Critical interfaces in random media: random bond Potts model and logarithmic CFTs

Raoul Santachiara

LPTMS (Orsay)

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In collaboration:
Jesper Jacobsen, Pierre Le Doussal, Kay Wiese: LPTENS,Paris
Marco Picco: LPTHE,Paris

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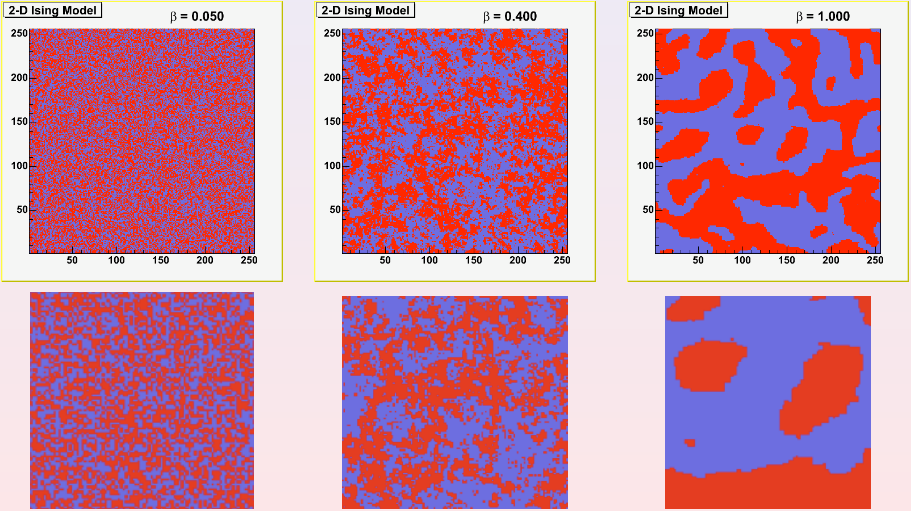
Outline

1. Pure critical Ising and 3−states Potts model: geometrical exponents
2. Random bond Potts Model: perturbed CFT approach
3. Geometric exponents in the random Potts model: perturbative CFT computation and logarithmic correlation functions
4. Numerical studies:Montecarlo and Transef Matrix methods
5. Conclusions

ISING MODEL: $H = -J \sum_{<ij>} \sigma_i \sigma_j$



Critical point $\Rightarrow$ Local Scale Invariance $\Rightarrow$ CFT

ISING MODEL, Local observables:

Minimal M_3, Unitary grid

$$\Delta_{n,m} = \frac{-1 + 16m^2 - 24mn + 9n^2}{48}$$

$1 \leq n \leq 3 \quad 1 \leq m \leq 2$

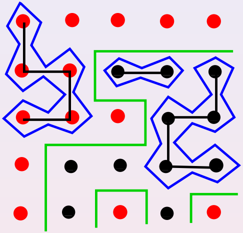
1/2 ε	0 Id
1/16 σ	1/16 σ
0 Id	1/2 ε

Energy and Spin-Spin correlation functions:

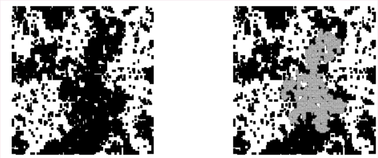
$$c = 1/2 \qquad \{\phi\} = \{I, \sigma, \varepsilon\} \qquad \{\Delta\} = \{0, 1/16, 1/2\}$$

$$<\sigma(z)\sigma(0)>= |z|^{-1/4} \qquad <\varepsilon(z)\varepsilon(0)>= |z|^{-2} \qquad \sigma\sigma \to I + \varepsilon,\ \varepsilon\varepsilon \to I$$

Geometrical objects..



- Stochastic (FK) clusters: Bond between equal spin with prob. $p = 1 - e^{-K}$
- Geometric (G) clusters: $p = 1$



Taken from Wolfhard Janke, KITP2006

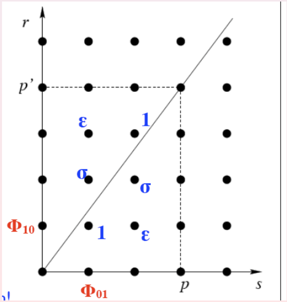
- show fractal behaviour and critical scaling
- Distribution of FK, G $\rightarrow$ Ising, $q = 1$ tricritical Potts critical exponent
- In $3D$ Ising: different percolation temperature..
- ..also in 2D non-minimal spin models? (M.Picco, A. Sicilia,RS, in progress)

...random interfaces and geometric exponents

Prob. two points belong to the perimeter of the same FK,G cluster:

H. Blote,Y. Knops,B. Nienhuis (1992)

$$\propto < \phi_{FK,G}(z_1)\phi_{FK,G}(z_2) > = \frac{1}{|z_1 - z_2|^{4\Delta_{FK,G}}}$$



- $\phi_{FK} = \phi_{1,0}$, $\phi_G = \phi_{0,1}$

  I. Rushkin, E. Bettelheim, I. A. Gruzberg, P. Wiegmann (2007)
- Extended Kac Table, logarithmic minimal model

  P. Pearce, J. Rasmussen, J.Zuber (2006),Y.Saint-Aubin,P. Pearce,

  J. Rasmussen (2008)
- fractal dimensions $d_f^{FK,G} = 2 - 2\Delta_{FK,G}$
- $d_f^{FK} = 5/3(SLE_{16/3})$, $d_f^{G} = 11/8\ (SLE_3)$

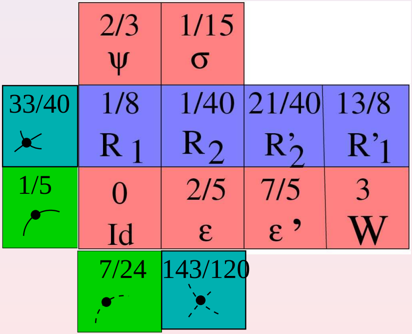
  A.Coniglio,A den Nijs, J. Cardy, B. Duplantier,

  B. Nienhuis , H. Saleur,C. Vanderzande

3−states Potts model, $H = -J \sum_{<ij>} \delta_{\sigma_i,\sigma_j}$:

Y. Deng,H. Blote, B. Nienhuis

Critical at β_c: $e^{\beta_c J} = 1 + \sqrt{3}$

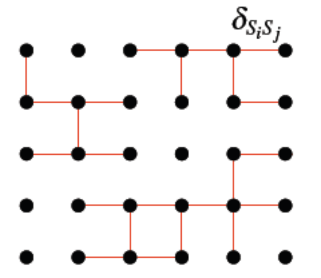


$$d_f^{FK} = 8/5(SLE_{24/5}),\ d_f^{G} = 17/12\ (SLE_{10/3})$$

General q-states Potts model:

$$\mathcal{Z} = \sum_{\{\sigma_i\}} e^{\beta \sum_{\langle ij \rangle} J_{ij} \delta_{\sigma_i \sigma_j}} \sim \sum_{\{\sigma_i\}} \prod_{\langle ij \rangle} \left[1 - p_{ij} + p_{ij} \delta_{\sigma_i \sigma_j} \right]$$

For $J_{ij} = J, p = 1 - e^{-\beta J}$:



$$\mathcal{Z} \sim \sum_{\mathcal{G}} p^{|\mathcal{G}|} (1-p)^{|\overline{\mathcal{G}}|} q^{||\mathcal{G}||} \ ,$$

Bond disorder and perturbed CFT

- $J_{ij} = \overline{J} + \delta J_{ij}$: Gaussian random variables: $\beta^2 \overline{\delta J_{ij}^2} = g_0$
- weak disorder: $\sqrt{g_0} \ll \beta \overline{J}$
- Near the β_c:

$$\mathcal{H} = \mathcal{H}_{\mathrm{pure}} + \int d^2x \, \varepsilon(x) \delta J(x)$$

- $\beta \mathcal{H}_{\mathrm{pure}} \rightarrow$ Minimal CFT with

$$c = 1 - \frac{3}{(2\epsilon + 3)(\epsilon + 2)} \quad \sqrt{q} = 2\cos(\pi/(2\epsilon - 4))$$

- ϵ : RG regularitation parameter. $\epsilon = 0, 1 \rightarrow$ Ising and 3−states Potts

Bond disorder: replica approach

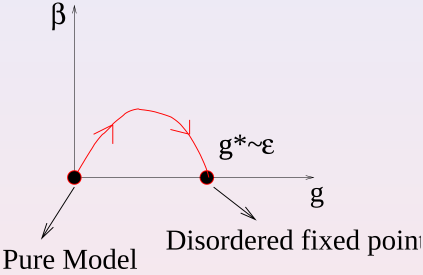
$$\overline{\exp\left(-\beta\sum_{a=1}^{n}\mathcal{H}^{a}\right)}$$

$$= \exp\Big(- \beta\sum_{a=1}^{n}\mathcal{H}^{a}_{\rm pure} + g_0\int d^2x \sum_{a,b=1}^{n}\varepsilon^{a}(x)\varepsilon^{b}(x)\Big)$$

$$4\Delta_{\varepsilon} = \frac{2\epsilon+6}{2\epsilon+3}$$

$\epsilon = 0$ (Ising)$\rightarrow$ $4\Delta_{\varepsilon} = 2$, disorder is marginal

$\epsilon = 1$ (3$-$state Potts)$\rightarrow$ $4\Delta_{\varepsilon} < 2$, disorder is relevant



$$\beta(g) = (2 - 4\Delta_\epsilon)g + 4\pi(n-2)g^2 + \cdots$$

- Replica limit: $n \to 0$, $g^* = \frac{1-2\Delta_\epsilon}{4\pi}$, conformal symmetry restored
- Perturbative computation in g and $\epsilon-$ expansion around the Ising model
- analogous to the ϵ-expansion for ϕ^4 scalar field theory around the gaussian model

Energy and Spin disordered average correlation functions

A. Ludwig 1987, Vl. Dotsenko, M. Picco and P. Pujol, 1995

$$< O(0)O(R) >=< O(0)O(R) >_0 + < S_I O(0)O(R) >_0 +$$

$$+\frac{1}{2} < S_I^2 O(0)O(R) >_0 + \cdots \quad S_I = g_0 \int d^2x \sum_{a,b=1}^{n} \varepsilon^a(x)\varepsilon^b(x)$$

$$\overline{< \varepsilon(0)\varepsilon(x) >} = \frac{1}{|x|^{4\Delta_\varepsilon^*}} \quad \overline{< \sigma(0)\sigma(x) >} = \frac{1}{|x|^{4\Delta_\sigma^*}}$$

$$2\Delta_\varepsilon^* = 2\Delta_\varepsilon + 0(\epsilon) \sim 2\Delta_\varepsilon + 0.36 + 0(\epsilon^3)$$

$$2\Delta_\sigma^* = 2\Delta_\sigma + 0(\epsilon^3) \sim 2\Delta_\sigma + 0.00264 + 0(\epsilon^4)$$

Renormalization of the operator $\Phi_{1,0}$

Second order diagrams:

$$\Phi^a_{10}(z_1)\frac{g_0^2}{2!}\left[\sum_{b\neq c}\int_{z_2}\varepsilon^b(z_2)\varepsilon^c(z_2)\right]\left[\sum_{d\neq e}\int_{z_3}\epsilon^d(z_3)\varepsilon^e(z_3)\right]\rightarrow\Phi^a_{10}(z_1)$$

We have to consider the following integral:

$$\int_{z_2,z_3}\langle\Phi_{10}(z_1)\,\varepsilon(z_2)\,\varepsilon(z_3)\Phi_{10}(\infty)\rangle\,\langle\varepsilon(z_2)\varepsilon(z_3)\rangle$$

Appearence of logarithmics..

$$< \Phi_{10}(z_1)\varepsilon(z)\varepsilon(z_2)\Phi_{10}(z_3) >= \cdots \eta^{c1}(\eta-1)^{c2}\mathcal{H}(\eta)$$

Hypergeometric differential equation:

$$\eta(1-\eta)\mathcal{H}^{''}(\eta)+(a(\Delta_{12},c_1)-b(c_1,c_2,\Delta_{12})\eta)\mathcal{H}^{'}(\eta)-c(\Delta_{12},c_1,c_2)\mathcal{H}(\eta)=0$$

$$\begin{aligned}\frac{3}{2(2\Delta_{12}+1)}(c_1(c_1-1)) &= \Delta_{10}-c_1\\ \frac{3}{2(2\Delta_{12}+1)}(c_2(c_2-1)) &= \Delta_{12}-c_2\end{aligned}$$

Solutions of the hypergeometric diff eq:

$$\mathcal{H}(\eta) = a_1 \mathcal{H}_1(\eta) + a_2(\ln(\eta)\mathcal{H}_1(\eta) + \mathcal{H}_2(\eta))$$

Consistent with the OPE:

Gurarie (1994)

$$\phi_{1,0}(\eta)\varepsilon(0) = \eta^{-\Delta_{1,2}-\Delta_{1,0}+1}\left(W(z)\ln(z) + W^{'}(z) + \frac{1}{z}\partial_z W^{'}(z)\right)$$

Imposing simple monodromy:

$$G(u)\Big|_{p=2} = \frac{\Gamma(\frac{1}{3})^6}{27\pi^2}\frac{|u|^{\frac{2}{3}}}{|1-u|^2}\left|{}_2F_1(-\tfrac{1}{3},\tfrac{2}{3};2;u)\right|^2 + \frac{\Gamma(\frac{1}{3})^8}{54\sqrt{3}\pi^3}\frac{|u|^{\frac{2}{3}}}{|1-u|^2}\Big[{}_2F_1(-\tfrac{1}{3},\tfrac{2}{3};2;u)\; G^{2,0}_{2,2}\left(\overline{u}\,\middle|\, \begin{matrix}\frac{1}{3},\frac{4}{3}\\ -1,0\end{matrix}\right) + c.c.\Big]\;,$$

Coulomb gas

$$\mathcal{S} = \mathcal{S}_0 + \mu_+ \int d^2 z V_+ + \mu_- \int d^2 z V_-$$

$$\begin{aligned} V_\pm &= \; : \exp\left(i\alpha_\pm \varphi(z)\right) : \quad \alpha_+\alpha_- = -1 \;\; \alpha_+ + \alpha_- = 2\alpha_0 \\ c &= 1 - 12\alpha_0^2 \quad < \varphi(z)\varphi(0) >= -4 \log|x/L| \end{aligned}$$

$$c = 1 - \frac{3}{(\epsilon+2)(2\epsilon+3)} \qquad \sqrt{q} = 2\cos(\pi/(2\epsilon-4))$$

Operators $\Phi_{n,m}(z)$ written in terms of vertex operators

$$\begin{aligned} \Phi_{nm}(z) &\rightarrow V_{nm}(z) =: \exp\left(i\alpha_{nm}\varphi(z)\right) : \\ \alpha_{nm} &= \frac{1-n}{2}\alpha_- + \frac{1-m}{2}\alpha_+ \end{aligned}$$

Results:

$$\int_{z_2,z_3} \langle \Phi_{10}(z_1)\,\epsilon(z_2)\,\epsilon(z_3)\Phi_{10}(\infty)\rangle\,\langle\epsilon(z_2)\epsilon(z_3)\rangle$$

Coulomb gas (+procedure of "regularitation" logarithmic cf) $\rightarrow$

$$I = \mathcal{N}\int_{z_2,z_3,u} \langle V_{10}(z_1)\,V_{1,2}(z_2)\,V_{1,2}(z_3)V_{+}(u)V_{\bar{10}}(\infty)\rangle\,|z_2-z_3|^{-4\Delta_{12}}$$

how to compute that? see Dotsenko, Picco, Pujol, (1995)!

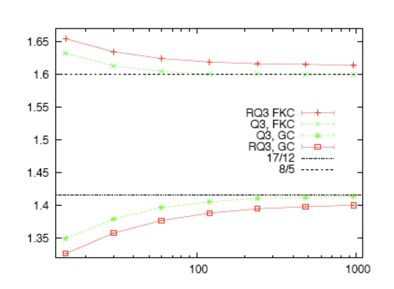
From RG:

$$2\Delta^*_{10} = 2\Delta_{10} + I\frac{9\tilde{\epsilon}^2}{16\pi} \quad \tilde{\epsilon} = \frac{-2\epsilon}{3(3+2\epsilon)}$$

$$\overset{p=3}{=} \frac{2}{5} - 0.01433 \rightarrow \quad d_f^{FK} = \frac{8}{5} + 0.01433$$

Montecarlo simulations:

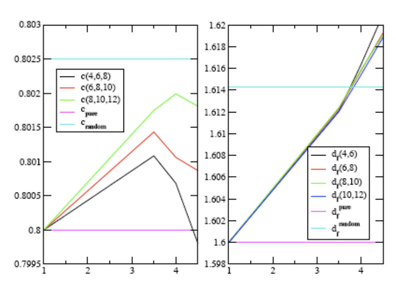
- Wolff algorithm: Prob. p that nn spins belong to the same cluster
- $p \longrightarrow$ symmetric bimodal distribution
  $\{p_1, p_2\} = \{1 - \exp(-\beta_c J_1), 1 - \exp(-\beta_c J_2)\}$
- Pure: $J_1 = J_2$, Random: $J_1/J_2 = 10$



$d_f^{FK} = 1.599 \pm 0.002$ $\quad d_f^{FK} = 1.614 \pm 0.003$

Transfer matrix approach:

- FK cluster in the equivalent loop formulation
- Top. sectors: enforcing $j = 0, 2, 4$ loops propagate$\rightarrow$ $\Delta_{j/2,0}$
- bimodal distribution $J_1/J_2 = \ln(1 + s\sqrt{q})/\ln(1 + \sqrt{q}/s)$
- Pure: $s = 1$, Random fixed point: $s^* = 4 \pm 0.3$

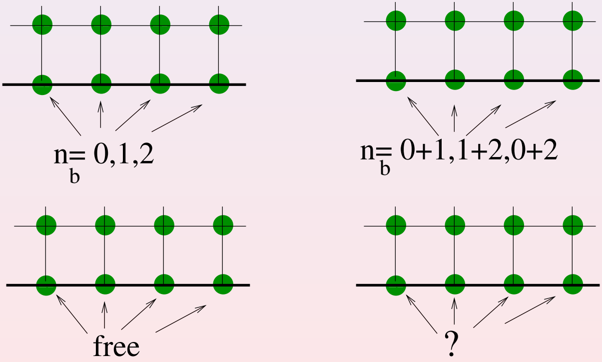


$$2 - 2\Delta_{1,0}$$

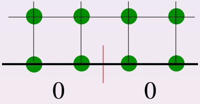
Pott Model on the Half-Plane

I. Affleck, M. Oshikawa and H. Saleur, (1998)

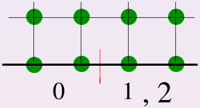
Conformally invariant boundary conditions:



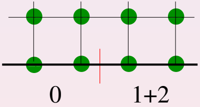
Associated boundary conditions changing operators:



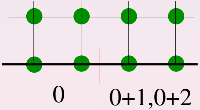
$$\psi(0) = [\mathrm{Id}]$$



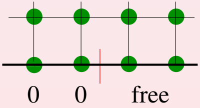
$$\psi(0) = [\ \psi\ ]$$



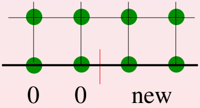
$$\psi(0) = [\ \varepsilon\ ]$$



$$\psi(0) = [\ \sigma\ ]$$



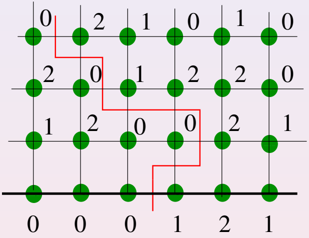
$$\psi(0) = [R_1]$$



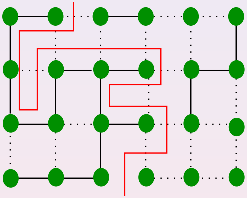
$$\psi(0) = [R_2]$$

Three state Potts SLE interfaces

A. Gamsa et J. Cardy (2007)



SLE 10/3



SLE 24/3

$d_f = 1 + \kappa/8$ Duplantier duality: $\kappa\tilde{\kappa} = 16$

MC results:

Pure : $d_f^G = 1.416 \pm 0.002$ Random : 1.401 ± 0.003

Pure : $\kappa\tilde{\kappa} = 15.95 \pm 0.13$ Random : $\kappa\tilde{\kappa} = 15.76 \pm 0.20$

Open problem?

What about the renormalization of $\Phi_{0,1}$?

$$< \Phi_{01}(z_1)\varepsilon(z)\varepsilon(z_2)\Phi_{01}(z_3) >=??$$

- Satisfy an hypergeometric diff eq. (no logaritmic solutions)
- Two solutions: one with simple monodromy, the other no
- One cannot build a monodromy invariant solution which satisfy the known OPEs
- Coulomb gas fails

- Critical interfaces: how disorder modifies their fractal dimensions
- (log)CFT powerfull tool to study this non-local objects (another test: red bond distribution..)
- Conformal symmetry+disordered systems: can SLE described disordered interfaces?
- Multi-scaling of the disordered correlation function?
- Disordered multi-fractal spectrum of the random critical curve?