The integrable structure of Liouville theory

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Starting point: Sinh-Gordon on the lattice

$$H_{\rm ShG} = \int_0^R \frac{dx}{4\pi} \left\{ \Pi^2 + (\partial_\sigma \Phi)^2 + 2\mu \cosh(2b\Phi) \right\} .$$

Discretize Sinh-Gordon variables as

$$\Pi_n \to \Pi(x)\,\Delta\,, \quad \Phi_n \to \Phi(x)\,, \quad x = n\Delta\,.$$

Quantize: $[\,\Phi_n\,,\,\Pi_n\,] = 2\pi i \delta_{n,m} \Rightarrow$ Hilbert space $\mathcal{H} \equiv (L^2(\mathbb{R}))^{\otimes \rm N}$.

$$L_n(u) \equiv \begin{pmatrix} L^n_{11}(u) & L^n_{12}(u) \\ L^n_{21}(u) & L^n_{22}(u) \end{pmatrix}, \quad \left[\begin{array}{l} L^n_{11}(u) = e^{+\frac{b}{4}(\Pi_n+2s)} \left(1 + e^{-2b(\Phi_n+s)}\right) e^{+\frac{b}{4}(\Pi_n+2s)} \\ L^n_{12}(u) = \sinh b\big(\pi u + \Phi_n\big) \\ L^n_{21}(u) = \sinh b\big(\pi u - \Phi_n\big) \\ L^n_{22}(u) = e^{-\frac{b}{4}(\Pi_n-2s)} \left(1 + e^{+2b(\Phi_n-s)}\right) e^{-\frac{b}{4}(\Pi_n-2s)} \end{array} \right]$$

$\mathsf{T}^{\rm ShG}(u) = tr(L_N(u) L_{N-1}(u) \dots L_1(u))$: Generating fctn. of conserved quantities.

Massless limits I

L-operator of lattice Sinh-Gordon model can be written as

$$L^{SG}(\mu,\bar{\mu}) \,=\, \begin{pmatrix} \mathsf{U}_n + \mu\bar{\mu}\mathsf{V}_n\mathsf{U}_n\mathsf{V}_n & \mu\mathsf{V}_n + \bar{\mu}\mathsf{V}_n^{-1} \\ \mu\mathsf{V}_n^{-1} + \bar{\mu}\mathsf{V}_n & \mathsf{U}_n^{-1} + \mu\bar{\mu}\mathsf{V}_n^{-1}\mathsf{U}_n^{-1}\mathsf{V}_n^{-1} \end{pmatrix},$$

where

$$\mathsf{U}_n \,=\, e^{\frac{b}{2}\Pi_n}, \quad \mathsf{V}_n \,=\, e^{-b\phi_n}, \quad \mu \equiv ie^{-\pi b(s-u)}, \quad \bar{\mu} \equiv -ie^{-\pi b(s+u)}.$$

There are two obvious massless $\big(m \propto e^{-\pi bs} \to 0\big)$ limits: $\mu \to 0$ and $\bar{\mu} \to 0$.

$$L_n^{\mathrm{KdV}}(\mu) \,\equiv\, \begin{pmatrix} \mathsf{U}_n & \mu\,\mathsf{V}_n \\ \mu\,\mathsf{V}_n^{-1} & \mathsf{U}_n^{-1} \end{pmatrix}, \quad \bar{L}_n^{\mathrm{KdV}}(\bar{\mu}) \,\equiv\, \begin{pmatrix} \mathsf{U}_n & \bar{\mu}\,\mathsf{V}_n^{-1} \\ \bar{\mu}\,\mathsf{V}_n & \mathsf{U}_n^{-1} \end{pmatrix},$$

Note: Two copies of lattice q-KdV theory, $(\mathrm{q}-\mathrm{KdV})^{+}$ and $(\mathrm{q}-\mathrm{KdV})^{-}$.

Lattice counterpart of **chiral massless free boson theory**.

Massless limits II

$$H_{\mathrm{ShG}} = \int_0^R \frac{dx}{4\pi} \left\{ \Pi^2 + (\partial_\sigma \varphi)^2 + 2m \cosh(2b\varphi) \right\} .$$

Let $m \to 0$, $\varphi \to \varphi + \xi$, $\xi \to \infty$ such that $me^{2b\xi} \to \mu \Longrightarrow$ Liouville theory

$$H_{\mathrm{Liouville}} = \int_0^R \frac{dx}{4\pi} \left\{ \Pi^2 + (\partial_\sigma \varphi)^2 + 2\mu e^{2b\varphi} \right\} .$$

The corresponding limit of L^{SG} exists if combined with a shift of spectral parameter and a gauge transformation:

$$L^{\mathrm{Liou}}(\mu) \equiv \lim_{s \to \infty} e^{-\frac{\pi}{2} bs\sigma_3} L^{\mathrm{SG}}(u+s) e^{\frac{\pi}{2} bs\sigma_3}$$

The result is

$$L^{\mathrm{Liou}}(\mu) = \begin{pmatrix} \mathsf{U}_n + \mathsf{V}_n \mathsf{U}_n \mathsf{V}_n & \mu \mathsf{V}_n \\ \mu \mathsf{V}_n^{-1} + \mu^{-1} \mathsf{V}_n & \mathsf{U}_n^{-1} \end{pmatrix} .$$

This is a lattice version of the L-matrix proposed by `Faddeev-Tirkkonen` as description of the integrable structure of Liouville theory.

Q-operators - the method I

Key instrument for determination of the **spectrum**: The Q-operator.

Defining relationship with $\mathsf{T}(u)$:

$$\mathsf{Q}(u)\mathsf{T}(u) = \big(a(u)\big)^{\mathrm{N}}\mathsf{Q}(u-ib) + \big(d(u)\big)^{\mathrm{N}}\mathsf{Q}(u+ib),$$

and furthermore

$$\left[\begin{array}{ll} \text{(a)} & \mathsf{Q}(u) \;\text{ is normal,}\;\; \mathsf{Q}(u)\mathsf{Q}^*(v) = \mathsf{Q}^*(v)\mathsf{Q}(u), \\ \text{(b)} & \mathsf{Q}(u)\,\mathsf{Q}(v) = \mathsf{Q}(v)\,\mathsf{Q}(u)\,, \\ \text{(c)} & \mathsf{Q}(u)\,\mathsf{T}(v) = \mathsf{T}(v)\,\mathsf{Q}(u)\,. \end{array}\right]$$

Eigenvalues $t(u)$, $q(u)$ of $\mathsf{T}(u)$, $\mathsf{Q}(u)$ must satisfy the **Baxter equation**

$$t(u)q(u) = \big(a(u)\big)^{\mathrm{N}}q(u-ib) + \big(d(u)\big)^{\mathrm{N}}q(u+ib)\,.$$

Q-operators - the method II

Eigenvalues $t(u)$, $q(u)$ of $\mathsf{T}(u)$, $\mathsf{Q}(u)$ must satisfy the **Baxter equation**

$$t(u)q(u) = \big(a(u)\big)^{\mathrm{N}} q(u-ib) + \big(d(u)\big)^{\mathrm{N}} q(u+ib)\,.$$

From explicit construction of Q-operator (**later!**) $\Rightarrow$ Supplementary conditions on solutions $q(u)$ of the Baxter equation of the form

$$\left[\begin{array}{ll} (\mathrm{an}) & q_t(u) \text{ is meromorphic in } \mathbb{C}, \text{ with poles } \textbf{known}, \\ (\mathrm{as}) & q_t(u) \sim \begin{cases} q_{t+}^{\mathrm{as}}(u) \text{ for } |u| \to \infty, \ |\mathrm{arg}(u)| < \frac{\pi}{2}, \\ q_{t-}^{\mathrm{as}}(u) \text{ for } |u| \to \infty, \ |\mathrm{arg}(u)| > \frac{\pi}{2}. \end{cases} \end{array}\right]$$

Conditions (as), (an) are **necessary** for solutions $t(u)$, $q(u)$ of Baxter eqn. to represent eigenvalues!

This set of conditions can be solved by means of nonlinear integral equations generalizing TBA.

Q: When does a solution to these conditions correspond to a state in the spectrum?

Q-operators - the method III

Sufficiency of these conditions from **Separation of variables**.

Main idea (**Sklyanin**): Diagonalize $\mathsf{B}(u)$, parametrize eigenvalues $b(u)$ as

$$b(u) \sim \prod \sinh 2\pi b(u - y_k)\,.$$

$\Rightarrow$ wave-functions $\Psi(y_1 \ldots y_{\mathrm{N}})$. Key observations:

- (`Sklyanin`) $\mathsf{T}(u)\Psi_t = t(u)\Psi_t(u) \Leftrightarrow$ Baxter equation:

  $$t(y_k)\Psi(\ldots y_k \ldots) = (a(y_k))^{\mathrm{N}}\Psi(\ldots y_k - ib \ldots) + (d(y_k))^{\mathrm{N}}\Psi(\ldots y_k + ib \ldots).$$

- Asymptotic behavior (as) $\overset{?}{\Rightarrow}$ Ansatz $\Psi_t = \prod_{k=1}^{N} q_t(y_k)$ yields **normalizable** eigenstates of $\mathsf{T}(u)$.

If so $\Rightarrow$ **Complete description of the spectrum**:

> **All** solutions $q(u)$ of the Baxter equation which satisfy (as) and (an) yield an eigenstate of $\mathsf{T}(u)$ via $\Psi_t = \prod_{k=1}^{N} q_t(y_k)$.

Q-operators - the construction I

Introduce **doubling of DOF**: Pairs of positive operators u_n, v_n, $\bar{\mathsf{u}}_n$, $\bar{\mathsf{v}}_n$ with relations $\mathsf{u}_n\mathsf{v}_n = q\mathsf{v}_n\mathsf{u}_n$, $\bar{\mathsf{u}}_n\bar{\mathsf{v}}_n = q^{-1}\bar{\mathsf{v}}_n\bar{\mathsf{u}}_n$

$$L_n^{\mathrm{KdV}}(\mu) \equiv \begin{pmatrix} \mathsf{u}_n & \mu\,\mathsf{v}_n \\ \mu\,\mathsf{v}_n^{-1} & \mathsf{u}_n^{-1} \end{pmatrix}, \quad \bar{L}_n^{\mathrm{KdV}}(\bar{\mu}) \equiv \begin{pmatrix} \bar{\mathsf{u}}_n & \bar{\mu}\,\bar{\mathsf{v}}_n \\ \bar{\mu}\,\bar{\mathsf{v}}_n^{-1} & \bar{\mathsf{u}}_n^{-1} \end{pmatrix},$$

We may then consider

$$\mathcal{L}^{\mathrm{ShG}}(\mu,\bar{\mu}) = L_n^{\mathrm{KdV}}(\mu)\bar{L}_n^{\mathrm{KdV}}(\bar{\mu}) = \begin{pmatrix} \mathsf{U}_n + \mu\bar{\mu}\mathsf{V}_n\mathsf{U}_n\mathsf{V}_n & \eta(\mu\mathsf{V}_n + \bar{\mu}\mathsf{V}_n^{-1}) \\ \eta^{-1}(\mu\mathsf{V}_n^{-1} + \bar{\mu}\mathsf{V}_n) & \mathsf{U}_n^{-1} + \mu\bar{\mu}\mathsf{V}_n^{-1}\mathsf{U}_n^{-1}\mathsf{V}_n^{-1} \end{pmatrix},$$

where $\mathsf{U}_n = \mathsf{u}_n\bar{\mathsf{u}}_n$, $\mathsf{V}_n = (\mathsf{u}_n^{-1}\mathsf{v}_n\bar{\mathsf{u}}_n^{-1}\bar{\mathsf{v}}_n^{-1})^{\frac{1}{2}}$, $\eta = (\mathsf{u}_n\mathsf{v}_n\bar{\mathsf{u}}_n^{-1}\bar{\mathsf{v}}_n)^{\frac{1}{2}}$. The corresponding transfer matrix,

$$\widehat{\mathsf{T}}^{\mathrm{ShG}}(\mu,\bar{\mu}) \equiv \mathrm{tr}\big[\hat{L}_{\mathrm{N}}^{\mathrm{ShG}}(\mu,\bar{\mu})\cdots\hat{L}_1^{\mathrm{ShG}}(\mu,\bar{\mu})\big],$$

does not depend on η, but only on $\mathsf{U}_n, \mathsf{V}_n$, $n = 1,\dots,\mathrm{N} \Rightarrow$ **doubling disappears:**

$$\boxed{\widehat{\mathsf{T}}^{\mathrm{ShG}}(\mu,\bar{\mu}) \simeq \mathsf{T}^{\mathrm{ShG}}(\mu,\bar{\mu})}$$

Q-operators - the construction II

Key building blocks for Q-operators: the **Fundamental R-operators**. It is defined as the solution to

$$\mathsf{R}_{nm}(\nu/\mu)L_n(\nu)L_m(\mu) \;=\; L_m(\mu)L_n(\nu)\mathsf{R}_{nm}(\nu/\mu)\,.$$

For the case of lattice KdV it may be represented explicitly as (`Faddeev, Volkov`)

$$\mathsf{R}^{\mathrm{KdV}}_{nm}(\mu) \;=\; \mathsf{P}_{nm}\,\rho(\mathsf{w}_{nm};\mu)\,, \qquad \mathsf{w}_{nm} \;\equiv\; \left(\mathsf{u}_m\mathsf{v}_m\mathsf{u}_n\mathsf{v}_n^{-1}\right)^{\frac{1}{2}}\,,$$

where $\rho(w;\mu)$ may be represented as $\rho(e^{\pi bx};e^{\pi bu}) = W_u(x)$,

$$W_u(x) \equiv e^{i\frac{\pi}{2}ux}\frac{e_b(x+u/2)}{e_b(x-u/2)}, \qquad \log e_b(x) = -\int\frac{dt}{4t}\frac{e^{-2\pi itx}}{\sinh bt\sinh b^{-1}t}\,.$$

$\Rightarrow$ Construction of $\mathsf{Q}^{\mathrm{KdV}}_{\pm}(u)$ as

$$\mathsf{Q}^{\mathrm{KdV}}_{+}(u) = \mathrm{tr}_{\mathcal{H}_a}(\mathsf{R}_{a\mathrm{N}}(u)\cdots\mathsf{R}_{a1}(u))\,,$$
$$\mathsf{Q}^{\mathrm{KdV}}_{-}(u) = \mathrm{tr}_{\mathcal{H}_a}(\mathsf{R}_{a\bar{\mathrm{N}}}(u)\cdots\mathsf{R}_{a\bar{1}}(u))\,,$$

Q-operators - the construction III

For $\mathsf{Q}^{\mathrm{ShG}}(u)$ we will need the solution to the equation

$$\mathcal{R}_{nm}(v,\bar{v};u,\bar{u})\,\mathcal{L}_{\imath n}(v,\bar{v})\mathcal{L}_{\imath m}(u,\bar{u}) \;=\; \mathcal{L}_{\imath m}(u,\bar{u})\mathcal{L}_{\imath n}(v,\bar{v})\mathcal{R}_{nm}(v,\bar{v};u,\bar{u})\,,$$

given that $\mathcal{L}_n(\mu,\bar{\mu}) = L_n(\mu)\bar{L}_n(\bar{\mu})$. It is quite clear that

$$\mathcal{R}_{nm}(v,\bar{v};u,\bar{u}) \;=\; \mathsf{R}_{n\bar{m}}(v,\bar{u})\mathsf{R}_{\bar{n}\bar{m}}(\bar{v},\bar{u})\mathsf{R}_{nm}(v,u)\mathsf{R}_{\bar{n}m}(\bar{v},u)$$

will do the job provided

$$\begin{aligned}
\mathsf{R}_{nm}(v,u)L_{an}(v)L_{am}(u) \;&=\; L_{am}(u)L_{an}(v)\mathsf{R}_{nm}(v,u)\,,\\
\mathsf{R}_{n\bar{m}}(v,u)L_{an}(v)\bar{L}_{am}(u) \;&=\; \bar{L}_{am}(u)L_{an}(v)\mathsf{R}_{n\bar{m}}(v,u)\,,\\
\mathsf{R}_{\bar{n}m}(v,u)\bar{L}_{an}(v)L_{am}(u) \;&=\; L_{am}(u)\bar{L}_{an}(v)\mathsf{R}_{\bar{n}m}(v,u)\,,\\
\mathsf{R}_{\bar{n}\bar{m}}(v,u)\bar{L}_{an}(v)\bar{L}_{am}(u) \;&=\; \bar{L}_{am}(u)\bar{L}_{an}(v)\mathsf{R}_{\bar{n}\bar{m}}(v,u)\,.
\end{aligned}$$

Q-operators - the construction IV

It is easy to show that

$$
\begin{aligned}
\mathsf{R}^{\mathrm{KdV}}_{nm}(v,u) &= \mathsf{R}^{\mathrm{KdV}}_{nm}(v-u)\,,\\
\mathsf{R}^{\mathrm{KdV}}_{n\bar{m}}(v,u) &= \varpi_n \cdot \mathsf{R}^{\mathrm{KdV}}_{nm}(v+u)\,,\\
\mathsf{R}^{\mathrm{KdV}}_{\bar{n}m}(v,u) &= \mathsf{R}^{\mathrm{KdV}}_{nm}(-v-u)\cdot \varpi_m\,,\\
\mathsf{R}^{\mathrm{KdV}}_{\bar{n}\bar{m}}(v,u) &= \varpi_n \cdot \mathsf{R}^{\mathrm{KdV}}_{nm}(-v+u)\cdot \varpi_m\,.
\end{aligned}
$$

satisfy the conditions above $\Rightarrow$

$$
\mathcal{R}^{\mathrm{ShG}}_{nm}(v,\bar{v};u,\bar{u}) = \mathsf{R}^{\mathrm{KdV}}_{n\bar{m}}(v,\bar{u})\mathsf{R}^{\mathrm{KdV}}_{\bar{n}\bar{m}}(\bar{v},\bar{u})\mathsf{R}^{\mathrm{KdV}}_{nm}(v,u)\mathsf{R}^{\mathrm{KdV}}_{\bar{n}m}(\bar{v},u)
$$

Q-operators - the construction V

Let us define generalized transfer matrices

$$\mathsf{T}_s^{\mathrm{ShG}}(w,\bar{w}) \;=\; \mathrm{tr}_{L^2(\mathbb{R}^2)}\left[\mathcal{R}_{A\mathrm{N}}^{\mathrm{ShG}}(w,\bar{w};s,s)\cdots\mathcal{R}_{A1}^{\mathrm{ShG}}(w,\bar{w};s,s)\right].$$

$\mathsf{T}_s^{\mathrm{ShG}}(w,\bar{w})$ gives us the Q-operator thanks to

$$\boxed{\mathsf{T}_s^{\mathrm{ShG}}(w,\bar{w}) = \mathsf{Q}_s^{\mathrm{ShG}}(w)(\mathsf{Q}_s^{\mathrm{ShG}}(\bar{w}))^{\dagger}, \quad \mathsf{Q}_s^{\mathrm{ShG}}(w) \equiv \mathsf{T}_s^{\mathrm{ShG}}(w,0)}\,.$$

The result may also be represented as

$$\mathsf{Q}^{\mathrm{ShG}}(u) \;=\; \mathsf{C}\cdot\mathrm{tr}_{L^2(\mathbb{R})}\left[\mathsf{R}_{a\bar{\mathrm{N}}}(u)\mathsf{R}_{a\mathrm{N}}(u)\cdots\mathsf{R}_{a\bar{1}}(u)\mathsf{R}_{a1}(u)\right]\cdot\mathsf{R}_{\mathrm{N}\bar{\mathrm{N}}}(s)\cdots\mathsf{R}_{1\bar{1}}(s)\,,$$

where $\mathsf{C}\cdot\mathsf{O}_n = \mathsf{O}_n\cdot\mathsf{C}$ and $\mathsf{C}\cdot\mathsf{O}_{\bar{n}} = \mathsf{O}_{\bar{n}-1}\cdot\mathsf{C}$.

Compare also with `Bazhanov/Lukyanov/Zamolodchikov`:

Q-operators as trace over q-oscillator representation of $\mathcal{U}_q(\widehat{\mathfrak{sl}_2})$.

Q-operators - the construction VI

Important: By definition $\mathcal{R}^{\mathrm{ShG}}_{nm}(v,\bar{v};u,\bar{u}) : L^2(\mathbb{R}^4) \to L^2(\mathbb{R}^4)$. However, it is easy to show that it projects to an operator $\mathsf{R}^{\mathrm{ShG}}_{nm}(v,\bar{v};u,\bar{u}) : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$.

$\mathsf{R}^{\mathrm{ShG}}_{nm}(v,\bar{v};u,\bar{u})$, for example, may be represented by the kernel

$$\begin{aligned} R^{v,\bar{v};u,\bar{u}}_{nm}(x'_n, x'_m | x_n, x_m) \; &= W_{v+\bar{u}}(x'_m + x'_n)\overline{W}_{v-u}(x'_m - x_n) \\ &\times \overline{W}_{\bar{u}-\bar{v}}(x'_n - x_m) W_{-\bar{v}-u}(x_m + x_n)\,, \end{aligned}$$

where $\overline{W}_u(y) = \int dx\, e^{2\pi i x y} W_u(x)$ — `Compare with Bazhanov-Stroganov`

Liouville theory I

Problem: If $\mathrm{N} = 2L+1$, the transfer matrix constructed from Faddeev-Tirkkonen L-matrix gives only $L+1$ commuting operators.

Key observation: There are actually two **commuting** transfer matrices $\mathsf{T}_{\pm}^{\mathrm{Liou}}(\mu)$ that can be obtained from

$$L_{\pm}^{\mathrm{Liou}}(\mu) \;\equiv\; \lim_{s\to\infty} e^{\mp\frac{\pi}{2}bs\sigma_3} L^{\mathrm{SG}}(u\pm s) e^{\pm\frac{\pi}{2}bs\sigma_3}\,.$$

Key result: There are two Q-operators $\mathsf{Q}_{\pm}^{\mathrm{Liou}}(v)$ satisfying Baxter-equations of the form

$$\begin{aligned}
\mathsf{T}_{+}^{\mathrm{Liou}}(v)\,\mathsf{Q}_{+}^{\mathrm{Liou}}(v) \;&=\; \mathsf{Q}_{+}^{\mathrm{Liou}}(v-ib) + \big(d(v)\big)^{\mathrm{N}}\mathsf{Q}_{+}^{\mathrm{Liou}}(v+ib)\,,\\
\mathsf{T}_{-}^{\mathrm{Liou}}(v)\,\mathsf{Q}_{-}^{\mathrm{Liou}}(v) \;&=\; \big(a(v)\big)^{\mathrm{N}}\mathsf{Q}_{-}^{\mathrm{Liou}}(v-ib) + \mathsf{Q}_{-}^{\mathrm{Liou}}(v+ib)\,,
\end{aligned}$$

that **commute** with each other,

$$\mathsf{Q}_{+}^{\mathrm{Liou}}(v)\mathsf{Q}_{-}^{\mathrm{Liou}}(v) \;=\; \mathsf{Q}_{-}^{\mathrm{Liou}}(v)\mathsf{Q}_{+}^{\mathrm{Liou}}(v)\,.$$

Liouville theory II

Indeed, introduce truncated L-operators

$$K_n^{\mathrm{KdV}}(\mu) \equiv \begin{pmatrix} \mathsf{u}_n & \mu\,\mathsf{v}_n \\ \mu\,\mathsf{v}_n^{-1} & 0 \end{pmatrix}, \quad \bar{K}_n^{\mathrm{KdV}}(\bar{\mu}) \equiv \begin{pmatrix} \bar{\mathsf{u}}_n & \bar{\mu}\,\bar{\mathsf{v}}_n \\ \bar{\mu}\,\bar{\mathsf{v}}_n^{-1} & 0 \end{pmatrix},$$

and observe

$$\mathcal{L}_+^{\mathrm{Liou}}(\mu,\bar{\mu}) = L_n^{\mathrm{KdV}}(\mu)\bar{K}_n^{\mathrm{KdV}}(\bar{\mu}), \qquad \mathcal{L}_-^{\mathrm{Liou}}(\mu,\bar{\mu}) = K_n^{\mathrm{KdV}}(\mu)\bar{L}_n^{\mathrm{KdV}}(\bar{\mu})$$

$$\Rightarrow$$

Liouville theory III

Similar Lego-game as before can be used for construction of Q-operators, need operators

$$
\begin{aligned}
\mathsf{R}_{\check{n}m}(\nu/\mu)K_{an}(\nu)L_{am}(\mu) \;&=\; L_{am}(\mu)K_{an}(\nu)\mathsf{R}_{\check{n}m}(\nu/\mu)\,.\\
\mathsf{R}_{\check{n}\check{m}}(\nu/\mu)K_{an}(\nu)K_{am}(\mu) \;&=\; K_{am}(\mu)K_{an}(\nu)\mathsf{R}_{\check{n}\check{m}}(\nu/\mu),
\end{aligned}
$$

etc., which may be represented explicitly as

$$
\mathsf{R}_{\check{n}m}(\mu) \;=\; \mathsf{P}_{nm}\,\sigma(\mathsf{w}_{nm};\mu)\,, \qquad \mathsf{R}_{\check{n}\check{m}}(\mu) \;=\; \mathsf{P}_{nm}\,\tau(\mathsf{w}_{nm};\mu)\,,\,,
$$

etc., where $\sigma(e^{\pi bx};e^{\pi bu}) = V_u(x)$, $\tau(e^{\pi bx};e^{\pi bu}) = U_u(x)$ with

$$
\begin{aligned}
&V_u(x) \equiv \frac{e^{-\frac{\pi i}{2}x^2}}{e_b(x-u/2)}, \qquad \log e_b(x) = -\int \frac{dt}{4t}\frac{e^{-2\pi i t x}}{\sinh bt \sinh b^{-1}t}\,,\\
&U_u(x) \equiv e^{-\pi i u x}\,.
\end{aligned}
$$

$$
\Rightarrow \quad \mathsf{Q}_{-}^{\rm Liou}(u) \;=\; \mathsf{C}\cdot \mathrm{tr}_{L^2(\mathbb{R})}\big[\mathsf{R}_{a\bar{\mathrm{N}}}(u)\mathsf{R}_{a\check{\mathrm{N}}}(u)\cdots\mathsf{R}_{a\bar{1}}(u)\mathsf{R}_{a\check{1}}(u)\big]\cdot\mathsf{R}_{\check{\mathrm{N}}\bar{\mathrm{N}}}(s)\cdots\mathsf{R}_{\check{1}\bar{1}}(s)\,, \text{ etc..}
$$

Liouville theory IV

The eigenvalues $q_\pm^{\rm Liou}$ of the operators $\mathsf{Q}_\pm^{\rm Liou}(v)$ have the following properties:

$$\left[\begin{array}{rl}
\text{(i)} & q_\pm^{\rm Liou}(v) \ \text{are meromorphic with poles of maximal order } \mathrm{N} \text{ in } \ \pm\Upsilon_0, \\[2ex]
\text{(ii)} & q_{p,\pm}^{\rm Liou}(v) \sim \exp\big(i\pi N(s+iQ/2)v - i\tfrac{\pi}{2}\mathrm{N}v^2\big) \quad \text{for } \ |v|\to\infty, \quad |\arg(\mp v)|<\tfrac{\pi}{2}\,, \\[2ex]
\text{(iii)} & q_{p,\pm}^{\rm Liou}(v) \sim c_p e^{2\pi i p v} + d_p e^{-2\pi i p v} \quad \text{for } \ |v|\to\infty, \quad |\arg(\pm v)|<\tfrac{\pi}{2}\,. \\[2ex]
\text{(iv)} & t_+^{\rm Liou}(v)\, q_+^{\rm Liou}(v) \ = \ q_+^{\rm Liou}(v-ib) + \big(d(v)\big)^{\mathrm{N}} q_+^{\rm Liou}(v+ib)\,, \\
& q_-^{\rm Liou}(v)\, q_-^{\rm Liou}(v) \ = \ \big(a(v)\big)^{\mathrm{N}} q_-^{\rm Liou}(v-ib) + q_-^{\rm Liou}(v+ib)\,, \\
& \text{where } \ d(v) = a(-v) = 1 + e^{-\pi b(2v+ib)}\,,
\end{array}\right]$$

Same conditions satisfied by $q_\pm^{\rm KdV}(v)$!!!

Liouville theory V

Note we now have two monodromy matrices $\mathsf{M}_\pm(u)$,

$$\mathsf{M}_\pm(u) = L_{N,\pm}(u) \cdot L_{N-1,\pm}(u) \cdots L_{1,\pm}(u) \equiv \begin{pmatrix} A_\pm(u) & B_\pm(u) \\ C_\pm(u) & D_\pm(u) \end{pmatrix}.$$

Separation of variables: Sklyanins recipe now gives $L+1$ variables $y_{-L}, \dots, y_{-1}, y_0$ from $C_-(u)$, $L+1$ variables $y_0, y_1, \dots, y_L$ from $B_+(u)$, with same variable y_0.

Eigenstates of $\mathsf{T}_\pm^{\rm Liou}$ can be represented in the SOV-representation as

$$\boxed{\Psi_{t,p}(\mathbf{y}) \;=\; \prod_{k=-L}^{-1} q_{p,-}^{\rm Liou}(y_k)\, e^{2\pi i y_0 p} \prod_{k=1}^{L} q_{p,+}^{\rm Liou}(y_k)\,.}$$

Liouville theory V

$\Rightarrow$ **Main conclusion:**

Claim 1. *There exists a unitary operator* U *such that*

$$\mathsf{U}^{-1} \cdot \mathsf{Q}_{\pm}^{\mathrm{Liou}}(v) \cdot \mathsf{U} \, = \, \mathsf{Q}_{\pm}^{\mathrm{KdV}}(v) \, .$$

- The unitary operator U is a highly nontrivial object: It is quantum analog of the **Bäcklund transformation** which relates Liouville theory (interacting) to KdV theory (essentially free field theory).

- The operator U can be understood as a lattice version of the Moeller operator that relates initial values to scattering data in the quantum mechanical scattering theory.

Liouville theory VI

- It follows from the description above that $\Psi_{t,p}$ satisfies a reflection relation of the form

$$\Psi_{t,p}(\mathbf{y}) \,=\, R_{t,p}\Psi_{t,-p}(\mathbf{y})\,.$$

- Let $\mathcal{F}^{\pm}$ be the Fock spaces generated by the harmonic oscillators $(\mathsf{a}_n^{\pm}, \mathsf{a}_{-n}^{\pm})$ for $n = 1, \dots, L$ with commutation relations:

$$[\,\mathsf{a}_n^{+}\,,\,\mathsf{a}_m^{-}\,] \,=\, 0\,, \qquad [\,\mathsf{a}_n^{\pm}\,,\,\mathsf{a}_m^{\pm}\,] \,=\, \delta_{n+m,0}\,\frac{\sin 2\rho n}{\rho}\,, \qquad \rho \,\equiv\, \frac{\pi}{N}\,.$$

  The Hilbert space $\mathcal{H}^{\mathrm{Liou}}$ may then be represented as

$$\mathcal{H}^{\mathrm{Liou}} \,\simeq\, \mathcal{H}^{\mathrm{FF}} \,\equiv\, \int_0^\infty dp\;\mathcal{F}_p^{+}\otimes\mathcal{F}_p^{-}\,.$$

Liouville theory VII

The continuum limit of the Q-functions exists. It satisfies a reflection relation of the form

$$q^{\pm}_{t,p}(\mathbf{y}) \;=\; R_{t,p} q^{\pm}_{t,-p}(\mathbf{y})\,.$$

For the case of the Fock vacuum $\mathcal{F}_p$, $R(P) \equiv R_{t,p}$ can be calculated exactly (using results of `Dorey/Tateo, Fateev/Lukyanov`)

$$R(P) \;=\; -\rho^{-8i\delta P}\,\frac{\Gamma(1+2ibP)\Gamma(1+2ib^{-1}P)}{\Gamma(1-2ibP)\Gamma(1-2ib^{-1}P)}\,,$$

$$\rho \;\equiv\; \frac{R}{2\pi}\,\frac{m}{4\sqrt{\pi}}\,\Gamma\biggl(\frac{1}{2+2b^2}\biggr)\Gamma\biggl(1+\frac{b^2}{2+2b^2}\biggr)\,.$$

This is the **Liouville reflection amplitude**, previously calculated by CFT techniques!

Conclusions and outlook

- This is the first example of a nonrational CFT where one has seen the emergence of **chiral factorization** from the **integrable structure**.

$$\Psi_{t,p}(\mathbf{y}) \;=\; \prod_{k=-L}^{-1} q_p^-(y_k)\, e^{2\pi i y_0 p} \prod_{k=1}^{L} q_p^+(y_k)\,,$$

  where $q_p^+(u)$: wave-function of left-movers, $q_p^-(u)$: wave-function of right-movers.

- My intention is to apply similar techniques to CFT (like Sigma models) whose chiral symmetries are too weak for solving them.