SUSY INTEGRABLE SPIN CHAINS AND DISCRETE HIROTA DYNAMICS

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GGI, Florence, 29 October 2008

based on

V.Kazakov, A.Sorin, A.Z. Nucl.Phys. B790 (2008) 345;

A.Z. Theor. Math.Phys. 155 (2008) 567

Motivation

- Classical versus quantum integrability:

  Classical integrable equations take part in **quantum** problems even at $\hbar \neq 0$

  Quantum spin chains

  Discrete classical Hirota dynamics

  (I.Krichever, O.Lipan, P.Wiegmann, A.Z. 1996)

- Clarifying SUSY Bethe ansatz, explaining variety of Bethe equations, “duality” transformations, etc.

Plan

1. Integrable SUSY GL(K|M)-invariant spin chains: commuting transfer matrices

2. *TT*-relation (Hirota equation) for quantum transfer matrices

3. Boundary and analytic conditions

4. Solving the *TT*-relation by classical methods
   - Auxiliary linear problems
   - Backlund transformations
   - "Undressing" procedure: nested Bethe ansatz as a chain of Backlund transformations

5. Generalized Baxter's *TQ*-relations

6. *QQ*-relation *(*Hirota equation for *Q's)* and Bethe equations

$GL(K|M)$ spin chains

(P.Kulish 1985)

$GL(K|M)$-invariant R-matrix acts in the tensor product $\pi_0 \otimes \pi_1$ of two irreps and

- obeys the graded Yang-Baxter equation

(P.Kulish, E.Sklyanin 1980)

- for any $g \in GL(K|M)$ obeys the condition

$$\pi_0(g) \otimes \pi_1(g)\, R_{01}(u) = R_{01}(u)\, \pi_0(g) \otimes \pi_1(g)$$

π_0 arbitrary, $\pi_1 = \pi_{\square}$

vector representation
in the space $\mathbb{C}^K \oplus \mathbb{C}^M$

spectral parameter

parity

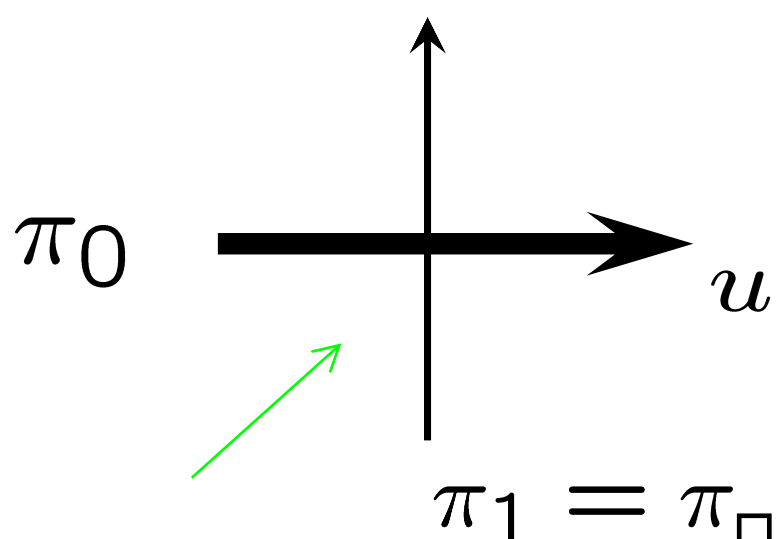
generators of $gl(K|M)$

$$(E_{\alpha\beta})_{\alpha'\beta'} = \delta_{\alpha\alpha'}\delta_{\beta\beta'}$$

$$\alpha, \beta = 1, \ldots, K+M$$

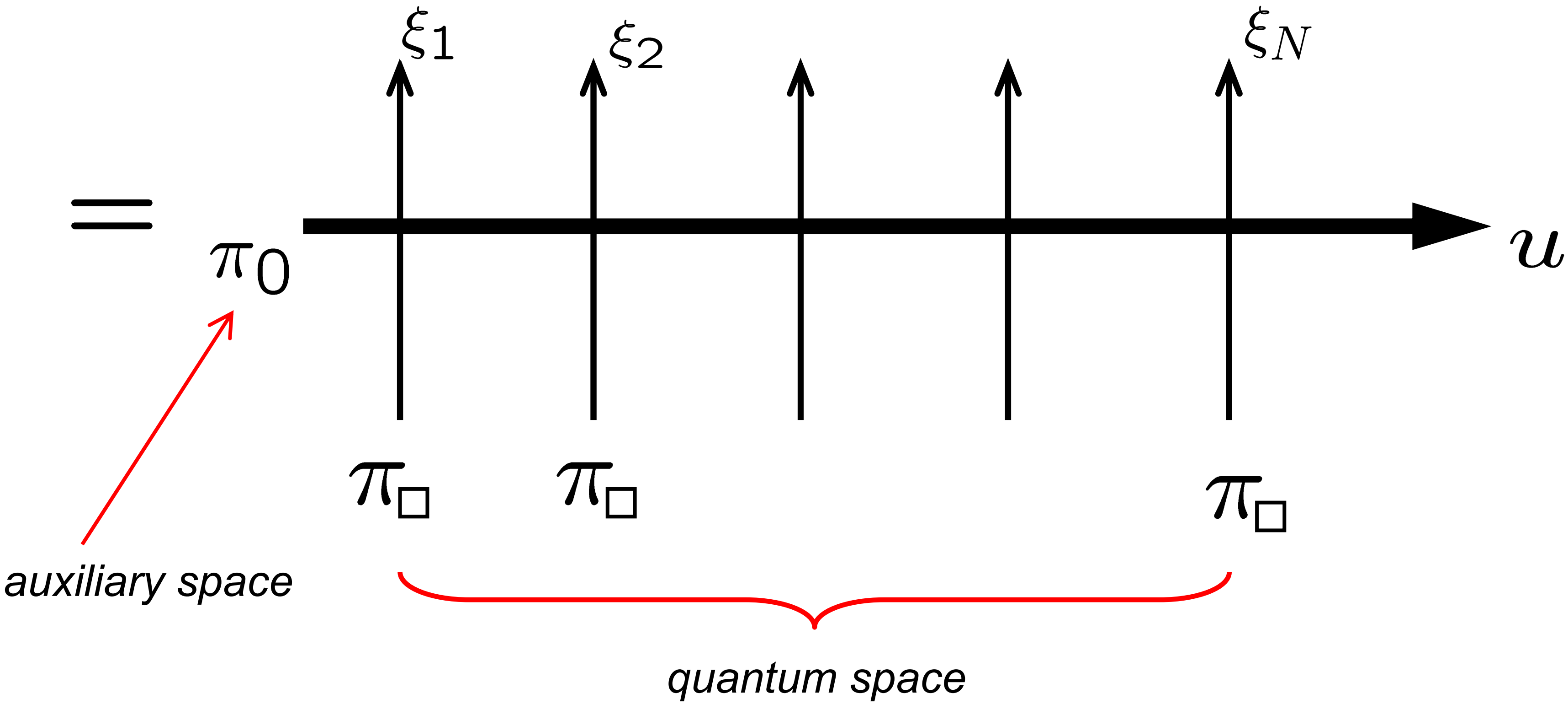
$$R_{01}(u) = u + 2\sum_{\alpha\beta} (-)^{p(\beta)} \pi_0(E_{\alpha\beta}) \otimes E_{\beta\alpha}$$

$=$



Quantum monodromy matrix

$$\mathcal{T}(u) = R_{01}(u - \xi_1) R_{02}(u - \xi_2) \dots R_{0N}(u - \xi_N)$$

$=$ 

$$\mathcal{T}(u): \; \pi_0 \otimes (\pi_\Box)^{\otimes N} \longrightarrow \pi_0 \otimes (\pi_\Box)^{\otimes N}$$

Quantum transfer matrix (periodic b.c.)

$$T^{(\pi_0)}(u) = \mathrm{str}_{\pi_0}\left(\mathcal{T}(u)\right)$$

Yang-Baxter equation $\Longrightarrow$ commutativity

$$[T^{(\pi_0)}(u),\ T^{(\pi_0')}(u')] = 0$$

Generalization: twisted b.c.

$$g = \mathrm{diag}\,(x_1, \ldots, x_K, y_1, \ldots, y_M) \in GL(K|M)$$

insert before taking **str**

$$T^{(\pi_0)}(u; g) = \mathrm{str}_{\pi_0}\,(\pi_0(g)\mathcal{T}(u))$$

YB equation + $GL(K|M)$-invariance

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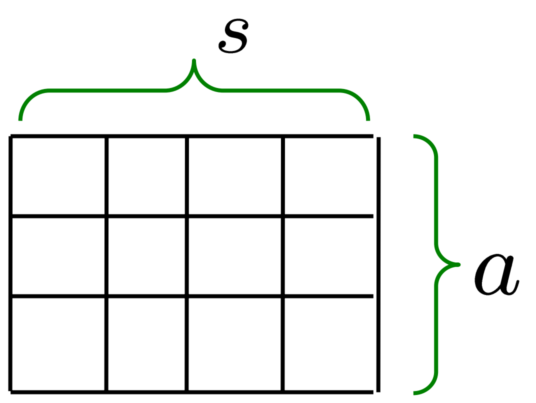
$$[T^{(\pi_0)}(u; g),\ T^{(\pi_0')}(u'; g)] = 0$$

Rectangular representations

(We consider covariant representations only)

$$\pi_0 = \pi_s^a =$$



$$T(a,s,u) = T^{(\pi_s^a)}(u-s+a;g)$$

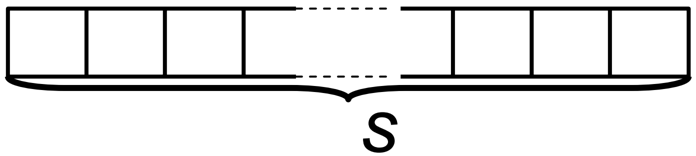
$$= \mathsf{str}_{\pi_s^a}\left(\pi_s^a(g)\mathcal{T}(u-s+a)\right)$$

Functional relations for transfer matrices

P.Kulish, N.Reshetikhin 1983;
V.Bazhanov, N.Reshetikhin 1990;
A.Klumper, P.Pearce 1992;
A.Kuniba, T.Nakanishi, J.Suzuki 1994;
Z.Tsuboi 1997 (for SUSY case)

$T^{(\pi_0)}(u)$ for general irreps π_0 can be expressed

through $T(1,s,u)$ for the rows



In particular,

$$T(a,s,u) \propto \det_{1\le i,j\le a} T(1,\, s+i-j,\, u+a+1-i-j)$$

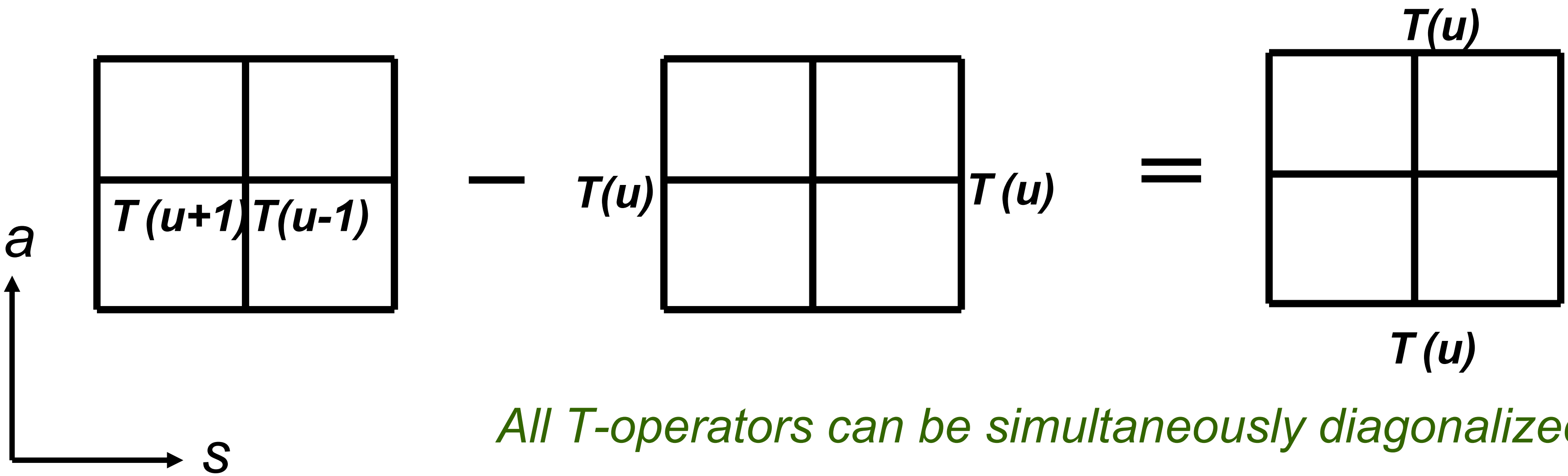
This is the Bazhanov-Reshetikhin det-formula for rectangular irreps.

- *Analog of Weyl formula for characters*
- *The same for SUSY and ordinary case*

TT-relation

- directly follows from the BR det-formula
- is identical to the ***HIROTA EQUATION***

$$T(a,s,u{+}1)T(a,s,u{-}1) - T(a,s{+}1,u)T(a,s{-}1,u) = T(a{+}1,s,u)T(a{-}1,s,u)$$



All T-operators can be simultaneously diagonalized

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The same relation holds for any of their eigenvalues

Hirota difference equation *(R.Hirota 1981)*

- Integrable difference equation solvable by classical inverse scattering method

- A ''master equation" of the soliton theory:
  - provides universal discretization of integrable PDE's
  - generates infinite hierarchies of integrable PDE's

- We use it to find all possible Baxter's TQ-relations and nested Bethe ansatz equations for super spin chains

(similar method for ordinary spin chains: I.Krichever, O.Lipan, P.Wiegmann, A.Zabrodin 1996)

Analytic and boundary conditions

$T(a,s,u)$ are polynomials in *u* of degree *N* for any *a,s*

length of spin chain

What is $T(0,s,u)$? And $T(a,0,u)$?

Transfer matrix for the trivial irrep is proportional to the identity operator:

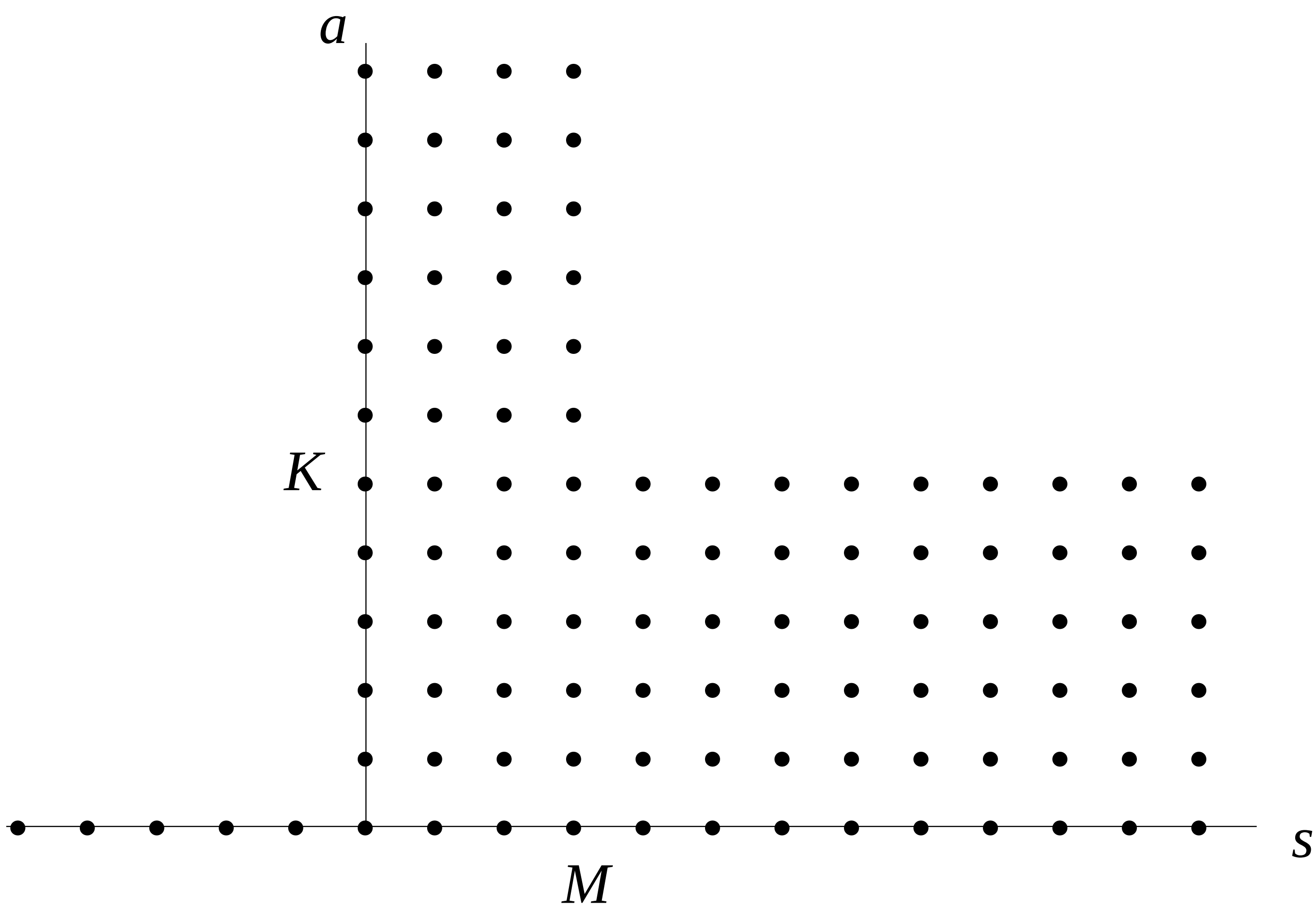
$$T(0,s,u) = \phi(u-s)\ ,\quad T(a,0,u) = \phi(u+a)$$

$$\phi(u) = \prod_{i=1}^{N} (u-\xi_i)$$

Here a,s > 0. Extension to negative values?

Fat hook

The domain where T(a,s,u) for GL(K|M) spin chains does not vanish identically

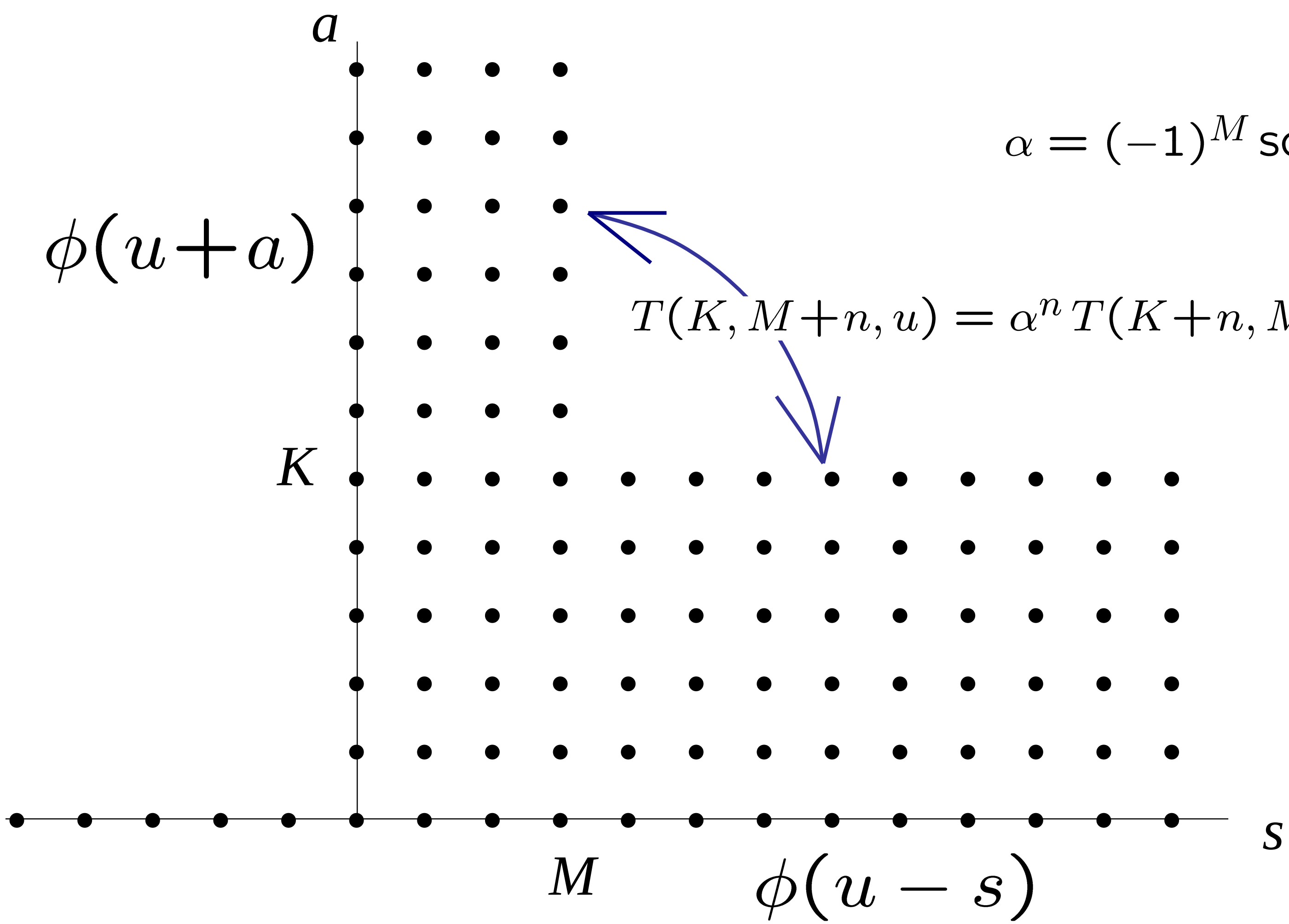


SUSY boundary conditions

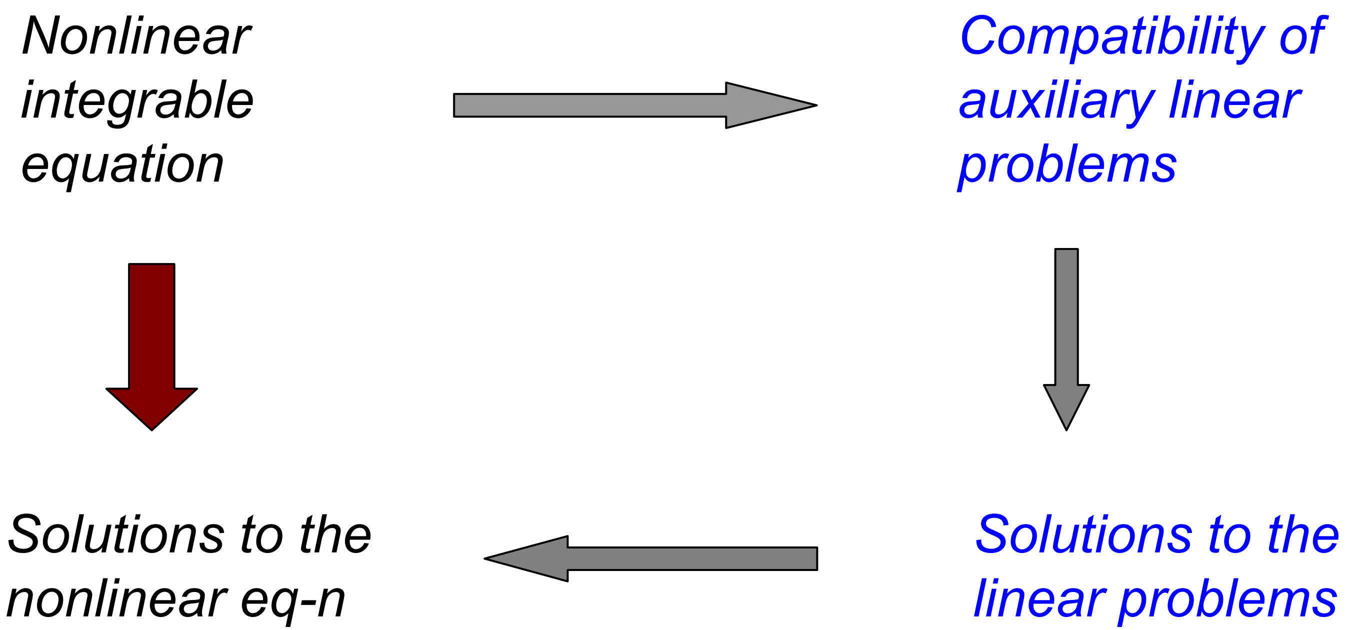
Z.Tsuboi 1997
(at g = id)

$\alpha = (-1)^M \,\mathrm{sdet}\, g$

$$T(K, M+n, u) = \alpha^n\, T(K+n, M, u)$$



The standard classical scheme



We are going to apply this scheme to the Hirota equation for transfer matrices

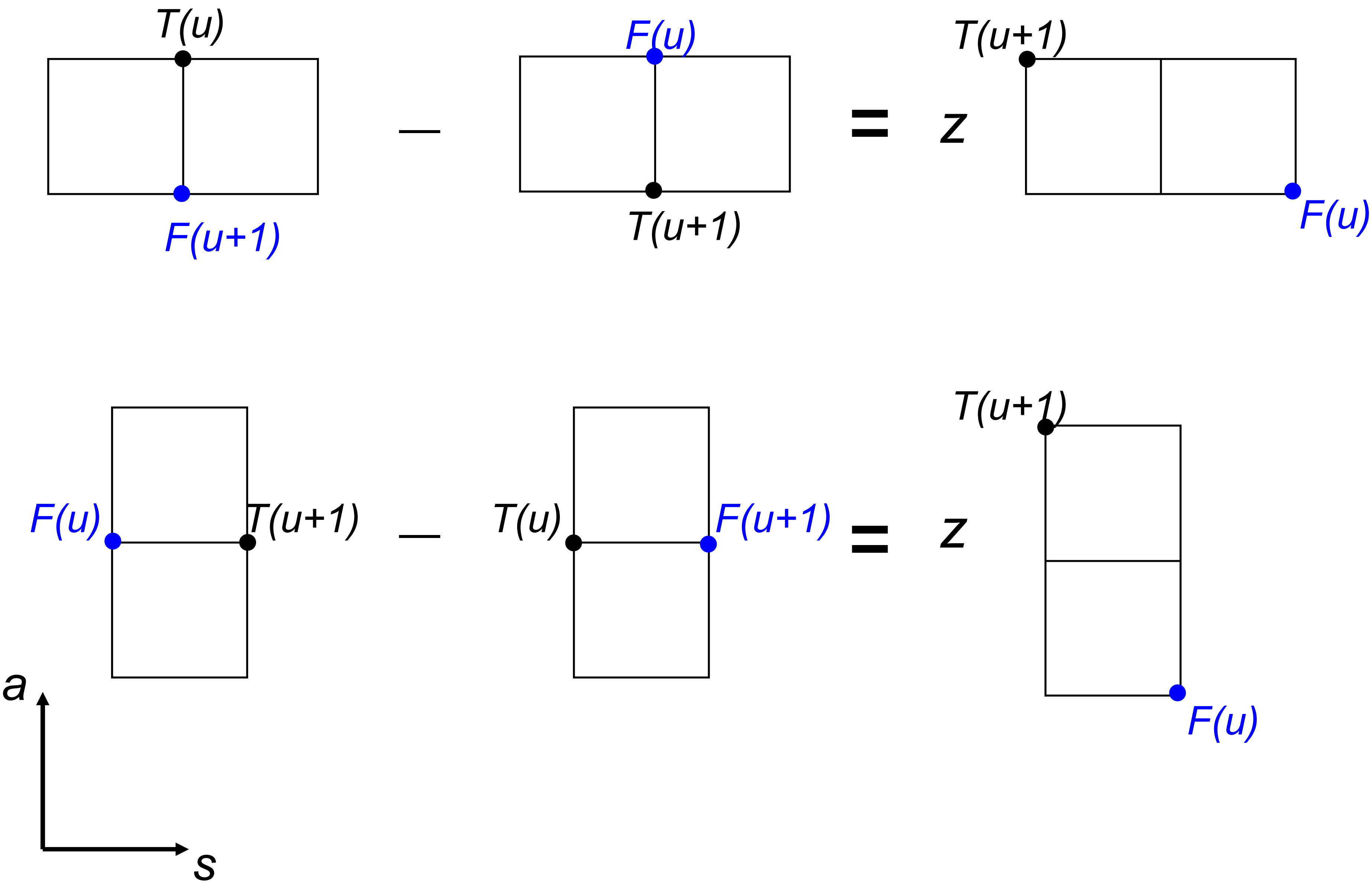
Auxiliary linear problems (Lax pair)

$$T(a+1,s,u)F(a,s,u+1) - T(a,s,u+1)F(a+1,s,u) = zT(a+1,s-1,u+1)F(a,s+1,u)$$

$$T(a,s+1,u+1)F(a,s,u) - T(a,s,u)F(a,s+1,u+1) = zT(a+1,s,u+1)F(a-1,s+1,u)$$

for any z
(classical spectral parameter)

The Hirota equation for *T(a,s,u)*



T(u)
F(u+1)
—
F(u)
T(u+1)
=
z
T(u+1)
F(u)
F(u)
T(u+1)
—
T(u)
F(u+1)
=
z
T(u+1)
F(u)
a
s

Backlund transformations

Simple but important fact:

F obeys the same Hirota equation

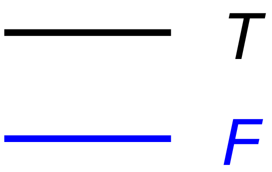
Therefore,

$$T \longrightarrow F$$

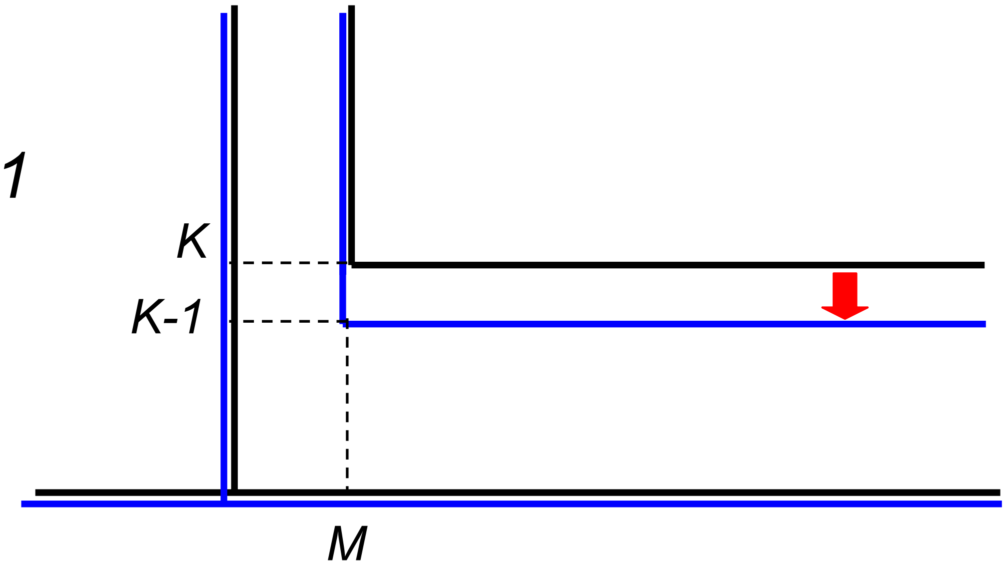
is a Backlund transformation

Boundary conditions for F - ?

There are two possibilities:

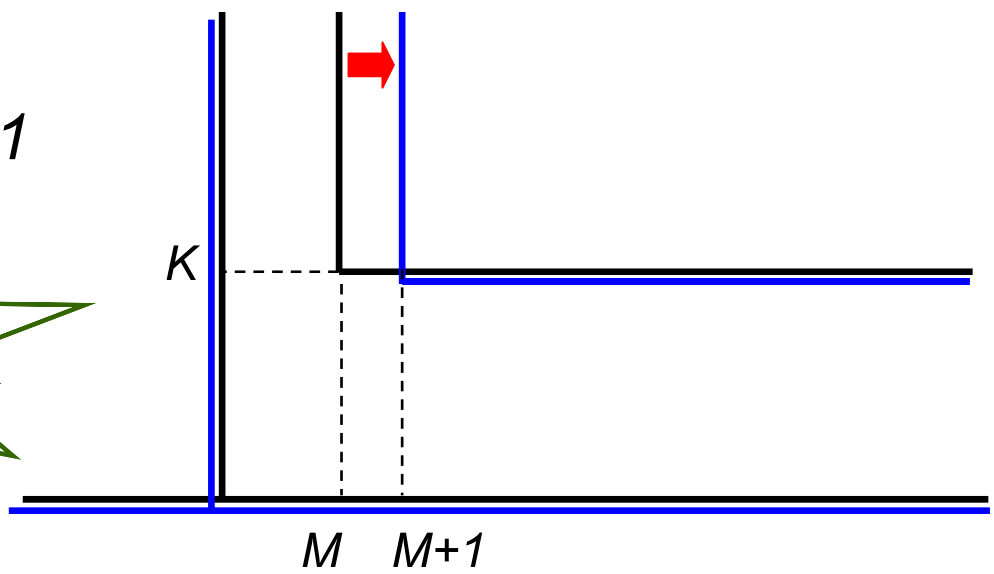


1) $K \rightarrow K\text{-}1$



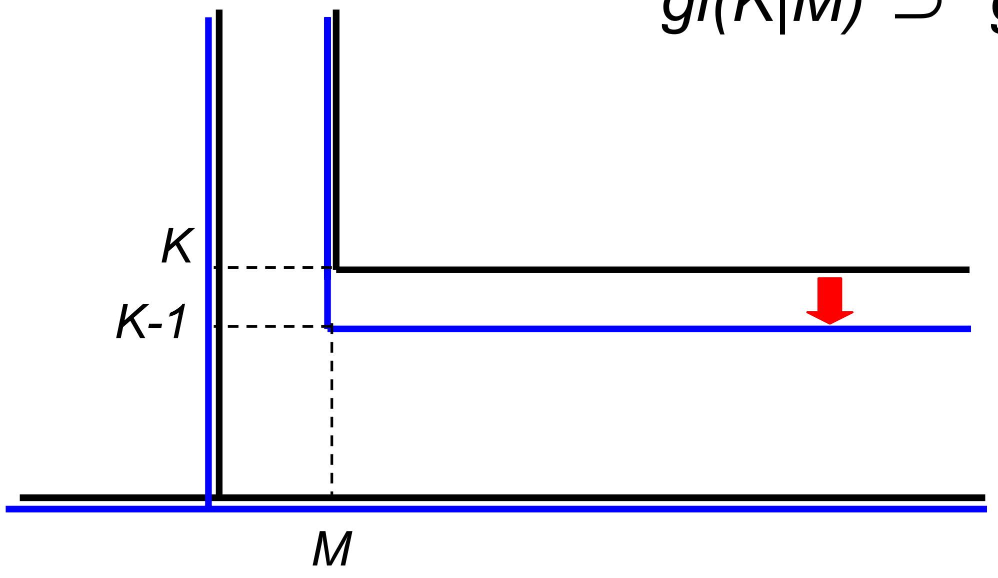
2) $M \rightarrow M+1$

We need inverse transform!



Undressing by BT-I: vertical move

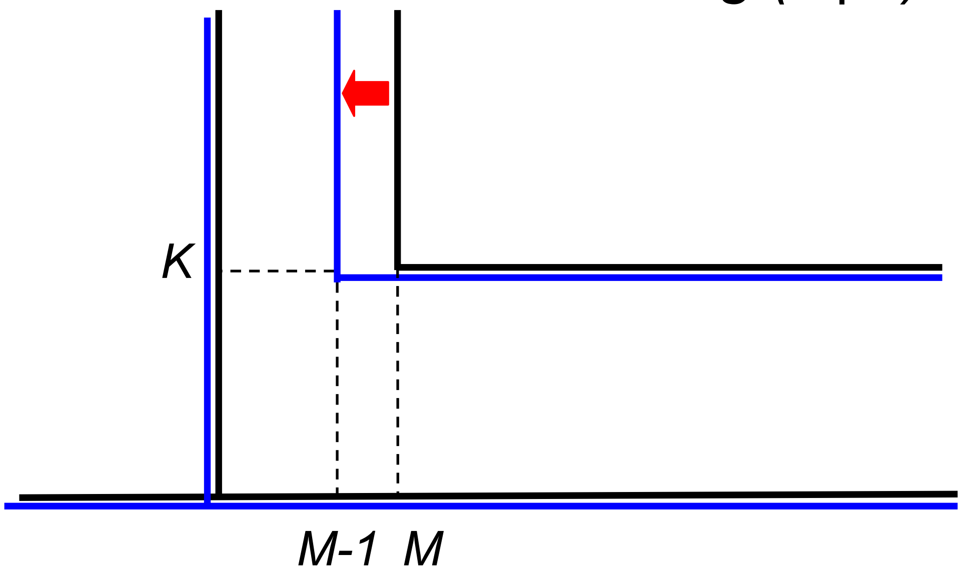
$gl(K|M) \supset gl(K\text{-}1|M)$



Notation: $T(a,s,u) \equiv T_{K,M}(a,s,u) \rightarrow F(a,s,u) \equiv T_{K\text{-}1,M}(a,s,u)$

Undressing by BT-II: horizontal move

$gl(K|M) \supset gl(K|M\text{-}1)$



Notation: $T(a,s,u) \equiv T_{K,M}(a,s,u) \rightarrow F(a,s,u) \equiv T_{K,M-1}(a,s,u)$

Repeating BT-I and BT-II several times, we introduce the hierarchy of functions

$$T_{k,m}(a,s,u)$$

k = 1,… , K;
m = 1,… , M

such that

- They obey the Hirota equation at any "level" *k,m* and are polynomials in *u* for any *a, s, k, m*

- They are connected by the Backlund transfromations

$$\text{BT-I}: \; T_{k,m}(a,s,u) \longrightarrow T_{k-1,m}(a,s,u)$$

$$\text{BT-II}: \; T_{k,m}(a,s,u) \longrightarrow T_{k,m-1}(a,s,u)$$

- At the highest level $T_{K,M}(a,s,u) = T(a,s,u)$

- At the lowest level $T_{0,0}(a,s,u) = 1$ at $a = 0$ or at $s = 0$, $a \geq 0$, and 0 otherwise

trivial solution in the degenerate domain

The idea is to "undress" the problem to the trivial one using a chain of Backlund transformations

At each step we have a continuous parameter *z*

Put $\begin{cases} z = x_k & \text{for BT-I} \quad (k,m) \to (k-1,m) \\ z = y_m & \text{for BT-II} \quad (k,m) \to (k,m-1) \end{cases}$

$$
\begin{array}{ccccccccc}
\cdots & \longleftarrow & T_{K,M-3} & \xleftarrow{y_{M-2}} & T_{K,M-2} & \xleftarrow{y_{M-1}} & T_{K,M-1} & \xleftarrow{y_M} & T_{K,M} \equiv T \\
 & & \downarrow x_K & & \downarrow x_K & & \downarrow x_K & & \downarrow x_K \\
 & & \cdots & \xleftarrow{y_{M-2}} & T_{K-1,M-2} & \xleftarrow{y_{M-1}} & T_{K-1,M-1} & \xleftarrow{y_M} & T_{K-1,M} \\
 & & & & \downarrow x_{K-1} & & \downarrow x_{K-1} & & \downarrow x_{K-1} \\
 & & & & \cdots & \xleftarrow{y_{M-1}} & T_{K-2,M-1} & \xleftarrow{y_M} & T_{K-2,M} \\
 & & & & & & \downarrow x_{K-2} & & \downarrow x_{K-2} \\
 & & & & & & \cdots & \xleftarrow{y_M} & T_{K-3,M} \\
 & & & & & & & & \downarrow \\
 & & & & & & & & \cdots
\end{array}
$$

"Undressing" to $T_{0,0}$:

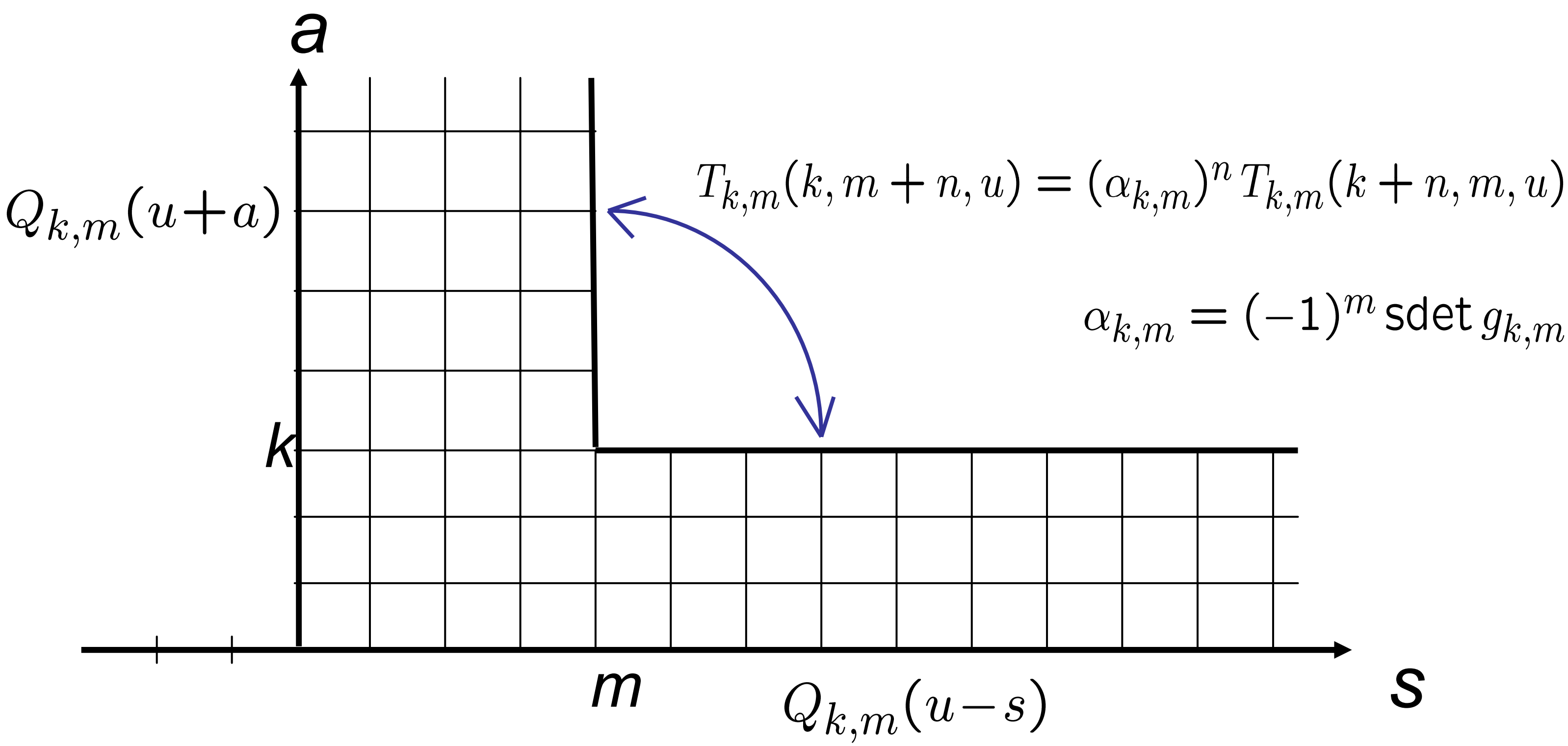
BT-I:

$$T_{k,m}(a+1,s,u)T_{k-1,m}(a,s,u+1) - T_{k,m}(a,s,u+1)T_{k-1,m}(a+1,s,u)$$
$$= \; x_k\, T_{k,m}(a+1,s-1,u+1)T_{k-1,m}(a,s+1,u),$$
$$T_{k,m}(a,s+1,u+1)T_{k-1,m}(a,s,u) - T_{k,m}(a,s,u)T_{k-1,m}(a,s+1,u+1)$$
$$= \; x_k\, T_{k,m}(a+1,s,u+1)T_{k-1,m}(a-1,s+1,u)$$

BT-II:

$$T_{k,m-1}(a+1,s,u)T_{k,m}(a,s,u+1) - T_{k,m-1}(a,s,u+1)T_{k,m}(a+1,s,u)$$
$$= \; y_m\, T_{k,m-1}(a+1,s-1,u+1)T_{k,m}(a,s+1,u),$$
$$T_{k,m-1}(a,s+1,u+1)T_{k,m}(a,s,u) - T_{k,m-1}(a,s,u)T_{k,m}(a,s+1,u+1)$$
$$= \; y_m\, T_{k,m-1}(a+1,s,u+1)T_{k,m}(a-1,s+1,u)$$

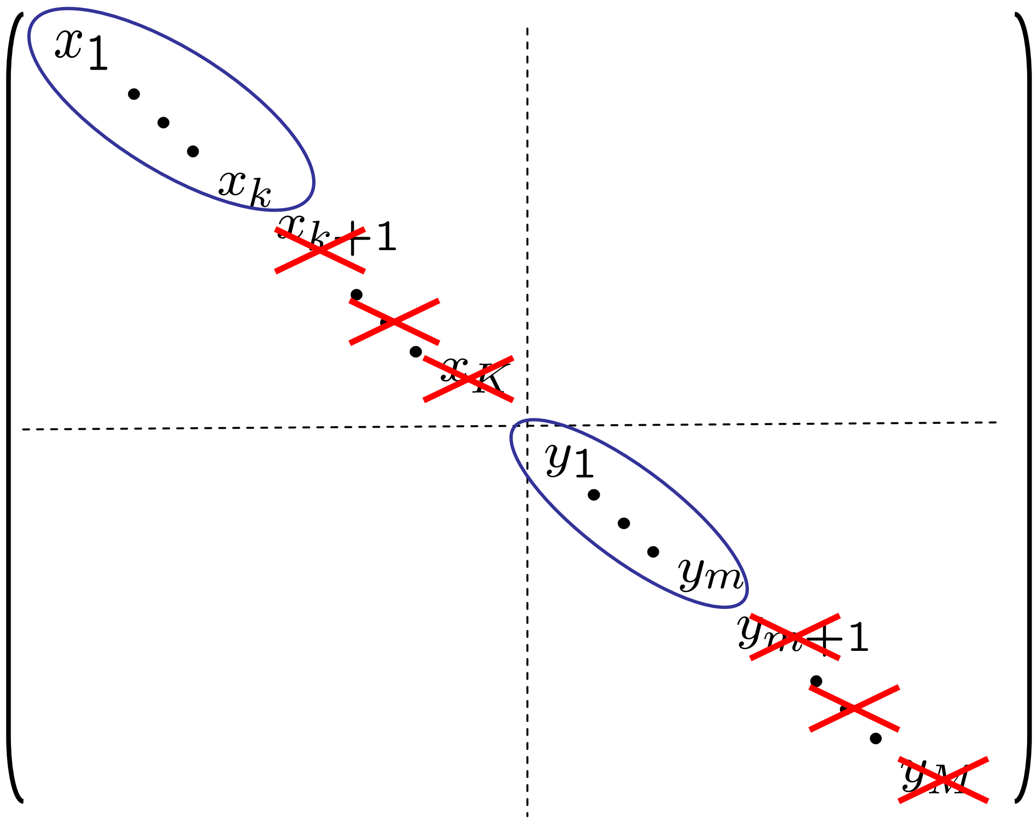
Boundary conditions at intermediate levels *k=1, … , K, m=1, … , M*



$$Q_{K,M}(u) = \phi(u) = \prod_{j=1}^{N} (u - \xi_j)$$ *fixed polynomial*

$$Q_{k,m}(u) = \prod_{j=1}^{N_{k,m}} (u - u_j^{(k,m)})$$ *eigenvalues of Baxter's Q-operators*

$$g_{k,m} =$$



$$\operatorname{sdet} g_{k,m} = \frac{x_1 \dots x_k}{y_1 \dots y_m}$$

Operator generating series

Pseudo-difference operator

$$\mathcal{W}(u) = \sum_{s\geq 0} \frac{T(1,s,u+s-1)}{\phi(u)}\, e^{2s\partial_u}$$

for normalization

Similar object at any level *k,m:*

$$\mathcal{W}_{k,m}(u) = \sum_{s\geq 0} \frac{T_{k,m}(1,s,u+s-1)}{Q_{k,m}(u)}\, e^{2s\partial_u}$$

It is clear that

$$\mathcal{W}_{0,0}(u) = 1$$

Operator form of the Backlund transformations

Backlund transformations BT-I and BT-II can be represented as recurrence relations for the $\mathcal{W}_{k,m}(u)$

$$\mathcal{W}_{k-1,\,m}(u) = \left(1 - X_{k,m}(u)e^{2\partial_u}\right)\mathcal{W}_{k,m}(u)$$

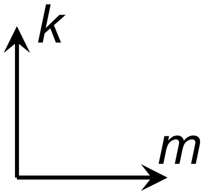
$$\mathcal{W}_{k,\,m+1}(u) = \left(1 - Y_{k,m+1}(u)e^{2\partial_u}\right)\mathcal{W}_{k,m}(u)$$

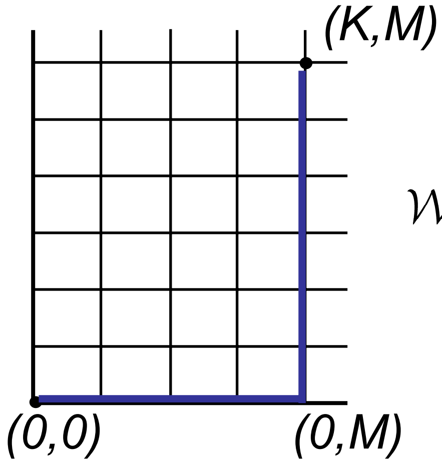
where

$$X_{k,m}(u) = x_k\,\frac{Q_{k,m}(u+2)\,Q_{k-1,m}(u-2)}{Q_{k,m}(u)\;Q_{k-1,m}(u)}$$

$$Y_{k,m}(u) = y_m\,\frac{Q_{k,m-1}(u+2)\,Q_{k,m}(u-2)}{Q_{k,m-1}(u)\;Q_{k,m}(u)}$$

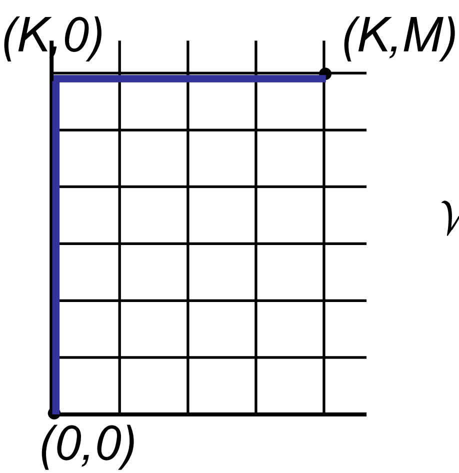
Factorization formulas





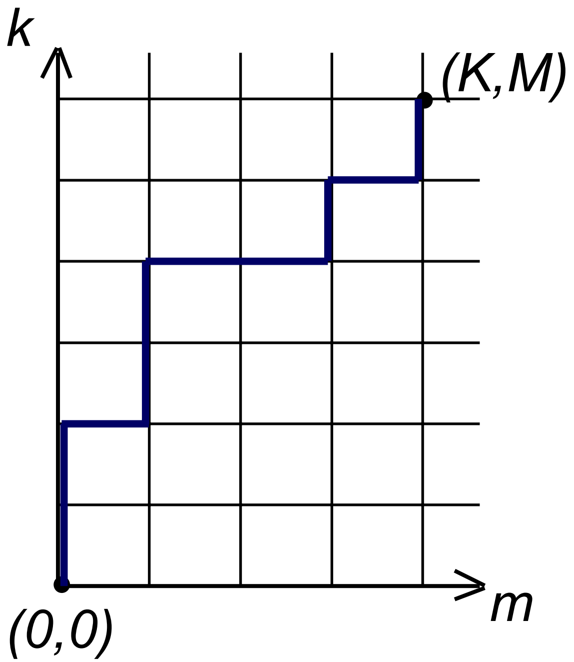
$$\mathcal{W}_{K,M}(u) = \prod_{K\geq k\geq 1}^{\leftarrow} \left(1 - X_{k,M}(u) e^{2\partial_u}\right)^{-1} \cdot \prod_{M\geq m\geq 1}^{\leftarrow} \left(1 - Y_{0,m}(u) e^{2\partial_u}\right)$$

Ordered product: $\prod_{J\geq i\geq I}^{\leftarrow} A_i = A_J\, A_{J-1} \ldots A_{I+1}\, A_I$



$$\mathcal{W}_{K,M}(u) = \prod_{M\geq m\geq 1}^{\leftarrow} \left(1 - Y_{K,m}(u) e^{2\partial_u}\right) \cdot \prod_{K\geq k\geq 1}^{\leftarrow} \left(1 - X_{k,0}(u) e^{2\partial_u}\right)^{-1}$$

Arbitrary undressing path



Each step $(k,m)\to(k+1,m)$ *brings*

$$\left(1 - X_{k+1,m}(u)e^{2\partial_u}\right)^{-1}$$

Each step $(k,m)\to(k+1,m)$ *brings*

$$\left(1 - Y_{k,m+1}(u)e^{2\partial_u}\right)$$

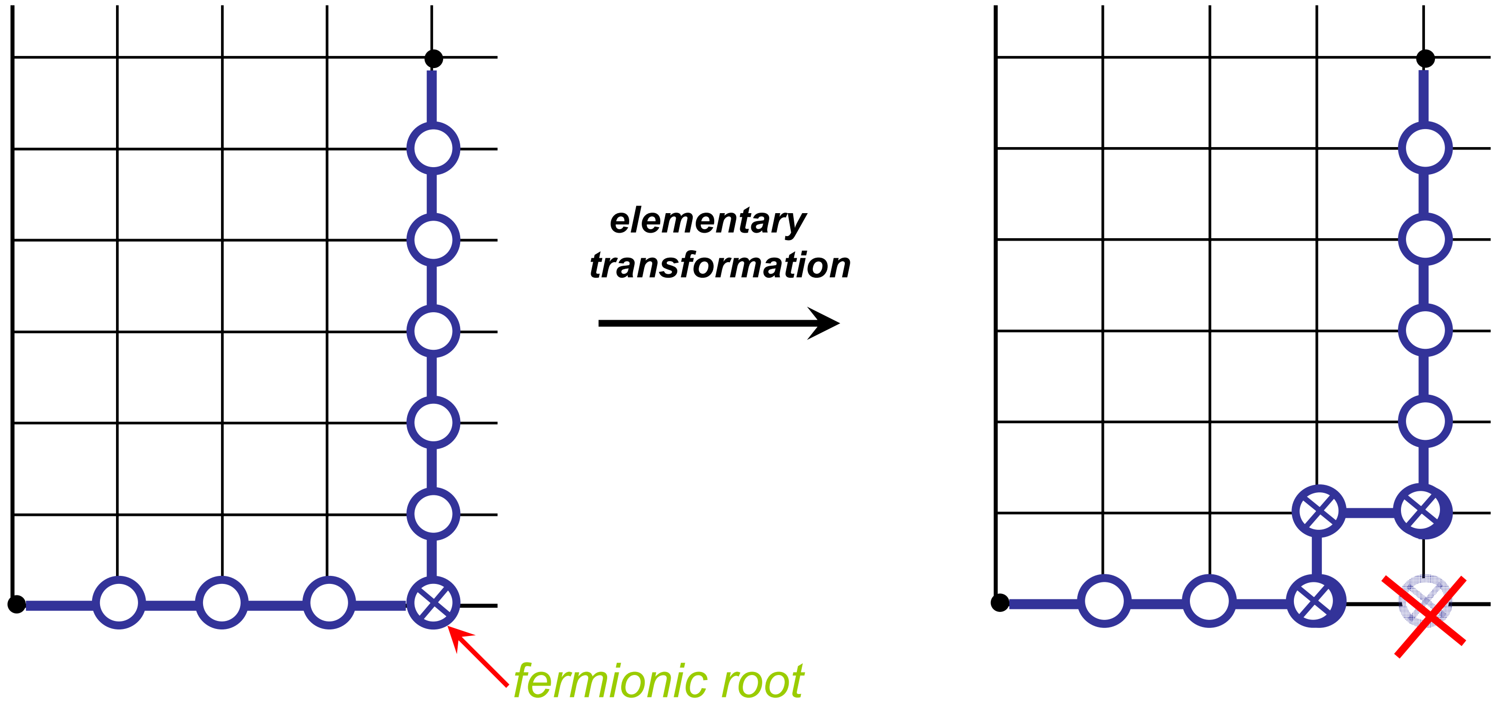
$\mathcal{W}_{K,M}(u) =$ *product of these factors along the path according to the order of the steps*

Equivalence of these representations follows from a discrete zero curvature condition

"Duality" transformations

Z.Tsuboi 1997;
N.Beisert, V.Kazakov,
K.Sakai, K.Zarembo 2005

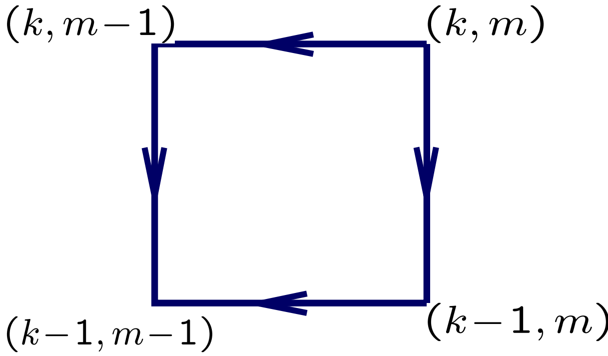
Undressing path ⟷ SUSY Dynkin diagram



The "duality" transformations can be most clearly understood in terms of **zero curvature condition** on the *(k,m)* lattice and the **QQ-relation** *(V.Kazakov, A.Sorin, A.Zabrodin 2007)*

"Zero curvature"

$$\left(1 - Y_{k-1,m+1}(u)e^{2\partial_u}\right)\left(1 - X_{k,m}(u)e^{2\partial_u}\right)$$
$$= \left(1 - X_{k,m+1}(u)e^{2\partial_u}\right)\left(1 - Y_{k,m+1}(u)e^{2\partial_u}\right)$$



The QQ-relation

$$x_k Q_{k-1,m-1}(u) Q_{k,m}(u+2) - y_m Q_{k,m}(u) Q_{k-1,m-1}(u+2)$$
$$= \quad (x_k - y_m)\, Q_{k-1,m}(u) Q_{k,m-1}(u+2)$$

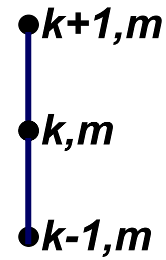
Again the Hirota equation!

We need polynomial solutions $Q_{k,m}(u) = \prod_{j=1}^{N_{k,m}} (u - u_j^{(k,m)})$

with the "boundary conditions" $Q_{0,0}(u) = 1, \; Q_{K,M}(u) = \phi(u)$

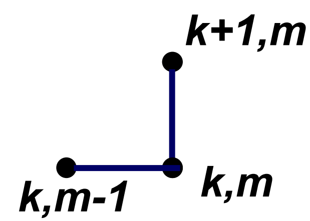
Building blocks for Bethe equations

$$\frac{Q_{k-1,m}\left(u_j^{(k,m)}\right) Q_{k,m}\left(u_j^{(k,m)}-2\right) Q_{k+1,m}\left(u_j^{(k,m)}+2\right)}{Q_{k-1,m}\left(u_j^{(k,m)}-2\right) Q_{k,m}\left(u_j^{(k,m)}+2\right) Q_{k+1,m}\left(u_j^{(k,m)}\right)} = -\frac{x_k}{x_{k+1}}$$

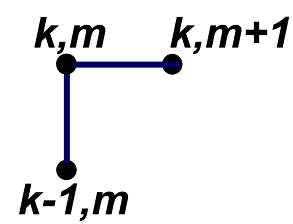


$$\frac{Q_{k,m+1}\left(u_j^{(k,m)}\right) Q_{k,m}\left(u_j^{(k,m)}-2\right) Q_{k,m-1}\left(u_j^{(k,m)}+2\right)}{Q_{k,m+1}\left(u_j^{(k,m)}-2\right) Q_{k,m}\left(u_j^{(k,m)}+2\right) Q_{k,m-1}\left(u_j^{(k,m)}\right)} = -\frac{y_{m+1}}{y_m}$$

$$\frac{Q_{k+1,m}\left(u_j^{(k,m)}\right) Q_{k,m-1}\left(u_j^{(k,m)}+2\right)}{Q_{k+1,m}\left(u_j^{(k,m)}+2\right) Q_{k,m-1}\left(u_j^{(k,m)}\right)} = \frac{x_{k+1}}{y_m}$$

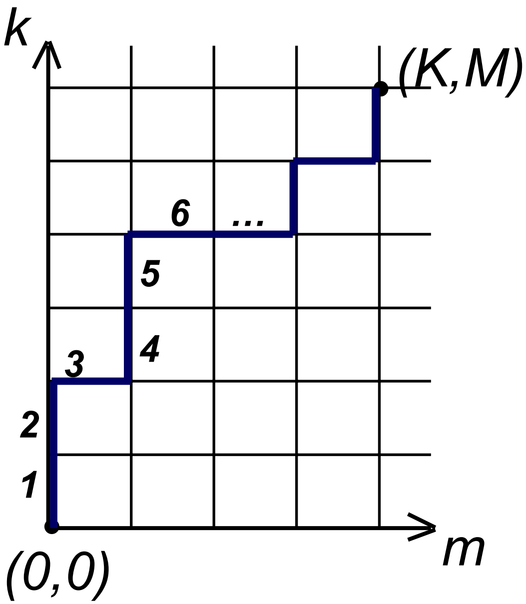


$$\frac{Q_{k,m+1}\left(u_j^{(k,m)}\right) Q_{k-1,m}\left(u_j^{(k,m)}-2\right)}{Q_{k,m+1}\left(u_j^{(k,m)}-2\right) Q_{k-1,m}\left(u_j^{(k,m)}\right)} = \frac{y_{m+1}}{x_k}$$



Bethe equations for arbitrary path

(with $x_k = y_m = 1$ for simplicity)



$$p_n = \begin{cases} 1 \text{ for } n \\ -1 \text{ for } n \end{cases}$$

$$Q_{k,m}(u) = \check{Q}_{k+m}(u - k + m)$$

$$\check{u}_j^{(n)} = u_j^{(n)} - k + m$$

$$\prod_{b=1}^{K+M} \frac{\check{Q}_b\left(\check{u}_j^{(a)} - K_{ab}\right)}{\check{Q}_b\left(\check{u}_j^{(a)} + K_{ab}\right)} = (-1)^{\frac{1+p_a p_{a+1}}{2}}$$

a=1, …, K+M-1

$K_{ab} = (p_a + p_{a+1})\delta_{a,b} - p_{a+1}\delta_{a+1,b} - p_a\delta_{a,b+1}$ *is the Cartan matrix*

Conclusion

- A lesson: SUSY case looks more natural and transparent

- Our approach provides an alternative to algebraic Bethe ansatz

- Generalizations and problems:

  - mixed (covariant + contravariant) irreps, infinite dimensional (non-compact) irreps

  - models with other types of *R*-matrices including non-standard ones like Hubbard or *su(2|2) R*-matrix in AdS/CFT