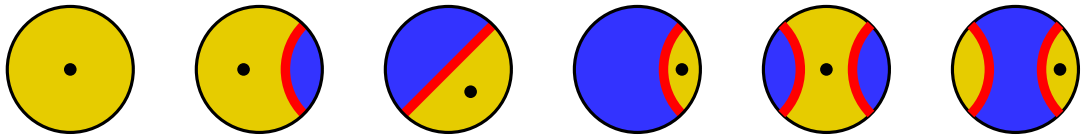
In the following example we choose to 'remember' in which cluster the infinite end of the cylinder is embedded.

$L = 3$



9 5 4

$L = 4$



32 19 16 13 10 3

Fusing two half cylinders together we can find e.g.:

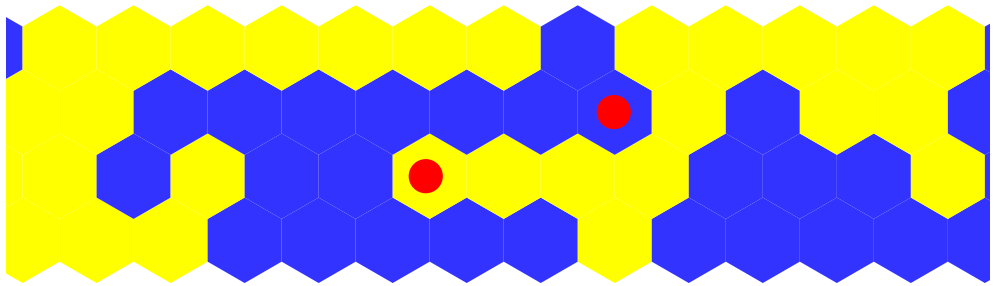
Correlations for site percolation

While eigenvector likewise can be written with integer elements, but the numbers grow faster.
It turns out to be much more difficult to guess the exact form of correlations.

One notable exception is the number of signed crossing domain walls between two points on an infinite strip or cylinder. This is as simple as

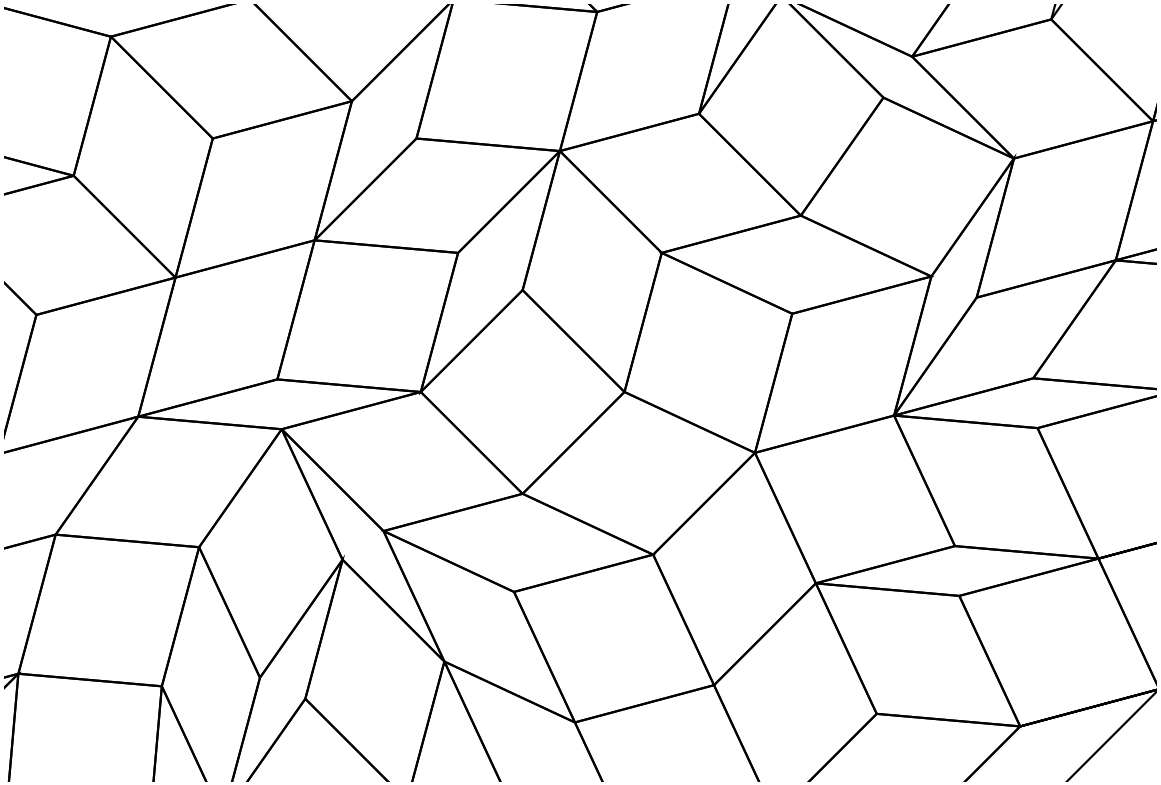
$$f(z_1, z_2) = \frac{\mathrm{Re}(z_2 - z_1)}{(L+1)} \quad \text{and} \quad \frac{\mathrm{Re}(z_2 - z_1)}{3L}$$

for the strip and cylinder respectively. $\mathrm{Re}(z)$ distance projected on axis measured in faces, L is number of rows.



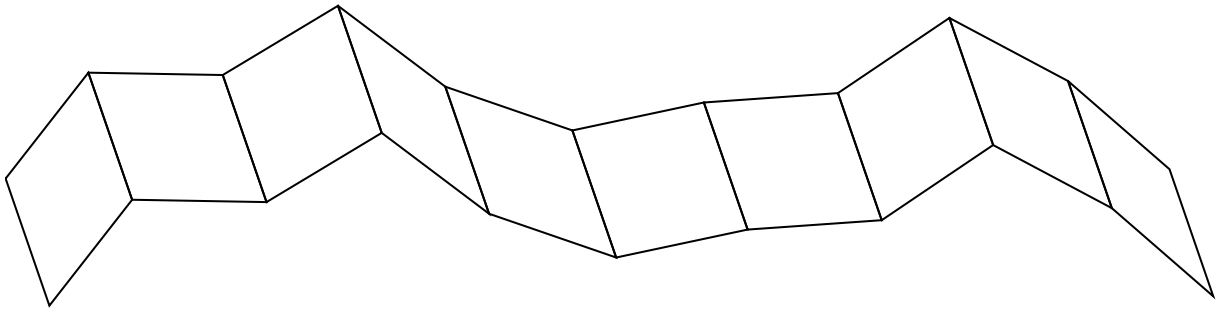
As yet there is no example of a correlation function with scaling behavior, known exactly for both bond and site percolation.

Generalizing the regular lattice to an arbitrary rhombus tiling seems a feasible approach.



The Boltzmann weights for each rhombus depends on its shape. These are chosen such they satisfy the Yang Baxter equations (YBE).

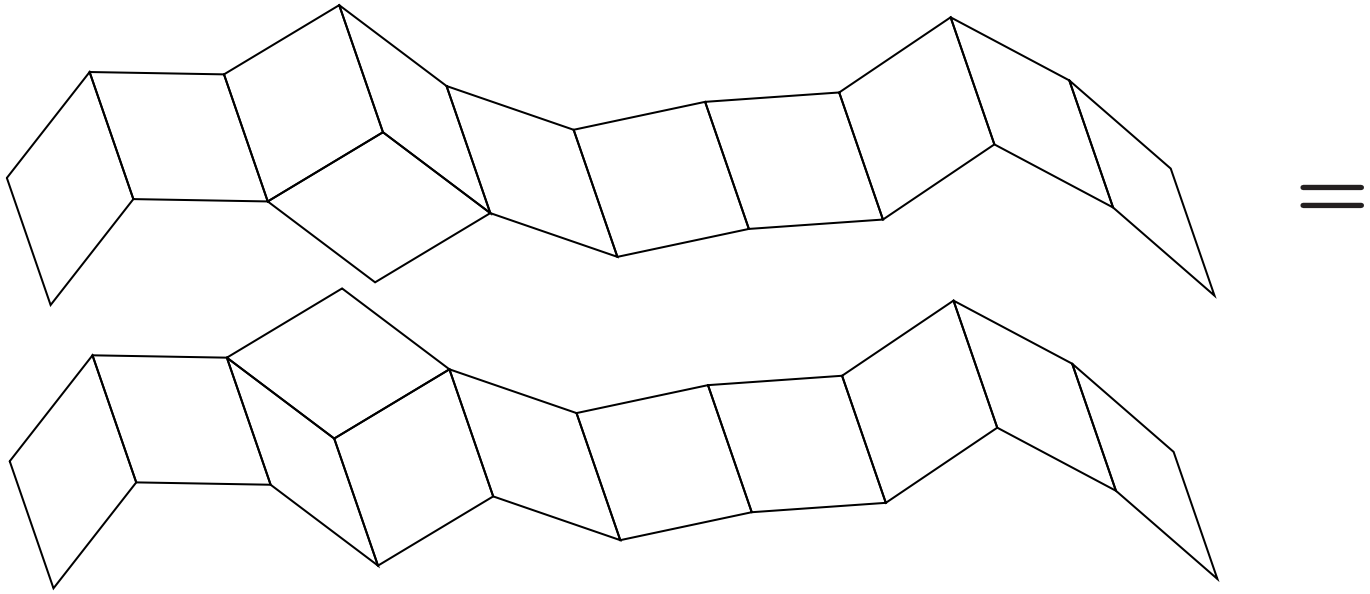
Also on such arbitrary lattice one can define a transfer matrix:



The YBE assures we have a family of commuting transfer matrices parametrized by the angle of the 'verticals'.

Because these transfer matrices commute, the eigenvector depends only on the relative angles of the quasi horizontals, but not on the verticals.

In such a rhombus tilings it is natural to apply the Yang Baxter equations also in another way:



(In the above pictures, summation over the internal lines is implied.)

$$R_i(u_i, u_{i+1}) \cdot T(v, \cdots, u_i, u_{i+1}, \cdots) \;=$$

$$T(v, \cdots, u_{i+1}, u_i, \cdots) \cdot R_i(u_i, u_{i+1})$$

The operator $R_i(u_i, u_{i+1})$ is represented by the lying rhombus.

Therefore the matrices $T(v, \cdots, u_i, u_{i+1}, \cdots)$ and $T(v, \cdots, u_{i+1}, u_i, \cdots)$ have the same eigenvalues, while the eigenvectors can be transformed into each other by $R_i(u_i, u_{i+1})$.

For the ground state eigenvector this implies:

$$R_i(u_i, u_{i+1}) \cdot \Psi(\cdots, u_i, u_{i+1}, \cdots) \;=\; W(u_i, u_{i+1})\; \Psi(\cdots, u_{i+1}, u_i, \cdots)$$

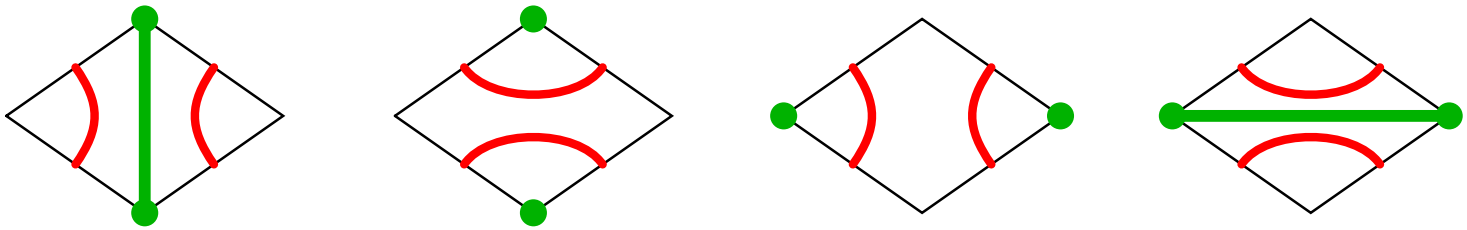
for some scalar W, presumed to depend on u_i and u_{i+1} only.
This is the main qKZ equation.

Ψ, being the ground state eigenvector, is the probability distribution of the configuration on the rim of a half infinite cylinder, and is therefore capable of giving all correlation functions on this line.

To apply this to the percolation models we need the Boltzmann weights (and R operators) as a function of the angles u.

A special property of percolation is that not only R but also W can be written as a polynomial in $z = \exp(\mathrm{i}\,\alpha\,u)$ for some α.

For bond percolation these weights $R(z, y)$ are relatively well known:



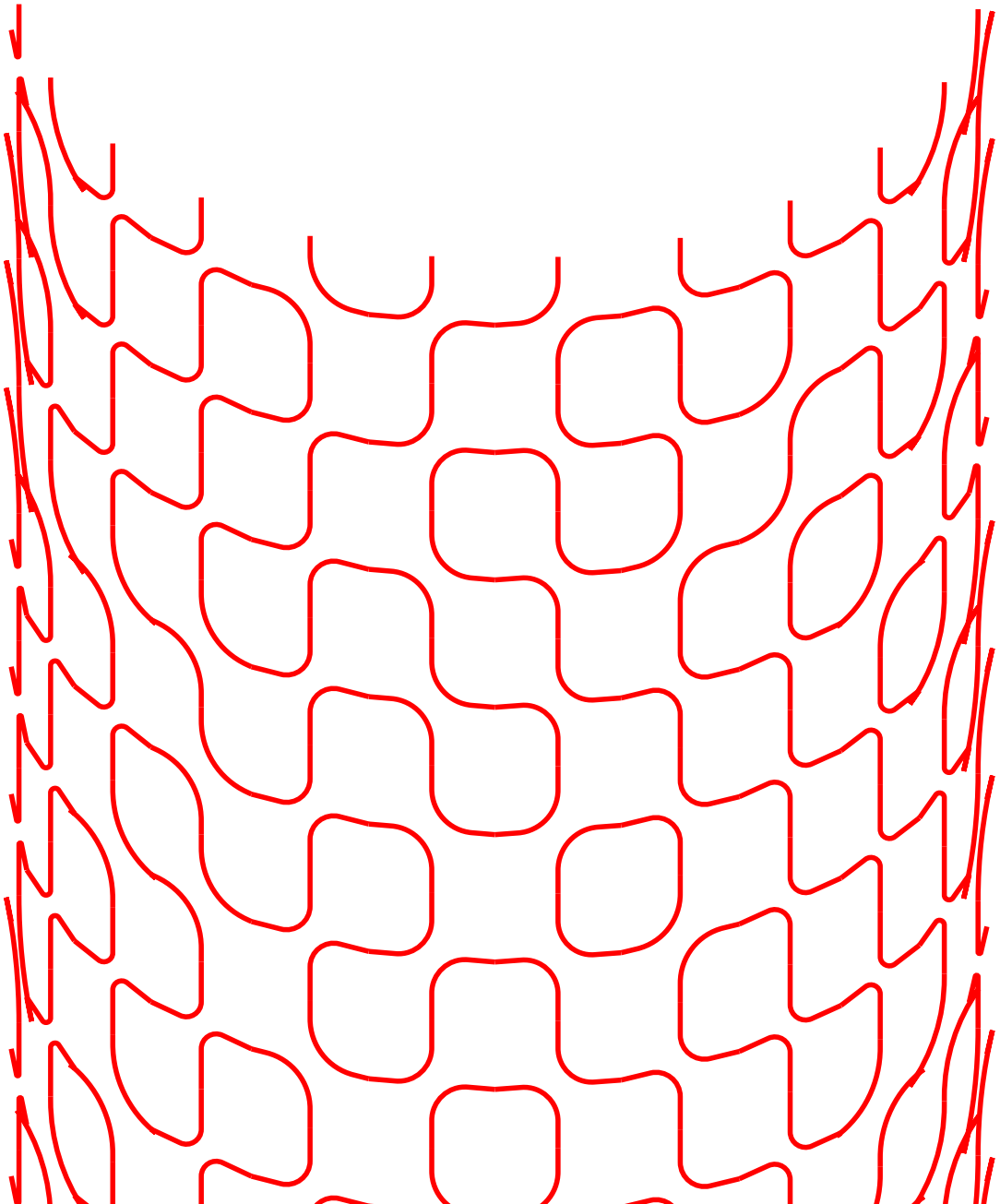
$(\omega^2 y + \omega z)$ $(y - z)$ $(\omega^2 y + \omega z)$ $(y - z)$

Here $\omega = \mathrm{e}^{\mathrm{i}\pi/3}$. The left angle of the rhombus is $\frac{3}{2}\arg(y/z)$. (i.e. $\alpha = 2/3$)
The normalization is $W(y, z) = (\omega^2 z + \omega y)$.

When $y = \omega z$ the square symmetry is restored.

The red lines separate the clusters from their dual counterparts, and can fully represent the cluster configuration.

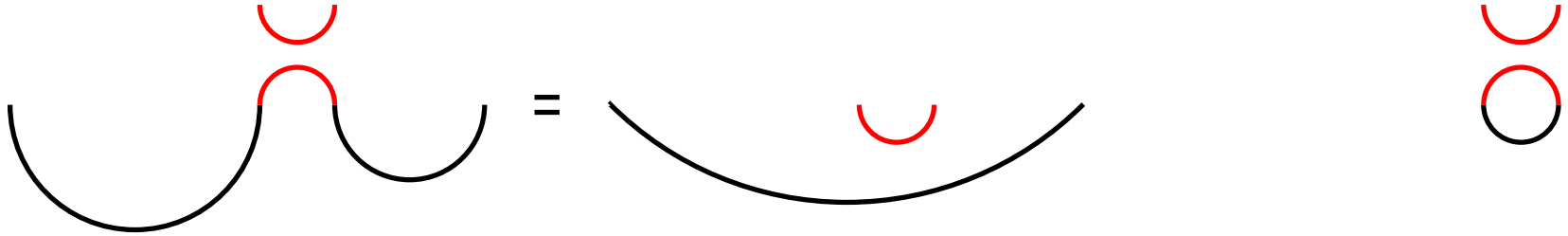
These red lines are closed, or terminate on the boundary.



Now consider the qKZ equation.

$$R_i(z_i, z_{i+1}) \cdot \Psi(\cdots, z_i, z_{i+1}, \cdots) \;=\; W(z_i, z_{i+1})\, \Psi(\cdots, z_{i+1}, z_i, \cdots)$$

In bond percolation, the operator R is a linear combination of the identity (| |) and a 'reconnection'.



Suppose we consider in the equation an element of Ψ in which i and $i+1$ are not mutually connected.

The only contribution to this in the left hand side is from the same element of Ψ and the 'identity term' of R.

The equation then reads:

$$(\omega^2 y + \omega z)\, \Psi^\gamma(\cdots, y, z, \cdots) \;=\; (\omega y + \omega^2 z)\, \Psi^\gamma(\cdots, z, y, \cdots)$$

for each configuration γ which does not connect the points with labels y and z.

Since the left- and right hand side differ only by interchange of y and z, they must be symmetric functions of these variables.

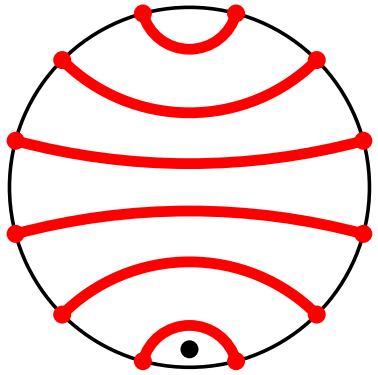
Supposing $\Psi^\gamma(\cdots, y, z, \cdots)$ be a polynomial, it must contain the factor $(\omega y + \omega^2 z)$, and be symmetric otherwise.

By induction this leads to the conclusion that a sequence of consecutive domain walls not connected to each other has the factor

$$\prod_{i=n}^{m} \quad \prod_{j=i+1}^{m} (\omega z_j + \omega^2 z_i)$$

and is otherwise symmetric.

The same reasoning applies to neighboring edges which are connected with a line winding the cylinder. (supposing this is information is retained in the indices of the transfer matrix.



As a result the most nested element:

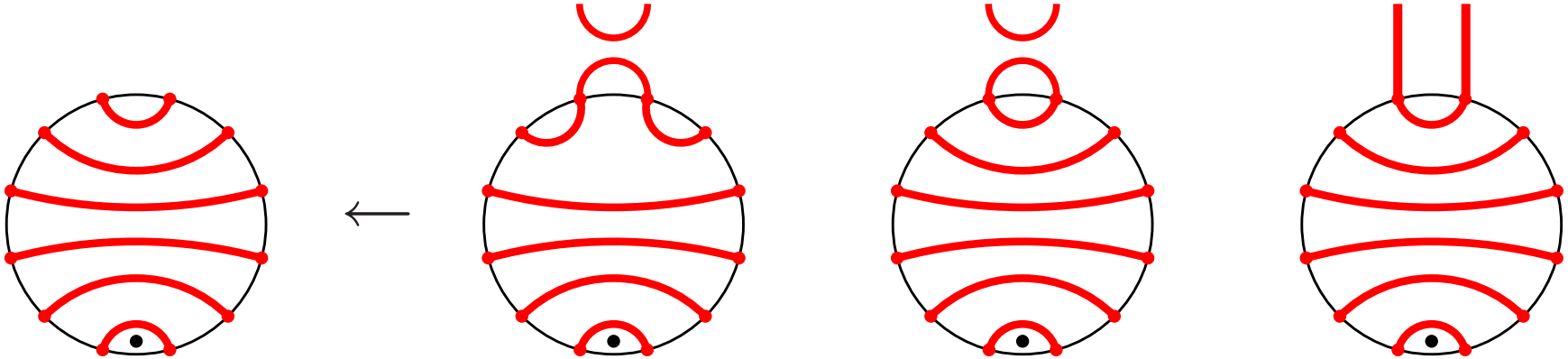
has the value:

$$\prod_{i=1}^{L} \prod_{j=i+1}^{L} (\omega z_j + \omega^2 z_i)$$

up to a factor symmetric in all variables.

The other elements can be constructed from it, by considering in the qKZ equations two adjacent positions that are connected.

Now consider an element in which the action of R results in two adjacent lines being connected.



This results in an equation involving the most nested element three times (once with arguments interchanged) and one other element, which can thus be computed.

In this way, the whole groundstate vector can be constructed recursively.

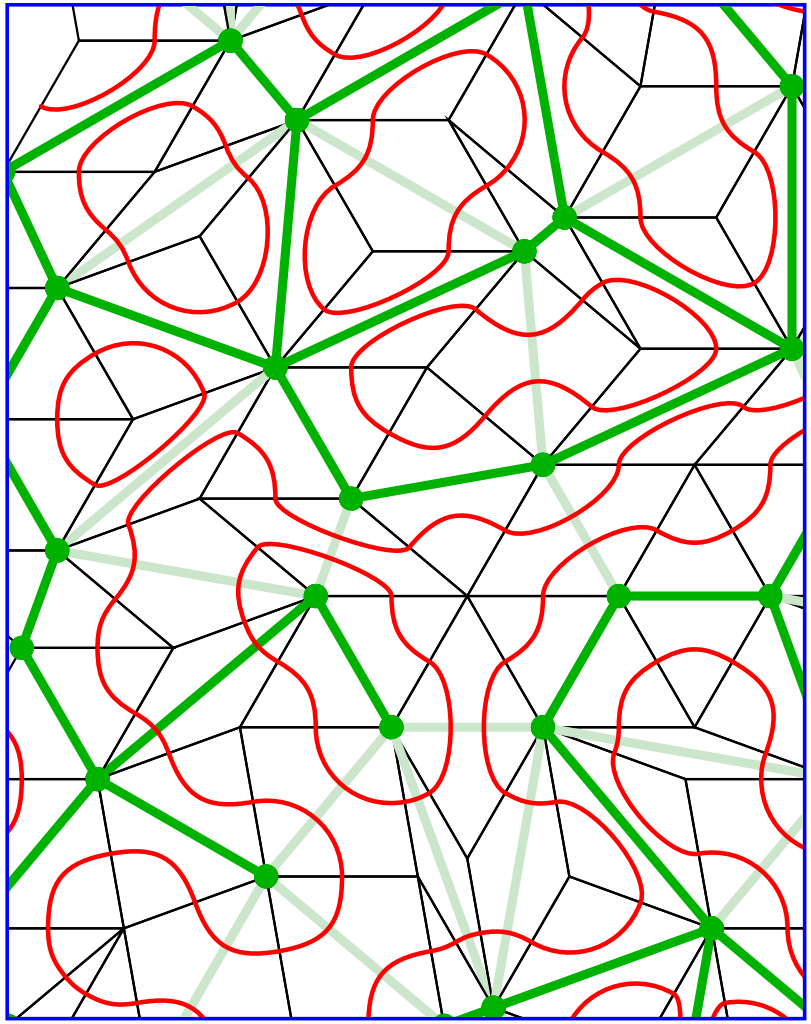
It turns out that the overall symmetric factor may be 1 for the vector elements to be polynomials.

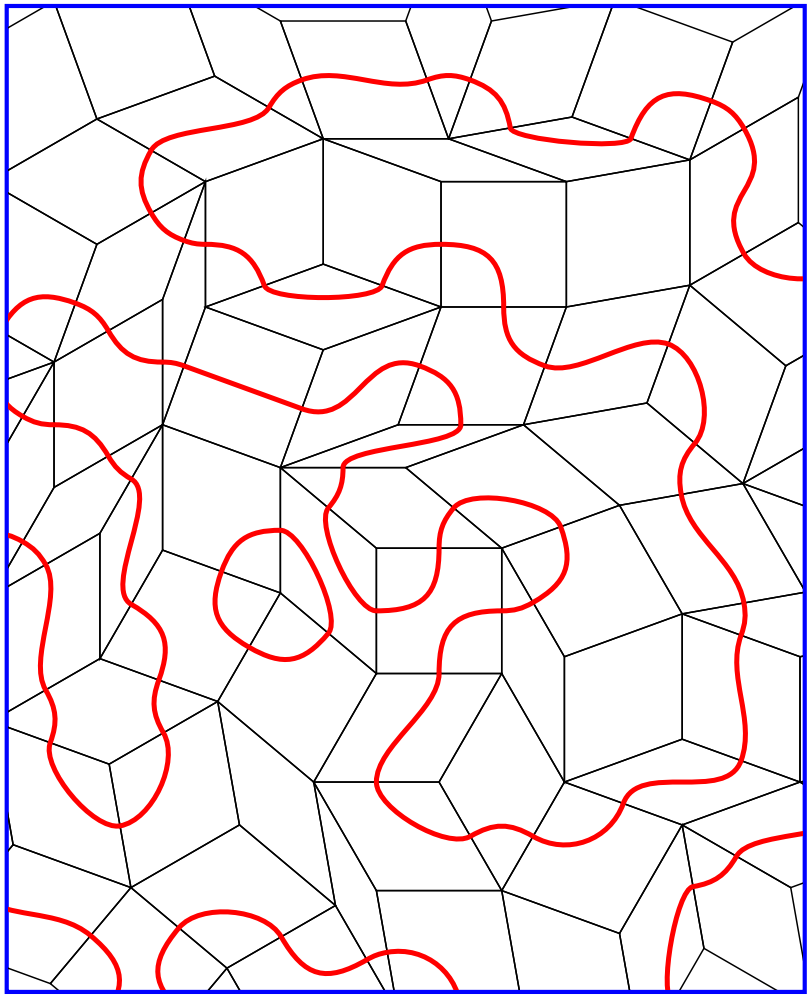
From the ground state we can calculate any correlation function.

Heuristically we find extrapolations of these to arbitrary size.

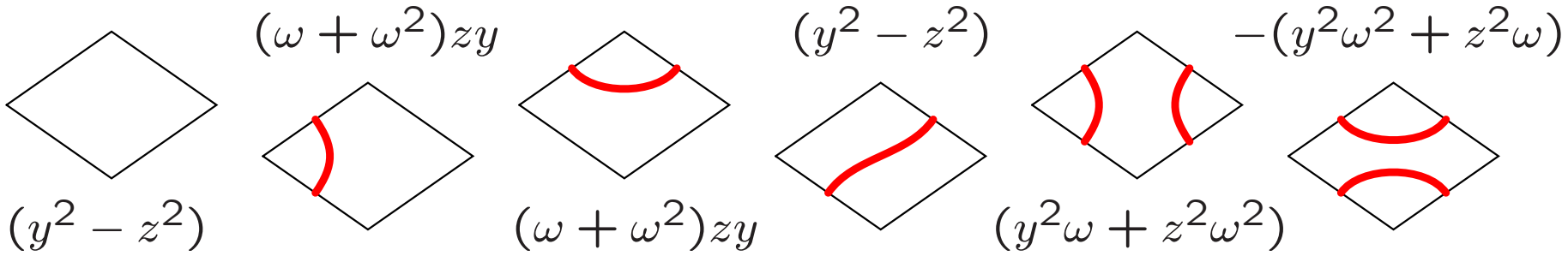
Can this be done for site percolation as well?

the two percolation models:





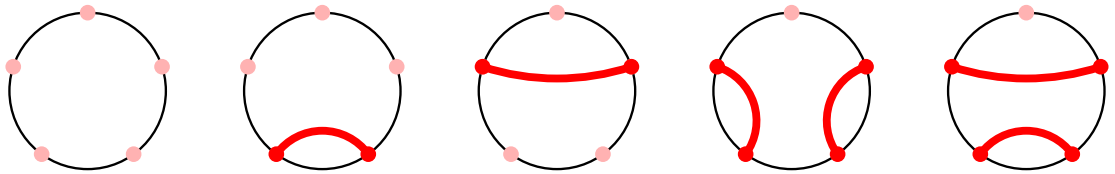
For site percolation the following weights $R(z, y)$ satisfy the YBE.



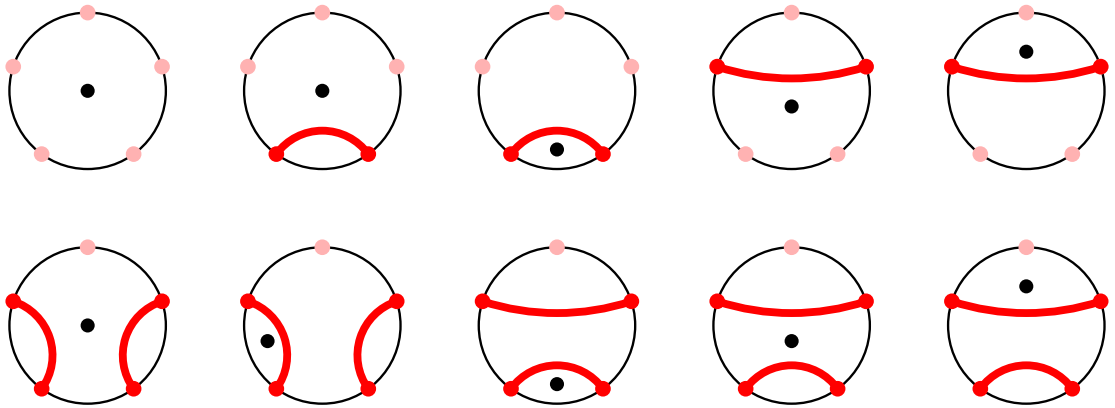
The left angle of the rhombus is $\arg(y/z)$, and again $\omega = e^{i\pi/3}$
The normalization is $W(y, z)(z\omega + y)(y + z\omega^2)$

When $y = \omega z$, it reduces to triangular site percolation: the rhombus factors into two equilateral triangles

For site percolation, some edges carry a domain wall, but others do not. Thus the configuration space is a perfect matching of all possible subsets of external edges.
E.g. for $L = 5$ we have
(the sites of the lattice are in between the dots.)



We may or may not distinguish on which side of the cylinder links are connected.

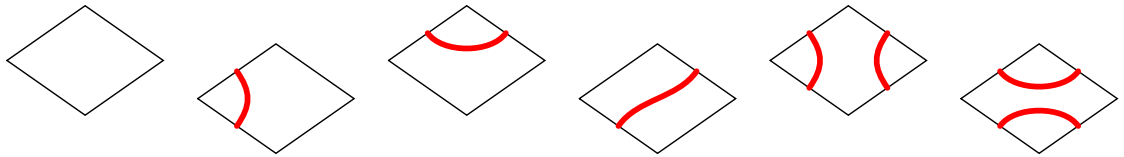


The black dot represents the infinite end of the cylinder.

Now consider the qKZ equation.

$$R_i(z_i, z_{i+1}) \cdot \Psi(\cdots, z_i, z_{i+1}, \cdots) \;=\; W(z_i, z_{i+1})\, \Psi(\cdots, z_{i+1}, z_i, \cdots)$$

In site percolation, the operator R is a linear combination of diagonal terms and a 'reconnection' as before, but now also annihilation and creation of a pair of domain walls, as well as a hop.



Like before, an element of Ψ in which i and $i+1$ are not mutually connected, can only result from the corresponding diagonal element.

The only contribution to this in the left hand side is from the same element of Ψ and the diagonal element of R.

The equation now reads:

$$(\omega y^2 + z^2\omega^2)\ \Psi^\gamma(\cdots, y, z, \cdots) \ = \ (\omega z + y)(\omega^2 z + y)\ \Psi^\gamma(\cdots, z, y, \cdots)$$

where the configuration γ does not connect the points with labels y and z.

After the common factor $(\omega^2 z + y)$ is divided out, the left- and right hand side differ only by interchange of y and z.

Supposing $\Psi^\gamma(\cdots, y, z, \cdots)$ be a polynomial, it must contain the factor $(\omega z + y)$, and be symmetric otherwise.

By induction this leads to the conclusion that a sequence of consecutive domain walls not connected to each other has the factor

$$\prod_{i=n}^{m} \prod_{j=i+1}^{m} (\omega z_j + z_i)$$

and is otherwise symmetric.

This leads the conclusion that the most nested element is

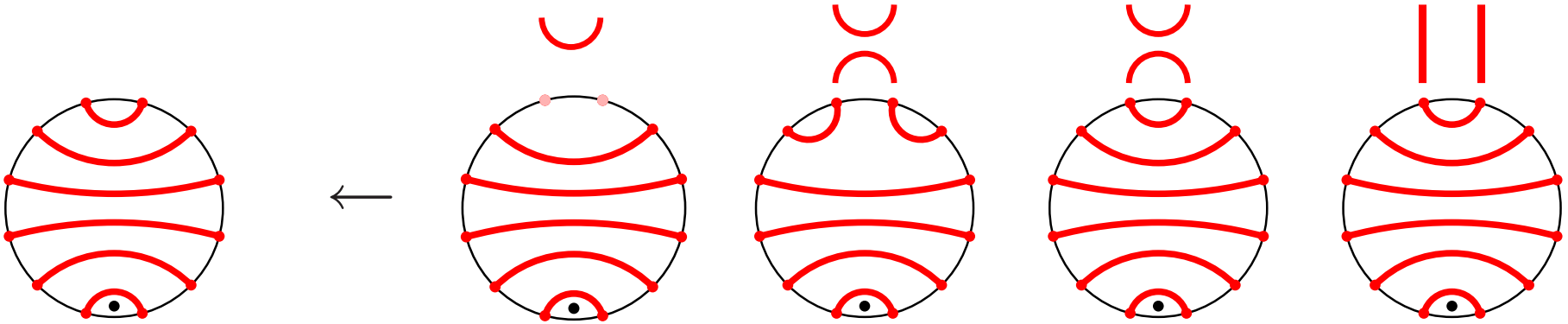
$$\prod_{i=1}^{n} \prod_{j=i+1}^{n} (z_j + \omega z_i)$$

Again, up to an overall symmetric factor

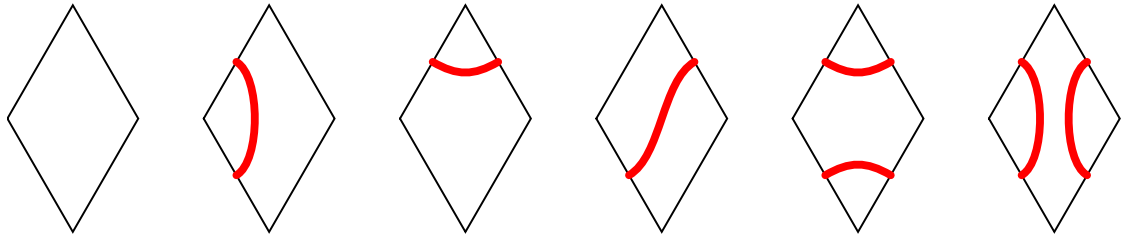
Can we take this proposal to calculate the other elements as with bond percolation?

Unfortunately not.

The first step in the 'qKZ recursion' has **two** new elements:



What saves the scheme is the fact that at the point $y = z\omega^2$ the R operator has a special form:



At this special point the matrix elements are all equal to $-(1+\omega)z^2$, except the last one, which is zero.

This implies that the rhombus factorizes into two triangles:
In other words $R = a \cdot b$ in which the operator b joins the two edges coming in, and a splits them again.

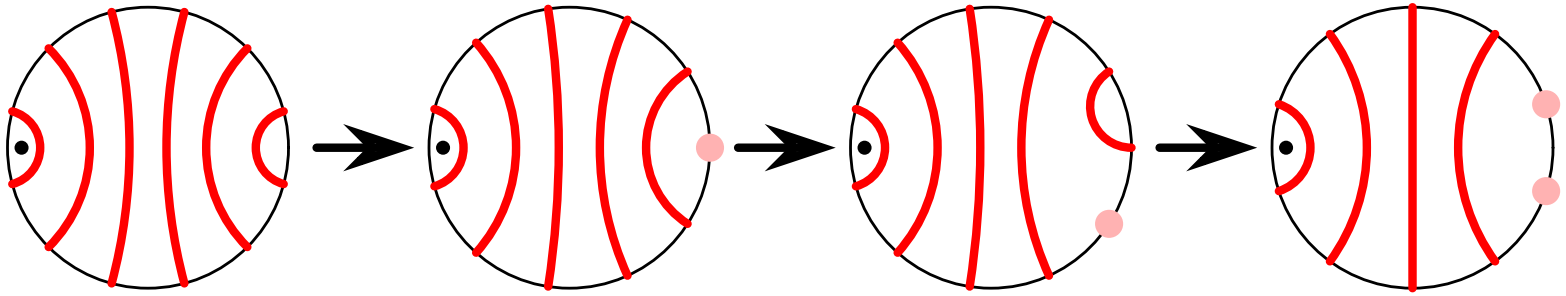
By letting the operator b act on the ground state vector we have the equation:

$$\Psi^{\gamma}_{L+1}(\cdots, z/\omega, z\omega, \cdots) = f_{\gamma}(z)\; F(z, \cdots)\; \Psi^{\tilde{\gamma}}_{L}(\cdots, z, \cdots)$$

The configuration $\tilde{\gamma}$ is obtained from the configuration c by joining the edges with z/ω and $z\omega$ into a single edge.

Choosing the values z/ω and $z\omega$ for two consecutive edges, fuses them into one. If the two original edges carry no domain wall, neither will the fused one. A single domain wall will be carried over to the fused edge. A single domain wall joining the two edges is replaced by an empty fused edge.

The function F does not depend on the configuration and is symmetric in all its arguments except the first.
It follows directly from the proposal for the most nested state.

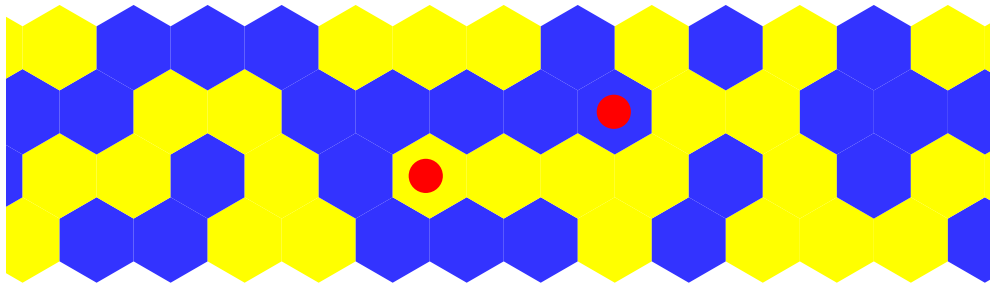


These states can be constructed from the leftmost by application of size recursions and R respectively.

Thus all states can be found constructively from the single proposal above.
Since they result in a polynomial solution all prerequisites are satisfied.

The eigenvector being formed we can use it to calculate various correlation function. They contain so much information for small L that extrapolation to arbitrary L is possible.
The result is then checked numerically for a sequence of larger systems.

We calculated for both bond and site percolation, the number of signed crossing domain walls between two points on an infinite cylinder $f(z_1, z_2)$.



This function is additive:

$$f(z_1, z_3) = f(z_1, z_2) + f(z_2, z_3)$$

so that it needs to be calculated through single edges only.

Two kinds of edges, depending on the closing or not closing of the De Bruijn lines.

Site percolation

Through a generic edge (i.e. its De Bruijn line does not close):

$$\frac{\mathrm{i}\sqrt{3}}{6}\left(\frac{1}{\sum_j z_1/z_j}-\frac{1}{\sum_j z_j/z_1}\right)$$

This is precisely the simple geometric result that we have seen in the regular lattice.

Through a 'vertical' edge:

$$\frac{\sum_{j=1}^{L}\frac{\mathrm{i}}{\sqrt{3}}\left(z_j-\frac{1}{z_j}\right)}{2(\sum_{i=1}^{L}z_i)(\sum_{i=1}^{L}\frac{1}{z_i})}\times\frac{\left(\prod_{i=1}^{n}\frac{\mathrm{i}}{\sqrt{3}}\left(z_j-\frac{1}{z_j}\right)+1\right)-\left(\prod_{i=1}^{n}\frac{\mathrm{i}}{\sqrt{3}}\left(z_j-\frac{1}{z_j}\right)-1\right)}{\left(\prod_{i=1}^{n}\frac{\mathrm{i}}{\sqrt{3}}\left(z_j-\frac{1}{z_j}\right)+1\right)+\left(\prod_{i=1}^{n}\frac{\mathrm{i}}{\sqrt{3}}\left(z_j-\frac{1}{z_j}\right)-1\right)}$$

Considerably more complicated obviously.

For bond percolation:

$$z_1 \frac{d}{dz_1} \log \left(z_1 \frac{S(p\,; z_1, z_2, ..., z_L)^2}{S(q\,; z_1, z_2, ..., z_L)^2} \right)$$

Where S is the Schur function of the partitions:

$$p = \{\cdots, 2, 2, 1, 1, 0\} \quad \text{and} \quad q = \{\cdots, 2, 2, 1, 1, 0, 0\}$$

$$S(p\,; z_1, z_2, \cdots, z_L) = \frac{\det z_i^{p_j + L - j}}{\det z_i^{L-j}}$$

For 'vertical' edges:

$$v \frac{d}{dv} \log \left(v \frac{S(p\,; v, y\,\omega^2, z_1, z_2, ..., z_L)^2}{S(q\,; v, y\,\omega^2, z_1, z_2, ..., z_L)^2} \right)$$

where $y \to v$ after differentiation.

Summary

- Solvable models on arbitrary rhombus tilings are quite tractable.
- It is possible to guess general size correlation functions from finite lattice results.
- We can now analyse the difference between site and bond percolation in their corrections to scaling.
- Hopefully more correlation functions will follow.