The (non-)linear integral equation for $sl(2)$ superconformal dimensions: the *status quæstionis* within integrability.

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Based on:
D. Bombardelli, D. F., M. Rossi, arXiv:0711.2934 and arXiv:0802.0027;
D. F., P. Grinza and M. Rossi, arXiv:0804.2893, arXiv:0805.4407 and arXiv:0808.1886;
F. Buccheri, D.F., arXiv:0805.4410.

Outline

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The Correspondence and Integrability

- AdS/CFT correspondence [Maldacena; Gubser, Klebanov, Polyakov; Witten]
  - $\mathcal{N} = 4$ SYM (SU(N) gauge theory)$\leftrightarrow$ IIB string theory on $AdS_5 \times S^5$
  - Operator dimension $\Delta = E_{string}$ energy of string configuration
- Integrability in planar limit ($N \to \infty$)
  - QCD: scattering of reggeised gluons and one-loop twist operators (the same as in SYM)[Lipatov; Braun; Korchemsky; ...]
  - $\mathcal{N}$=4 SYM scalar sector: 1-loop [Minahan, Zarembo]
  - $\mathcal{N}$=4 SYM 'full' theory [Beisert, Kristjansen, Staudacher....]
  - string σ-model [Bena, Polchinski, Roiban; Kazakov, Marshakov, Minahan, Zarembo....]
- Bethe Ansatz Equations (no clear physical Model!)
  - gauge/string bridge: **asymptotic** Bethe Ansatz equations with string dressing. Wrapping? [Staudacher, Arutyunov, Frolov, Serban, Kostov, Janik, DF....]
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A single object for many: the non-linear integral equation (NLIE)

- *Usual* NLIE [Destri, de Vega; DF, Ravanini....]
  - Scattering matrices depending on **difference** of rapidities: *gauge* $su(2)$ sector [Feverati, DF, Grinza, Rossi]
  - Bethe roots on the **whole** real axis: *string* $su(2)$ sector [DF, Rossi]
  - Yet, the method enforceable: $sl(2)$ sector with dressing [Freyhult, Rej, Staudacher]
  - Exact results for arbitrary size and parameters (e.g. L, s, g,....)
  - Very useful for
    - analytical computations, specifically in some important limits: e.g. **s**$\rightarrow \infty$, L$\rightarrow \infty$
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- *New* NLIE [Bombardelli, DF, Rossi]
  - Scattering matrices generally depending on rapidities: string $sl(2)$ sector (dressing factor)
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  - **Advantage**: non-linear terms $\sim 1/s$, $s \rightarrow \infty \Rightarrow$ **linear integral equation**

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- Gauge invariant (single trace) sl(2) operators:

$$\mathrm{O} = \mathrm{Tr}(\mathcal{D}^s \mathcal{Z}^L) + ...$$

- The dilatation operator

$$D\,\mathrm{O} = \Delta\,\mathrm{O}$$

- We would like to evaluate the anomalous part

$$\Delta = L + s + \gamma(g, s, L)$$

- at the **high spin** ($s \to \infty$) correction $O(s^0)$

$$\gamma(g, s, L) = f(g) \ln s + f_1(g, L) + O\left(\frac{1}{\ln s}\right)$$

- and also for the **high** spin, **long** operators $L \to \infty$, $j = L/\ln s =$**fixed**

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- Cusp anomalous dimension $f(g)$ **universal** (no wrapping): $f_1(g, L)$? Linearity.
- $f(g, j)$ allows for semi-classical computations ($g \to \infty$)[Frolov, Roiban, Tseytlin...]. In a particular limit: **integrable** $O(6)$ **sigma model**[Alday,Maldacena; DF, Grinza, Rossi;....].

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One loop dilatation operator and its Bethe Ansatz

- Dilatation operator = non-compact isotropic *XXX* spin $-\frac{1}{2}$ chain [Lipatov;..]
- Diagonalisation via a wave function Ansatz, which ought to be periodic after multi-scatering
- Bethe Ansatz equations (exploited alternative for the $\ln s$ scaling: Baxter Q operator and $T-Q$ relation[Belitsky, Gorsky, Korchemsky;...])

$$\left(\frac{u_k-\frac{i}{2}}{u_k+\frac{i}{2}}\right)^L=\prod_{\substack{j=1\\ j\neq k}}^{s}\frac{u_k-u_j+i}{u_k-u_j-i}$$

- Take the logarithm and thus define the **counting function**

$$Z_0(u)\equiv iL\ln\left(\frac{\frac{i}{2}-u}{\frac{i}{2}+u}\right)-i\sum_{j=1}^{s}\ln\left(\frac{i+u-u_j}{i-u+u_j}\right)\equiv\Phi_0(u)-\sum_{j=1}^{s}\phi_0(u,u_j)$$

- BAEs become

$$Z_0(u_k)=\pi(2n_k-L-s+1)\,,\quad n_k\in\mathbb{Z}$$
$$Z_0(u_h)=\pi(2n_h-L-s+1)\,,\quad n_h\in\mathbb{Z}$$

- All the Bethe roots concentrate on a finite symmetric interval ($-b_0$,b_0)
- The twist $L=H$ is the number of holes
- Two of these are external to the interval and scale as s: $u_h^{1,2}=\frac{s}{\sqrt{2}}+O(s^0)$
- $L-2$ internal holes $u_h=0+O(1/\ln s)$, $h=3,...,L$ for finite L ($s\to\infty$)

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The NLIE for the counting function

- The *usual* NLIE basic idea: logarithmic indicator formula

$$\sum_{k=1}^{s} O(u_k) = -\int_{-\infty}^{\infty} \frac{dv}{2\pi} O(v) Z_0'(v) + \int_{-\infty}^{\infty} \frac{dv}{\pi} O(v) \frac{d}{dv} \mathrm{Im} \ln \left[1 + e^{iZ_0(v-i0)}\right] - \sum_{h=1}^{H} O(u_h)$$

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$$Z_0(u) = f_0(u) - \int_{-b_0}^{b_0} \frac{dv}{2\pi} \frac{d}{dv}\phi_0(u,v)\, Z_0(v) + 2\int_{-b_0}^{b_0} \frac{dv}{2\pi} \frac{d}{dv}\phi_0(u,v) \,\mathrm{Im} \ln \left[1 + e^{iZ_0(v-i0)}\right]$$

$$= \underbrace{F_0(u)}_{forcing\ term} + 2 \underbrace{(G_0 \star L_0)(u)}_{non-linear\ convolution}$$

where (these are general formulæ, though!)

$$L_0(u) = \mathrm{Im} \ln \left[1 + e^{iZ_0(u+i0)}\right] , \quad G_0(u,v) = \varphi_0(u,v) + \sum_{k=2}^{\infty} (-1)^{k-1} (\varphi_0^{\star k})(u,v) ,$$

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The fate of the Non-Linear Terms

Any conserved charge

$$\sum_{k=1}^{s} O(u_k) = \int_{-b_0}^{b_0} \frac{dv}{2\pi} O'(v) F_0(v) - \sum_{h=1}^{L-2} O(u_h^{(i)}) + $$
$$+ \quad 2\int_{-b_0}^{b_0} \frac{dv}{2\pi} O'(v) \int_{-b_0}^{b_0} dw [G_0(v,w) - \delta(v-w)] L_0(w)$$

- Advantage: non-linear terms go *rapidly* to zero (faster than any $(1/\ln s)^n$)

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where $b_0 = b_0(s)$ is a separator: $Z_0(b_0) = -\pi(s+L-2)$

- $\Longrightarrow$ They do not affect the logarithmic scaling $\ln s$ at high spin s !
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The Linear Integral Equation for one loop

$$Z_0 = F_0(u) + 2(G_0 \star L_0)(u) \approx F_0(u)$$

- about which, if Fourier transform is **not** enforceable,

$$F_0(u) = f(u) - \int_{-b_0}^{b_0} dv \varphi_0(u,v) F_0(v)$$

- In terms of the root density $\sigma_0(u) = \frac{d}{du} F_0(u)$

$$\sigma_0(u) = -\frac{4L}{1+4u^2} + 2\sum_{h=1}^{L-2} \frac{1}{1+(u-u_h^{(i)})^2} + \int_{-b_0}^{b_0} \frac{dv}{\pi} \frac{1}{1+(u-v)^2} \sigma_0(v)$$

- Large s, up to $O(1/\ln s)$ – coming **only** from the internal holes –, finite L

$$\hat{\sigma}_0(k) = -4\pi \frac{\frac{L}{2} - e^{-\frac{|k|}{2}} \cos \frac{ks}{\sqrt{2}}}{2 \sinh \frac{|k|}{2}} + 2\pi(L-2) \frac{e^{-\frac{|k|}{2}}}{2 \sinh \frac{|k|}{2}} - (4\pi \ln 2)\delta(k)$$

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One-loop conserved charges

- Single particle expression

$$q_r(v) = \frac{i}{r-1}\left[\frac{1}{\left(v+\frac{i}{2}\right)^{r-1}} - \frac{1}{\left(v-\frac{i}{2}\right)^{r-1}}\right]$$

- then the eigenvalues of the charges are given by

$$Q_r = -\int_{-b_0}^{b_0} \frac{dv}{2\pi} q_r(v)\sigma_0(v) - \sum_{h=1}^{H_i} q_r(u_h^{(i)}) + o(s^0)$$

- in particular for the energy, beyond the leading logarithm

$$Q_2 = E = 4\ln s + 4(\gamma_E - (L-2)\ln 2) + o(s^0)$$

- mathes exact Q results for $L = 2, 3$ [Beccaria; Kotikov,Lipatov,Rej,Staudacher,Velizhanin]

Many loops in blue, string factor in red

- The asymptotic sl(2) BAEs [Arutyunov, Frolov, Staudacher, Beisert, Eden...]

$$\left(\frac{u_k+\frac{i}{2}}{u_k-\frac{i}{2}}\right)^L \left(\frac{1+\frac{g^2}{2x_k^{-2}}}{1+\frac{g^2}{2x_k^{+2}}}\right)^L = \prod_{\substack{j=1 \\ j\neq k}}^{s} \frac{u_k-u_j-i}{u_k-u_j+i}\left(\frac{1-\frac{g^2}{2x_k^+x_j^-}}{1-\frac{g^2}{2x_k^-x_j^+}}\right)^2 e^{2i\theta(u_k,u_j)}$$

where $x_k^{\pm}=x(u_k\pm i/2)\,,\ x(u)=\frac{u}{2}\left[1+\sqrt{1-\frac{2g^2}{u^2}}\right]$ and $\theta(u,v)$ is the dressing phase

- A LIE for $F(u)=F_0(u)+F^H(u)$, 0=1-loop, H=higher loops, since F_0 solved
- $\Rightarrow$ LIE for higher loop density $\sigma^H(u)=\frac{d}{du}F^H(u)$

$$\sigma^H(u)=-iL\frac{d}{du}\ln\left(\frac{1+\frac{g^2}{2x^-(u)^2}}{1+\frac{g^2}{2x^+(u)^2}}\right)-\frac{i}{\pi}\int_{-b_0}^{b_0}dv\,\frac{d}{du}\left[\ln\left(\frac{1-\frac{g^2}{2x^+(u)x^-(v)}}{1-\frac{g^2}{2x^-(u)x^+(v)}}\right)+i\theta(u,v)\right](\sigma_0(v)+2\pi\delta(v))$$

$$+\int_{-\infty}^{+\infty}\frac{dv}{\pi}\left[\frac{1}{1+(u-v)^2}-i\frac{d}{du}\left[\ln\left(\frac{1-\frac{g^2}{2x^+(u)x^-(v)}}{1-\frac{g^2}{2x^-(u)x^+(v)}}\right)+i\theta(u,v)\right]\right]\sigma^H(v)$$

with the **same BES kernel** as for $f(g)$, but **different inhomogeneous term** (in the first line)

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The anomalous dimension or energy

- Only the leading term of the inhomogeneous term: the BES equation for the scaling function $f(g)$

$$\gamma(g,s,L) = g^2 E(g,s,L) = f(g) \ln s + f_1(g,L) + O\left(\frac{1}{\ln s}\right)$$

- Next to the leading order, i.e. $f_1(g,L)$

$$E(g,s,L) = h(g,s,L) - \int_{-\infty}^{+\infty} \frac{dv}{2\pi} \left[\frac{i}{x^+(v)} - \frac{i}{x^-(v)}\right] \sigma^H(v) + o(s^0)$$

whith the one loop contribution

$$h(g,s,L) = -\int_{-b_0}^{b_0} \frac{dv}{2\pi} \left[\frac{i}{x^+(v)} - \frac{i}{x^-(v)}\right] \sigma_0(v) - (L-2)\left[\frac{i}{x^+(0)} - \frac{i}{x^-(0)}\right]$$

- An all loop contribution proportional to the BES density $\sigma^{BES}(u)$

$$\sigma^H(u) = \frac{h(g,s,L)}{4\ln s}\sigma^{BES}(u) + \sigma^{extra}(u)$$

- entails, finally, the better form

$$E(g,s) = \frac{1}{4} h(g,s,L) f(g) + E^{extra}(g,s) + o(s^0)$$

The anomalous dimension at weak coupling

- No further structure beyond BES part, but small g expansion

$$E^{extra}(g,s) = -2g^2 L\zeta(3) + \frac{2}{3}\pi^2 g^4 L\zeta(3) - \pi^2 g^4\zeta(3) + 9Lg^4\zeta(5) + O(g^6)$$

$$h(g,s,L) = 4\ln s + 4\gamma_E - 4(L-2)\ln 2 + g^2(6L-14)\zeta(3) + g^4(62-30L)\zeta(5) + \dots$$

- Agreement with the results at weak coupling from the NLIE by [Freyhult, Rej, Staudacher]
- Kotikov-Lipatov property for the scaling function $f(g)$ can be extended for the whole range of g

$$\hat{\sigma}^H(0) = \pi g^2 E(g,s) = \pi\gamma(g,s)$$

- The LIE controls also all the $O(1/\ln s)$ corrections up to $O(\ln s/s)$ (determined by the non-linear integrals)

The other consercved charges

- Deformation of the XXX charges, $r = 3, 4, \ldots.$

$$Q_r(g,s) = \frac{i}{r-1} \sum_{k=1}^{s} \left[\left(\frac{1}{x^+(u_k)} \right)^{r-1} - \left(\frac{1}{x^-(u_k)} \right)^{r-1} \right]$$

- Weak coupling expansion

$$Q_r(g,s) = Q_{r,0}(s) + Q_{r,1}(s) g^2 + Q_{r,2}(s) g^4 + O(g^6)$$

$$Q_{r,0}(s) = \frac{2(-1)^{\frac{r}{2}-1}\zeta(r-1)}{r-1}[(2-2^{r-1})L - 2(1-2^{r-1})] + o(s^0)\,,$$

$$Q_{r,1}(s) = 4(-1)^{\frac{r}{2}}\zeta(r)[\ln s + \gamma_E - (L-2)\ln 2] + $$
$$+ L(-1)^{\frac{3r}{2}}(r+2-2^{r+1})\zeta(r+1) + 2(-1)^{\frac{3r}{2}}(2^{r+1}-1)\zeta(r+1) + o(s^0)\,,$$

$Q_{r,2}(s) =$ *…did not fit the page…..*

Outline

The generalised scaling functions

- Let us consider the double scaling limit [Belitski, Gorksy, Korchemksy; Frolov, Tseytlin]

$$s \to \infty \, , \; L \to \infty \, , \; j = \frac{L}{\ln s} \text{ fixed}$$

- The anomalous dimension enjoys the form

$$\gamma(g, s, L) = f(g, j) \ln s + \dots$$

- Let us study the generalised scaling function as this expansion

$$f(g, j) = \sum_{n=0}^{\infty} f_n(g) j^n = f_0(g) + f_1(g) j + f_2(g) j^2 + \dots \qquad (1)$$

- $f_0(g) = f(g)$. Instead $f_1(g)$ comes out by keeping only terms proportional to L, $f_2(g)$ to L^2, etc.:

$$\sigma_H(u) = \left[\sum_{n=0}^{\infty} \sigma_H^{(n)}(u) j^n \right] \ln s + \dots$$

- Let us focus on $f_1(g)$: then we have proved that in general for $f_n(g)$ similar situation.

The first generalised scaling function $f_1(g)$

[DF, Grinza, Rossi]

- For any $f_n(g)$: kernel still BES, i.e. Bessel functions (from dressing $\theta(u,v)$ and $x^{\pm}(u)$). Inhomogeneous term changes with n and depends on $\sigma_H^{(n-2)}(u)$ and $\sigma_0(u) \Rightarrow$ only one-loop density for $f_1(g)$!

$$s(k) = \frac{\sinh(|k|/2)}{\pi|k|} \hat{\sigma}_H^{(1)}(k) \Longrightarrow f_1(g) = 2s(0)$$

- $\Rightarrow$ solution as Neumann-Bessel series

$$s(k) = \sum_{p=1}^{\infty} s_p(g) \frac{J_p(\sqrt{2}gk)}{k}$$

- Equivalent to an infinite linear system

$$\frac{s_{2p}(g)}{2p} = -\bar{a}_{2p}(g) - 2\sum_{m=1}^{\infty} Z_{2p,2m}(g)(2+s_{2m}(g)) + 2\sum_{m=1}^{\infty} Z_{2p,2m-1}(g)s_{2m-1}(g)$$

$$\frac{s_{2p-1}(g)}{2p-1} = -a_{2p-1}(g) - 2\sum_{m=1}^{\infty} Z_{2p-1,2m}(g)(2+s_{2m}(g)) - 2\sum_{m=1}^{\infty} Z_{2p-1,2m-1}(g)s_{2m-1}(g)$$

where $a_r(g) = \int_{-\infty}^{+\infty} \frac{dh}{h} J_r(\sqrt{2}gh) \frac{1}{1+e^{\frac{|h|}{2}}}$, $\bar{a}_r(g) = \int_{-\infty}^{+\infty} \frac{dh}{|h|} J_r(\sqrt{2}gh) \frac{1}{1+e^{\frac{|h|}{2}}}$

$Z_{n,m}(g) = \int_0^{\infty} \frac{dh}{h} \frac{J_n(\sqrt{2}gh)J_m(\sqrt{2}gh)}{e^h - 1}$

Exact asymptotic expansions

- Strong coupling expansion

$$s_{2m}(g) \doteq \sum_{n=0}^{\infty} \frac{s_{2m}^{(n)}}{g^n} \,, \quad s_{2m-1}(g) \doteq \sum_{n=0}^{\infty} \frac{s_{2m-1}^{(n)}}{g^n}$$

- All-order solution

$$s_{2m}^{(2n)} = -2m \frac{\Gamma(m+n)}{\Gamma(m-n+1)} 2^{-n} \sum_{k=0}^{n} \frac{E_{2k} 2^{2k}}{(2k)!(2n-2k)!} \,,$$

$$s_{2m-1}^{(2n)} = 0 \,; \; n \geq 0 \,, \quad m \geq 1 \,,$$

$$s_{2m}^{(2n-1)} = 0 \,,$$

$$s_{2m-1}^{(2n-1)} = -(2m-1) \frac{\Gamma(m+n-1)}{\Gamma(m-n+1)} 2^{-n+\frac{1}{2}} \sum_{k=0}^{n-1} \frac{E_{2k} 2^{2k}}{(2k)!(2n-2k-1)!} \,; \; n \geq 1 \,, \quad m \geq 1$$

$$\Downarrow$$

$$f_1(g) \doteq -1$$

- Agreement with semi-classical ($g \to \infty$) string expansion ($j \sim g$), though different order of limits

$$\Delta - s - L = \gamma(g, s, L) = \Big(f(g) - j + O(j^3) \Big) \ln s + \ldots$$

Non-perturbative terms and numerical support

- Matrix system may be solved numerically and furnishes

$$f_1(g) = -1 + m(g) = -1 + \kappa\, g^{1/4}\, e^{-\frac{\pi}{\sqrt{2}}\, g}, \quad g \to \infty$$

  with $\kappa_{fit} = 2.26530 \pm 0.00015$
- More recent analytic work: $\kappa = 2^{5/8}\pi/\Gamma(5/4) = 2.265218666...$, perfect
- Alday-Maldacena proposal: world sheet theory of string associated with highly spinning operators (light-like Wilson loops) at low energy $\Leftrightarrow$ only massless bosonic excitations on a S^5, i.e. integrable $O(6)$ model when

$$j \ll g \to +\infty$$

- They analysed $O(6)$ small coupling $m \ll j$ for comparing with semi-classical string computations in the limit $y = j/g \ll 1$ fixing

$$m(g) = \kappa\, g^{1/4}\, e^{-\frac{\pi}{\sqrt{2}}\, g}(1 + O(1/g)), \quad g \to \infty$$

- $O(6)$ limit still valid in strong coupling $j \ll m$: S-matrix BA [Buccheri, DF]

$$f(j,g) = 2\epsilon_{O(6)}(j/2,g) = m(g)\, j + O(j^3)$$

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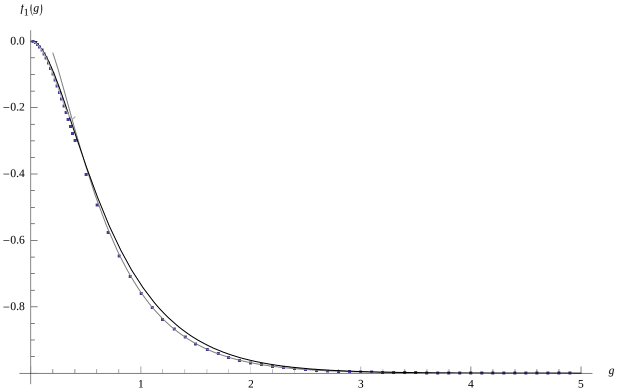
An attempt at summing

- A *wrong* summation for interpolating weak and strong coupling

$$\tilde{f}_1(g) = -2\sqrt{2}g\int_0^\infty \frac{dt}{t}\frac{\sinh t}{\cosh 2t}J_1(2\sqrt{2}gt)$$

- Yet, the correct exponential non-perturbative $e^{-\pi g/\sqrt{2}}$ at strong coupling (cf. $O(6)$ mass gap) and a good estimate at all g



Recent developments

[DF, Grinza, Rossi]

- As $L \to \infty$ internal holes open a gap both at 1-loop, $(-c_0, c_0)$ and at higher loops $(-c, c) \Rightarrow$ expansion of forcing terms and normalisations:

$$\int_{-c_0(j)}^{c_0(j)} du \sigma_0(u) = -2\pi(L-2) + \ldots ; \quad \int_{-c(j)}^{c(j)} du \left(\sigma_0(u) + \sigma^H(u)\right) = -2\pi(L-2) + \ldots$$

- Higher order generalised scaling functions from the iteration on n in

$$\sigma_H(u) = \left[\sum_{n=0}^{\infty} \sigma_H^{(n)}(u) j^n\right] \ln s + \ldots$$

- $f_2(g) = 0 \leftrightarrow$ [Freyhult, Rej, Staudacher]
- $f_3(g) = \frac{\pi^2}{24\, m(g)} \leftrightarrow$ [Basso, Korchemsky]
- $f_4(g) = -\frac{\pi}{24\, m(g)^2}(2\ln 2 + \pi) \leftrightarrow$ interaction and $O(6)$ and not reducible to BES
- Exact $O(6)$ energy expansion in the IR $j \ll m$: checks and predictions [Buccheri, DF]

$$\begin{aligned} f(j,g) &= 2\epsilon_{O(6)}(j/2, g) = m(g)\, j + \frac{\pi^2}{24\, m(g)} j^3 - \frac{\pi}{24\, m(g)^2}(2\ln 2 + \pi)\, j^4 + \\ &+ \frac{\pi^4}{16\, m(g)^3}\left[-\frac{1}{40} + \frac{1}{\pi^3}\left(\frac{2\ln 2 + \pi}{\pi}\right)^2\right] j^5 + \ldots \end{aligned}$$

- SYSTEMATICS on $f_5(g)$, $f_6(g)$, $f_7(g)$, $f_8(g)$ etc.: 1)recursive system; 2) reducible and not to BES.

Outline

Summary...

- A new kind of NLIE on finite intervals: necessary in SYM Bethe Ansatz
- Applied to the Bethe Ansatz equations of the sl(2) sector of $\mathcal{N} = 4$ SYM
- Calculations of high spin subleading corrections to the conserved charges, at weak and strong coupling
- Computation of the limit high spin, s, large twist, L, fixed ratio $j = L/\ln s$, at weak and strong coupling. And use of the S-matrix Bethe Ansatz.
- Comparisons with string results in the $O(6)$ limit and beyond (not available):

$$m_{i,SYM}(g) = m(g) + O(m^3) = \kappa\, g^{1/4}\, e^{-\frac{\pi}{\sqrt{2}}\, g}(1 + \frac{a_1}{g} + \ldots) + c_i\, g^{-3/4}\, e^{-\frac{3\pi}{\sqrt{2}}\, g} + \ldots.$$

  in other words:1) computation of a_1 (for the string two loops) and the rest of the Brezin–Zinn-Justin function (higher loops), 2)computation of the $O(6)$ departures, i.e. c_i.
- Hints for new *physics* (new eqs. not reducible to BES): is string theory a lattice theory (dispersion relation)?

...and Perspectives

- a_1 enters the constant part of the two-loop string result: two loop discrepancy resolved in favour of Bethe Ansatz. [Gromov]
- All loop predictions when $y = \frac{j}{g} \ll 1$ from $O(6)$ and beyond.
- String theory computations for fixed j: origin of the mass-gaps (non-analytic in g)?
- Analysis of regular structures for the anomalous dimension at weak coupling: reciprocity equation.
- Better systematics of the expansion of the generalised scaling function.
- Other sectors of $\mathcal{N} = 4$ SYM.
- ....