The SL(2) sector of at strong coupling

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with

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arXiv:hep-th/0703031, arXiv:0801.2542

GGI, Florence, 30 November 2008

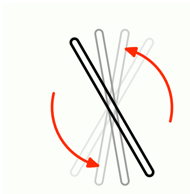
The *sl*(2) sector of *PSU*(2,2|4)

Excitations in the sl(2) sector:

Lorentz spin Twist

$$\mathrm{tr}\left(\mathcal{D}_+^M \mathcal{Z}^L\right) + \ldots \longleftrightarrow$$

Classical folded strings propagating in $AdS_3 \times S^1$



Gubser-Klebanov-Polyakov'02

Bethe Ansatz equations:

At one loop: $[XXX]_{-1/2}$ spin chain

$$\left(\frac{x_k^+}{x_k^-}\right)^L = \prod_{j\neq k}^{M} \left(\frac{u_k^- - u_j^+}{u_k^+ - u_j^-}\right) \left(\frac{1-1/x_k^+ x_j^-}{1-1/x_k^- x_j^+}\right)^2 e^{2\,i\,\theta(u_k,u_j)} \longleftarrow$$

Dressing phase

$$u^\pm = u \pm i\epsilon, \qquad x^\pm = x(u^\pm)$$

$$u(x) \equiv \frac{1}{2}\left(x + \frac{1}{x}\right)$$

$$x(u) = u\left(1 + \sqrt{1 - \frac{1}{u^2}}\right)$$

$$\epsilon \equiv \frac{1}{4g}$$

$$g^2 = \frac{g_{\mathrm{YM}}^2\,N}{16\,\pi^2}.$$

Large *M* limit:

Beisert-Eden-Staudacher'06 (BES) (L finite)

Freyhult-Rej-Staudacher'07 (FRS) (L~Log M)

Anomalous dimension for **large** M:

$$\Delta = M + L + f(g, L) \ln M + \ldots$$

universal scaling function
= cusp anomalous dimension

Korchemsky'89;
GKP'02

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Provides a critical test of AdS/CFT:

Weak coupling expansion:

$$f(g) = 8\, g^2 - \frac{8}{3}\, \pi^2 g^4 + \frac{88}{45}\, \pi^4 g^6 - 16 \left(\frac{73}{630}\, \pi^6 + 4\, \zeta(3)^2 \right) g^8 \pm \ldots .$$

From perturbative SYM up to g^8

3-loop guess
[Moch, Vermasseren, Vogt'04; Lipatov at al'04]

4-loop result
[Bern et al'06]

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Strong coupling expansion:

$$f(g) = 4\,g - \frac{3\,\log 2}{\pi} - \frac{\mathrm{K}}{4\,\pi^2}\frac{1}{g} + \ldots$$

From string perturbation theory

[Gubser,Klebanov, Polyakov'02]

Frolov,Tseytlin'02

Roiban,Tseytlin'07

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Both expansions should be reproduced from BA equations

[Klebanov *et al*'06, Kotikov,Lipatov'06, Alday *et al* '07; I.K., Serban, Volin'07]

Casteill, Kristjansen'07; Belitsky'07

Basso, Korchemsky, Kotanski'07; IK, Serban, Volin'08

(BES equation was taylored so that the weak coupling expans on is reproduced)

Functional Equation for resolvents at one loop

(x=2u, no dressing factor)

Baxter's equation for $Q(u) = \prod_{k=1}^{M} (u - u_k)$:

$$T(u) = \frac{Q(u+2i\epsilon)}{Q(u)}(u+i\epsilon)^L + \frac{Q(u-2i\epsilon)}{Q(u)}(u-i\epsilon)^L$$

For $M \longrightarrow \infty$ with u finite only one of the terms of the Baxter equation survives

=> linear equations for the magnon and hole resolvents

$$(1-D^2)R_m + R_h = \frac{j}{u+i\epsilon} \qquad (\Im u > 0)$$

$$(1-D^{-2})R_m + R_h = \frac{j}{u-i\epsilon} \qquad (\Im u < 0)$$

$$R_m(u) \sim \frac{d\log Q}{du}$$

$$R_h(u) \sim \frac{d\log T}{du}$$

- D is a shift operator: $D = e^{i\epsilon\partial_u}$: $Df(u) = f(u+i\epsilon)$
- j is related to L by $j = L/\log(M\epsilon)$

$1 << |u| << M\epsilon$: the density is constant, of order Log($M\epsilon$)

=> asymptotic conditions at infinity

$$R_h \to \frac{j}{u} \qquad (u \to \infty)$$

$$R_m \to \mp\frac{i}{\epsilon} \qquad (u \to \infty \pm i0)$$

Functional-integral equation at all orders (BES/FRS)

The universal scaling function can be extracted from the behavior of the magnon resolvent at infinity:

$$R_m(u) \to -\frac{i}{\epsilon} - \frac{j}{2u} - \frac{1}{2u} f(\epsilon, \ell) + \ldots$$

$$(1 - D^2 + \mathcal{K}) R_m + R_h = j\, D \frac{d \log x}{du} \qquad \text{(UHP)}$$

$$\mathcal{K} = D\left(K_- + K_+ + 2K_- \frac{D^2}{1-D^2} \mathrm{K}_+\right) D$$

-- the kernel is given by the “magic formula” of BES in terms of the even/odd kernels $K_\pm$

$$K_\pm(u,v) = -\frac{1}{2\pi i} \frac{d}{du} \left[\ln\left(1 - \frac{1}{xy}\right) \mp \ln\left(1 + \frac{1}{xy}\right) \right]$$

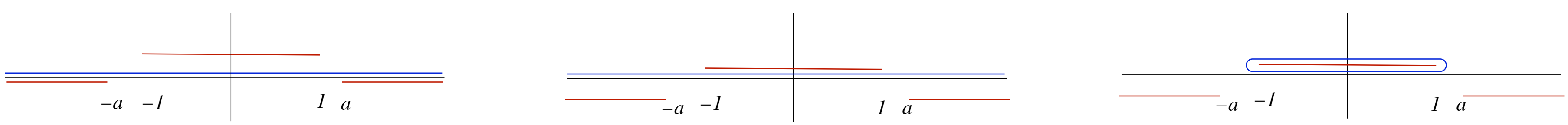
$$x = x(u + i0), \quad y = y(v - i0)$$

$$K_\pm F(u) = \int_{\mathbb{R} - i0} dv\; K_\pm(u,v) F(v)$$

For functions *F(u)* analytic in UHP and the real axis and decaying faster than *1/u*

IK, Serban, Volin’08

$$\mathrm{K}_\pm F(u) \equiv \int_{-1+i0}^{1+i0} \frac{dv}{2\pi i} \sqrt{\frac{v^2 - 1}{u^2 - 1}}\, \frac{F(v+i0) \pm F(-v+i0)}{v - u}$$



BES/FRS equation in the x-plane

Express magnon resolvent $R_m \sim \sum (u-u_i)^{-1}$ in terms of resolvent in x-space $S \sim \sum (x-x_i)^{-1}$

$$R_m(u) = S(x) + S(1/x)$$

and require that $(D-D^{-1})S(x)$ has at most a simple pole at $x = \pm 1$.

Then the action of K_+ drastically simplifies: to any order in ϵ,

$$\Leftrightarrow \qquad \mathrm{K}_+ D R_m = (D - D^{-1}) S(1/x)$$

and the BES/FRS equation becomes

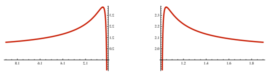
$$\Leftrightarrow \qquad (D^{-1} - D) S(x) + \mathrm{K}_- D [S(x) - S(1/x)] + D^{-1} R_h = j\, \partial_u \log x \qquad \text{(upper half plane } u\text{)}$$

$$(D - D^{-1}) S(x) - \mathrm{K}_- D^{-1} [S(x) - S(1/x)] + D R_h = j\, \partial_u \log x \qquad \text{(lower half plane } u\text{)}$$

- Solution in the leading order (first obtained by Casteil-Kristjansen'07)

$$S(x) = \frac{1}{\epsilon} \frac{\sqrt{b^2 - x^2} - j\epsilon}{x - \frac{1}{x}}, \qquad b = \sqrt{1 + (j\epsilon)^2}$$

$$\Delta = M + \sqrt{j^2 + 16 g^2}\, \log M$$



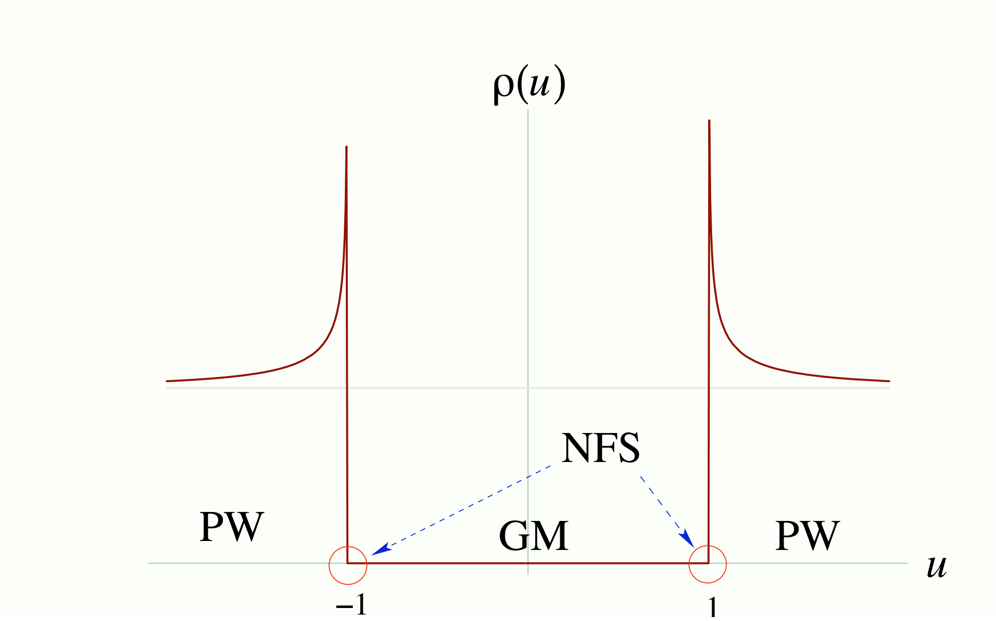
- Can be solved perturbatively in ϵ. The second order found by D. Volin'08 confirms the (formidable) calculation by N. Gromov'08.

The case j=0: BES equation

The ϵ expansion is not uniform: two different strong coupling limits [IK, Serban, Volin'07]

$\epsilon \to 0$ with u fixed (Plane Waves/ Giant Magnons)

$\epsilon \to 0$ with $z = (u-1)/\epsilon$ fixed (Near Flat Space)



BES equation (j=0): Complete perturbative (in ϵ) solution

Basso, Korchemsky, Kotanski'07; IK, Serban, Volin'08

At $j\to 0$: homogeneous equation:

$$(D - D^{-1}) S(1/x) = \mathrm{K}_+ D[S(x) + S(1/x)]$$

$$(D - D^{-1})\, S(x) = \mathrm{K}_- D[S(x) - S(1/x)]$$

(UHP)

=> $$S(x + i0) + S(x - i0) = 0$$

$$S(x) + S(-x) = 0$$

(valid perturbatively in ϵ)

$$S(x) \to \mp \frac{i}{\epsilon}, \qquad (x \to \infty \pm i0)$$

Solution in the leading order: $$S(x) = \frac{1}{\epsilon} \frac{\sqrt{1 - x^2}}{x - \frac{1}{x}}$$

Alday, Arutyunov, Benna, Eden, Klebanov'07

1) Solution in the PW regime ($|u|>1$)

General solution of the homogeneous equations:

$$S = \frac{1}{\epsilon}\frac{x}{\sqrt{1-x^2}}\sum_{k=0}^{\infty}\frac{\epsilon^{2k}c_k^+[\epsilon]}{(1-x^2)^{2k}} + \frac{\epsilon^{2k}c_k^-[\epsilon]}{(1-x^2)^{2k+1}}$$

The solution has 2 singular points: at $x = \pm 1$ or $u = \pm 1$ (NFS regime).

The coefficients can be fixed by comparing with the expansion near the singular points in the rescaled variable $z = \dfrac{u-1}{\epsilon}$

From the homogeneous equations:

$$G_\pm = \frac{1\pm i}{2}(D \mp iD^{-1})[S(x) \pm iS(1/x)]$$ -- analytic in $\mathbb{C}$ / [-∞,-1]∪[1,+∞]

$$g_\pm = \pm i(D - D^{-1})[S(x) \pm iS(1/x)]$$ -- analytic in $\mathbb{C}$ / [-1,1]

$$g_\pm = \frac{1\pm i}{D \mp i}(D-1)\,G_\pm$$

Inverse Laplace w.r.t. $z = \frac{u-1}{2\epsilon}$

$$\frac{\Gamma[\frac{s}{2\pi}]}{\Gamma[\frac{1}{2}+\frac{s}{2\pi}\mp\frac{1}{4}]}\tilde{g}_\pm(s) = \pm\sqrt{2}\frac{\Gamma[\frac{1}{2}-\frac{s}{2\pi}\pm\frac{1}{4}]}{\Gamma[1-\frac{s}{2\pi}]}\tilde{G}_\pm(s)\ .$$

analytic everywhere except the negative real axis

analytic everywhere except the positive real axis.

=> **no poles, only a branch cut [0, ∞]**

expansion of rhs at $S=\infty$ coincides with expansion of lhs at $S=0$ (known)

the coefs $c_k(\epsilon)$

3 different scaling regimes:

Extend the method for the case when both S and L are large.

$L \sim \log M$

Freyhult, Rej, Staudacher' 07

Three different regimes:

$L/(g \log M) \sim 1$

N. Gromov'08
D. Volin, 08

$L/(g \log M) \sim g^{-1/4}$

"Double scaling limit"

$L/(g \log M) \sim e^{-ag}$

B. Basso, G. Korchemsky'08
Fioravanti, Grinza,Rossi'08

O(6) **Alday, Maldacena'07**

Integral equation for the sl(2) sector (BES/FRS)

$$\left(\frac{x_k^+}{x_k^-}\right)^L = \prod_{j\neq k}^{M} \left(\frac{u_k^- - u_j^+}{u_k^+ - u_j^-}\right) \left(\frac{1-1/x_k^+ x_j^-}{1-1/x_k^- x_j^+}\right)^2 e^{2\,i\,\sigma(u_k,u_j)}$$

$$u^{\pm} = u \pm i\epsilon, \qquad x^{\pm} = x(u^{\pm})$$

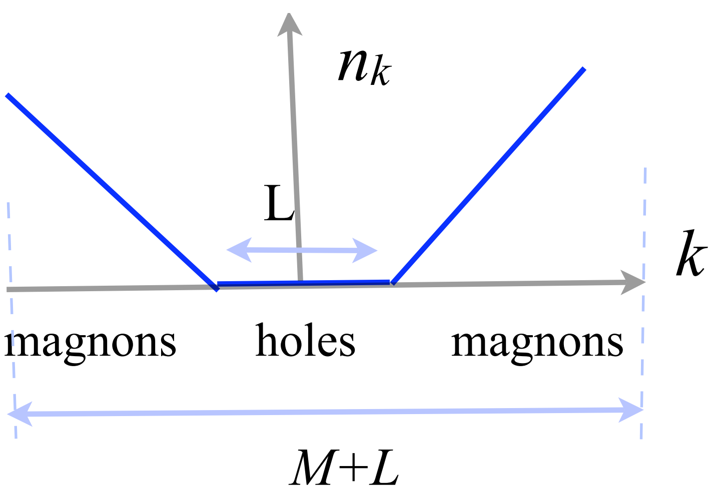
$$u(x) \equiv \frac{1}{2}\left(x + \frac{1}{x}\right)$$

$$x(u) = u\left(1 + \sqrt{1 - \frac{1}{u^2}}\right)$$

$$\epsilon \equiv \frac{1}{4g}$$

-- Repulsive interaction
=> Bethe roots on the real axis

Take log, specify the root (mode number n_k) for each u_k.



In the limit $M \to \infty$

=> Integral equation for the magnon density $\rho(u) = dk/du$

Derivation of the holomorphic kernels

Assuming that the function $F(v)$ is analytic in the upper half plane and decreases at least as $1/u$ at $u \to \infty$, we can express the action of the kernals $K_\pm$ as a contour integral

$$K_\pm F(u) = \int\limits_{\mathbb{R}-i0} dv\ K_\pm(u,v)F(v) \ = \oint\limits_{[-1,1]} dv K_\pm(v)F(v), \tag{4.5}$$

where the integration contour closes around the cut $[-1,1]$ of $K_\pm$. Then we represent the contour integral as a linear integral of the discontinuity of the integrand. Using the the definition of the kernels $K_\pm$ and the properties

$$\begin{aligned} x(v-i0) &= 1/x(v+i0)\ ,\ u \in [-1,1] \\ x(v-i0) &= x(v+i0)\ ,\ u \in \mathbb{R}\backslash[-1,1]\,, \end{aligned} \tag{4.6}$$

we obtain the following simple expressions for the continuous and the discontinuous part of the kernel

$$\begin{aligned} K_\pm F(u) &= \frac{2}{1-x^2} \int\limits_{-1+i0}^{1+i0} \frac{dv}{2\pi i} F(v) \left(\frac{-yx}{y-x} \pm \frac{yx}{y+x} - \frac{1}{y-\frac{1}{x}} \mp \frac{1}{y+\frac{1}{x}} \right) \\ &= \int\limits_{-1+i0}^{1+i0} \frac{dv}{2\pi i} F(v) \frac{y-\frac{1}{y}}{x-\frac{1}{x}} \left(\frac{1}{v-u} \mp \frac{1}{v+u} \right) . \end{aligned} \tag{4.7}$$

$$
\mathrm{K}_+ \cdot 1 = -\frac{1/x}{\sqrt{u^2-1}} = \frac{2}{1-x^2}, \quad \mathrm{K}_- \cdot 1 = 0\,.
$$

$$
\mathrm{K}_- \cdot \frac{x}{x^2-1} = \mathrm{K}_- \cdot \frac{1}{2u\sqrt{1-u^{-2}}} = 0.
$$

$$
\begin{aligned}
K_\pm F(u) &= \frac{2}{1-x^2} \int\limits_{-1}^{1} \frac{dv}{2\pi i} \left[F(v+i0) \left(\frac{-yx}{y-x} \pm \frac{yx}{y+x} \right) - F(v-i0) \left(\frac{1}{y-\frac{1}{x}} \pm \frac{1}{y+\frac{1}{x}} \right) \right] \\
&= \int\limits_{-1}^{1} \frac{dv}{2\pi i} \frac{\sqrt{\frac{v^2-1}{u^2-1}}\, [\not{F}(v) \pm \not{F}(-v)] + \hat{F}(v) \mp \hat{F}(-v)}{v-u} + \frac{1 \mp 1}{\sqrt{u^2-1}} \int\limits_{-1}^{1} \frac{dv}{2\pi i} \hat{F}(v).
\end{aligned}
$$