World-sheet duality for supersphere σ-models

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Miniworkshop: Integrability in String Theory

Galileo Galilei Institute Workshop on
"Low-dimensional Quantum Field Theories and Applications"

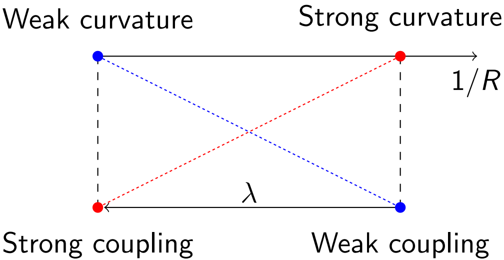
Based on arXiv:0809.1046 (with V. Mitev and V. Schomerus)

Outline

String theory/gauge theory dualities

String theory in $10D$
(σ-model with constraints)

Gauge theory

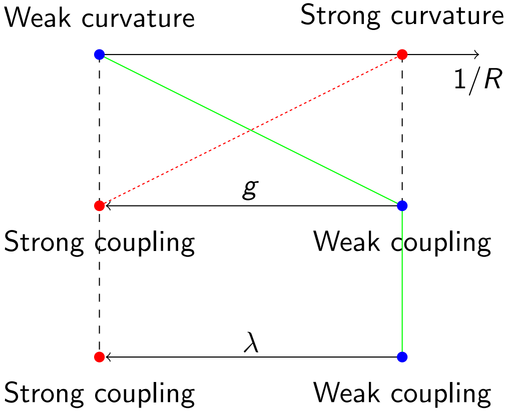


String theory/gauge theory dualities

String theory in $10D$
(σ-model with constraints)

“Some dual 2D theory”

Gauge theory



A prominent example: $AdS_5 \times S^5$

	$AdS_5 \times S^5$	$\mathcal{N} = 4$ super Yang-Mills theory
Symmetry	$PSU(2,2\|4)$	$PSU(2,2\|4)$
Parameters	α', g_s	Gauge group $SU(N)$
	Radius R	Coupling g_{YM}
		t'Hooft coupling $\lambda = N g_{\text{YM}}^2$

[Maldacena] [...]

[Metsaev,Tseytlin] [...]

[Minahan,Zarembo] [Beisert,Staudacher] [...]

Another prominent example: $AdS_4 \times \mathbb{CP}^3$

	$AdS_4 \times \mathbb{CP}^3$	$\mathcal{N} = 6$ Chern-Simons theory
Symmetry	$OSP(6\|2,2)$	$OSP(6\|2,2)$
Parameters	α', g_s	Gauge group $U(N) \times U(N)$
	Radius R	Level k
		t'Hooft coupling $\lambda = 2\pi^2 N/k$
Interpretation		N $M2$-branes probing $\mathbb{C}^4/\mathbb{Z}_k$

[Arutyunov,Frolov] [Stefanski] [Fre,Grassi] [...]

[Aharony,Bergman,Jafferis,Maldacena] [...]

And a common structure...

Both space are actually **supercosets** of the form

$$AdS_5 \times S^5 = \frac{PSU(2,2|4)}{SO(1,4) \times SO(5)}$$

$$AdS_4 \times \mathbb{CP}^3 = \frac{OSP(6|2,2)}{U(3) \times SO(1,3)}$$

The definition of these cosets is as follows:

$$G/H = \left\{ g \in G \middle| gh \sim g, h \in H \right\}$$

Note that G/H still admits an action of G: $\quad g = hg$

(Generalized) symmetric spaces

Let G be a Lie (super)group, $\Omega : G \to G$ an automorphism, $H = \mathrm{Inv}_\Omega(G) = \{h \in G | \Omega(h) = h\}$ the invariant subgroup. Ω being of finite order, $\Omega^L = \mathrm{id}$. Then the coset G/H is called a **generalized symmetric space**.

Theorem

If G has vanishing Killing form then the coset G/H is classically integrable and *quantum conformally invariant, at least to the lowest non-trivial order in perturbation theory.* *[Young] [Kagan, Young]*

Examples: Cosets of $PSL(N|N)$, $OSP(2S+2|2S)$, $D(2,1;\alpha)$.

A simple example: Superspheres

Superspheres $S^{M|2N} \subset \mathbb{R}^{M+1|2N}$ can be introduced as follows:

$$\vec{X} = \begin{pmatrix} \vec{x} \\ \vec{\eta}_1 \\ \vec{\eta}_2 \end{pmatrix} \qquad \rightarrow \qquad \vec{X}^2 = \vec{x}^2 + 2\vec{\eta}_1\vec{\eta}_2 = R^2$$

From this one derives their realization as a **symmetric space**:

$$S^{M|2N} = \frac{OSP(M+1|2N)}{OSP(M|2N)}$$

Superspheres: Conformal invariance

$(M, N) = (2S + 1, S) \Rightarrow$ Family of conformal σ-models

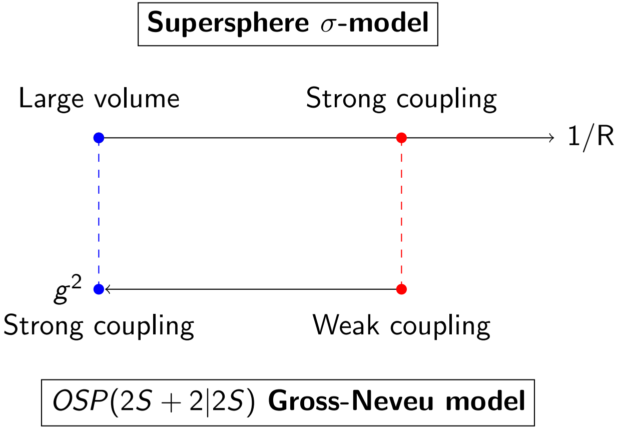
- Relation to $O\big(M - 2N\big) = O(2)$ σ-models
- There is no topological Wess-Zumino term
- There is one free parameter, the radius R

In this talk: Focus on $S^{3|2} = \frac{OSP(4|2)}{OSP(3|2)}$

Question: How can this theory be quantized?

[Read,Saleur] [Mann,Polchinski] [Candu,Saleur] [Mitev,TQ,Schomerus]

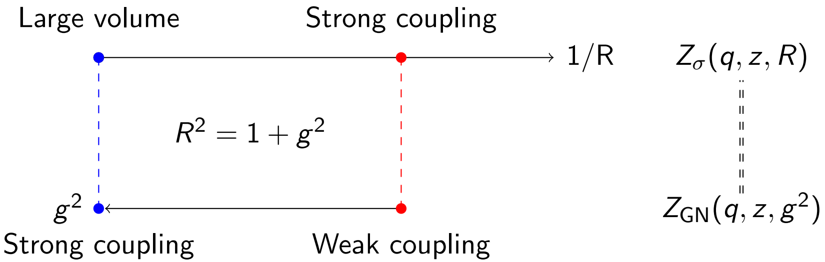
A new world-sheet duality?



[Candu,Saleur]^2 [Mitev,TQ,Schomerus]

A new world-sheet duality?

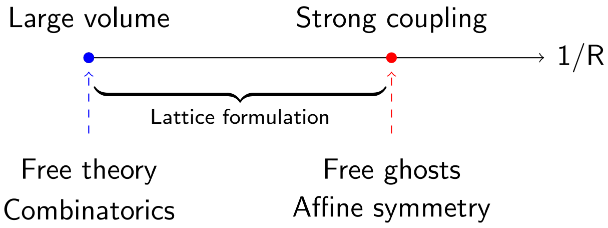
Supersphere σ-model



$OSP(2S+2|2S)$ **Gross-Neveu model**

[Candu,Saleur]^2 [Mitev,TQ,Schomerus]

Summary of existing evidence for the duality



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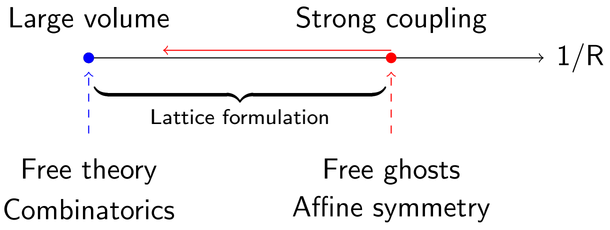


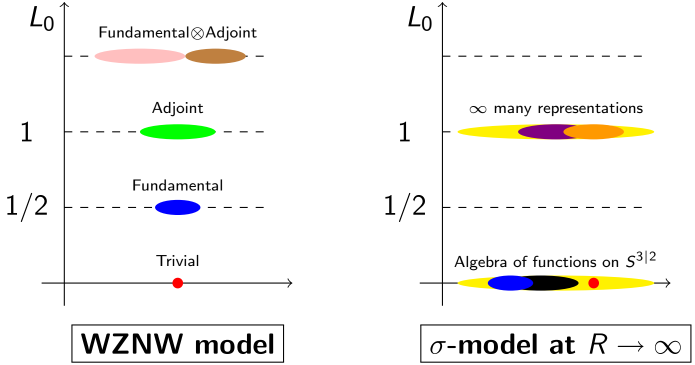
[Candu,Saleur]^2 [Mitev,TQ,Schomerus]

Certain partition functions can be determined for all R

$$Z_\sigma(q,z,R) \;=\; \sum_\Lambda \psi_\Lambda^\sigma(q,R)\,\chi_\Lambda(z)$$

Interpolation of the spectrum

We have to show that the following two spectra are continuously connected by the deformation:



Outline

Definition

The model is defined by the action

$$\mathcal{S}_\sigma = \int \partial\vec{X} \cdot \bar{\partial}\vec{X} \qquad \text{with} \qquad \vec{X}^2 = R^2$$

Properties of this σ-model:

- There is no topological term
- Conformal invariance for each value of R
- Central charge: $c = 1$
- Non-unitarity

[Read,Saleur] [Polchinski,Mann] [Candu,Saleur]^2 [Mitev,TQ,Schomerus]

The large volume limit

For $R \to \infty$ one has a **free field theory**...

- Coordinates $\vec{X} \to$ fields $\vec{X}(z)$
- Partition function is pure combinatorics
- Symmetry

$$OSP(4|2) \;\to\; \underbrace{SP(2)}_{SU(2)} \;\times\; \underbrace{SO(4)}_{SU(2)\times SU(2)}$$

The large volume partition function

State of states: (on a space-filling brane)

$$\prod X^{a_i} \prod \partial X^{b_j} \prod \partial^2 X^{c_k} \cdots \qquad \text{and} \qquad \vec{X}^2 = R^2$$

$\Rightarrow$ Products of coordinate fields and their derivatives.

How to count? $\rightarrow$ Want to keep representation content!

- Coordinates $\vec{X}$ form representation V of $OSP(4|2)$
- The state space is built from (symmetrized) tensor products

$$\big[\underbrace{V \otimes \cdots \otimes V}_{\sum a_i \text{ factors}}\big]_{\text{antisym}} \otimes \big[\underbrace{V \otimes \cdots \otimes V}_{\sum b_i \text{ factors}}\big]_{\text{antisym}} \otimes \cdots$$

- Each derivative contributes 1 to the energy

The ground states: Harmonic analysis

Let us look at the ground states first...

$$\prod_i X^{a_i} \qquad \text{and} \qquad \vec{X}^2 = R^2$$

Classify states according to $SU(2) \times SU(2) \times SU(2)$ symmetry:

$$V = \underbrace{(0,1/2,1/2)}_{\text{bosons}} \oplus \underbrace{(1/2,0,0)}_{\text{fermions}}$$

Naively, one thus obtains the partition function

$$\tilde{Z}_\sigma^{(0)}(q,z,R_\infty) = \frac{(1+z_1)(1+z_1^{-1})}{(1-z_2z_3)(1-z_2z_3^{-1})(1-z_2^{-1}z_3)(1-z_2^{-1}z_3^{-1})}$$

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Taking into account the contraint yields

$$Z_\sigma^{(0)}(q, z, R_\infty) = \frac{(1-t^2)(1+tz_1)(1+tz_1^{-1})}{(1+tz_2z_3)(1-tz_2z_3^{-1})(1-tz_2^{-1}z_3)(1-tz_2^{-1}z_3^{-1})}$$

The full σ-model partition function

Including all **derivative terms** one finds

$$Z_\sigma(q,z,R_\infty) \;=\; q^{-\frac{1}{24}} Z_\sigma^{(0)}(q,z,R_\infty) \prod_{n=1}^{\infty}(1-q^n)\times$$
$$\times \prod_{n=1}^{\infty} \frac{(1+z_1q^n)(1+z_1^{-1}q^n)}{(1-z_2z_3q^n)(1-z_2z_3^{-1}q^n)(1-z_2^{-1}z_3q^n)(1-z_2^{-1}z_3^{-1}q^n)}$$

Remarks:

- The constraints $\vec{X}^2 = R^2$ implies $\partial^n(\vec{X}^2) = 0$
- This is taken into account by the factor $\prod_{n=1}^{\infty}(1-q^n)$

Decomposition into representations of $OSP(4|2)$

Since the model is symmetric under $OSP(4|2)$ the partition function may be decomposed into characters of $OSP(4|2)$:

$$Z_\sigma(q,z,R_\infty) \;=\; \sum_{[j_1,j_2,j_3]} \psi^\sigma_{[j_1,j_2,j_3]}(q)\;\underbrace{\chi_{[j_1,j_2,j_3]}(z)}_{\text{Long multiplets}}$$

$$\psi^\sigma_{[j_1,j_2,j_3]}(q) \;=\; \frac{q^{-C_{[j_1,j_2,j_3]}/2}}{\eta(q)^4}\sum_{n,m=0}^{\infty}(-1)^{m+n}q^{\frac{m}{2}(m+4j_1+2n+1)+\frac{n}{2}+j_1-\frac{1}{8}}$$
$$\times\left(q^{(j_2-\frac{n}{2})^2}-q^{(j_2+\frac{n}{2}+1)^2}\right)\left(q^{(j_3-\frac{n}{2})^2}-q^{(j_3+\frac{n}{2}+1)^2}\right)$$

Remark: $C_{[j_1,j_2,j_3]}$ is the Casimir of $[j_1,j_2,j_3]$

The strong coupling limit: $OSP(4|2)$ Gross-Neveu model

Field content: 4 fermions ψ^i 2 ghosts β, γ

- All fields have conformal weight $h = 1/2$
- The fields form the $OSP(4|2)$-multiplet V

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The theory has **quartic interactions**

$$\mathcal{S}_{\text{GN}} = \mathcal{S}_{\text{free}} + g^2\,\mathcal{S}_{\text{int}} \quad \begin{cases} \mathcal{S}_{\text{free}} = \int \left[\psi\bar{\partial}\psi + 2\beta\bar{\partial}\gamma + h.c.\right] \\ \mathcal{S}_{\text{int}} = \int \left[\psi\bar{\psi} + \beta\bar{\gamma} - \gamma\bar{\beta}\right]^2 \end{cases}$$

Reformulation as a WZNW model

The $OSP(4|2)$ Gross-Neveu model has a nice reformulation

- At $g = 0$ there is an affine $\widehat{OSP}(4|2)_{-1/2}$ symmetry
- The interaction is of current-current type

$$\mathcal{S}_{\text{int}} \sim \int \mathrm{J}^a \bar{\mathrm{J}}_a$$

- Vanishing Killing form $\Rightarrow$ exact marginality
- There is a "bosonic" realization as an orbifold

$$\widehat{OSP}(4|2)_{-1/2}\ \text{WZNW} \ \cong \ \Big[\widehat{SU}(2)_{-1/2} \times \widehat{SU}(2)_1 \times \widehat{SU}(2)_1\Big]/\mathbb{Z}_2$$

A D-brane spectrum

The $OSP(4|2)$ WZNW model admits a number of symmetry preserving D-branes...

- **We use:** Exchange automorphism in $\widehat{SU}(2)_1 \times \widehat{SU}(2)_1$
- The associated spectrum is

$$\begin{aligned} Z_{\text{GN}}(g^2=0) &= \bigl(\chi_0 + \chi_{1/2}\bigr)_{k=-1/2} \times \bigl(\chi_0\chi_0 + \chi_{1/2}\chi_{1/2}\bigr)_{k=1} \\ &= \underbrace{\chi_{\{0\}}(q,z)}_{\text{vacuum}} + \underbrace{\chi_{\{1/2\}}(q,z)}_{\text{fundamental}} \end{aligned}$$

Decomposition into representations of $OSP(4|2)$

Plugging in concrete expressions, one obtains

$$
\begin{aligned}
Z_{\mathrm{GN}}(g^2=0) &= \frac{\eta(q)}{\theta_4(z_1)}\left[\frac{\theta_2(q^2,z_2^2)\theta_2(q^2,z_3^2)}{\eta(q)^2}+\frac{\theta_3(q^2,z_2^2)\theta_3(q^2,z_3^2)}{\eta(q)^2}\right] \\
&= \sum \psi^{\mathrm{WZNW}}_{[j_1,j_2,j_3]}(q)\chi_{[j_1,j_2,j_3]}(z)
\end{aligned}
$$

Decomposition into representations of $OSP(4|2)$

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$$Z_{\mathrm{GN}}(g^2=0) = \frac{\eta(q)}{\theta_4(z_1)}\left[\frac{\theta_2(q^2,z_2^2)\theta_2(q^2,z_3^2)}{\eta(q)^2}+\frac{\theta_3(q^2,z_2^2)\theta_3(q^2,z_3^2)}{\eta(q)^2}\right]$$

$$= \sum \psi^{\mathrm{WZNW}}_{[j_1,j_2,j_3]}(q)\chi_{[j_1,j_2,j_3]}(z)$$

$$\psi^{\mathrm{WZNW}}_{[j_1,j_2,j_3]}(q) = \frac{1}{\eta(q)^4}\sum_{n,m=0}^{\infty}(-1)^{n+m}q^{\frac{m}{2}(m+4j_1+2n+1)+j_1+\frac{n}{2}-\frac{1}{8}}$$

$$\times\left(q^{(j_2-\frac{n}{2})^2}-q^{(j_2+\frac{n}{2}+1)^2}\right)\left(q^{(j_3-\frac{n}{2})^2}-q^{(j_3+\frac{n}{2}+1)^2}\right)$$

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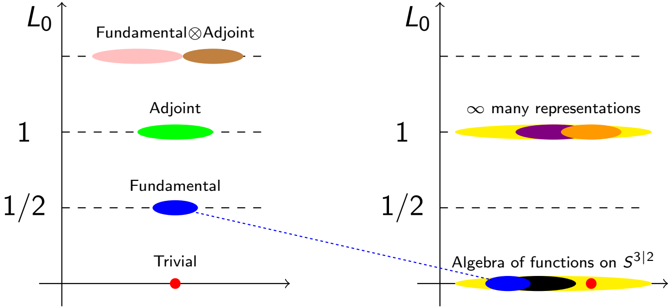

WZNW model

σ-model at $R \longrightarrow \infty$

Interpolation of the spectrum

- Vanishing Killing form $\Rightarrow$ the perturbation is abelian (for the purposes of calculation anomalous dimensions)

  [Bershadsky,Zhukov,Vaintrob] [TQ,Schomerus,Creutzig]

- An $OSP(4|2)$ representation Λ is shifted according to

$$h_\Lambda(g^2) \;=\; h_\Lambda(0) - \frac{1}{2}\frac{g^2 C_\Lambda}{1+g^2} \;=\; h_\Lambda(0) - \frac{1}{2}\Big(1 - 1/R^2\Big) C_\Lambda$$

- As a consequence one has

$$\psi_\Lambda^\sigma(q,R) \;:=\; q^{-\frac{1}{2}\left(1-1/R^2\right)C_\Lambda}\,\psi_\Lambda^{WZNW}(q)$$

- For $R \to \infty$ this correctly reduces to $\psi_\Lambda^\sigma(q)$!

Outline

Outlook

Several open issues on supersphere σ-models remain...

- Deformation of the bulk spectrum
- S-matrix approach
- Correlation functions
- Path integral derivation?
- Other examples: $\mathbb{CP}^{S-1|S}$, AdS-spaces, ...

Lessons about string theory on AdS_n?

- Not a **symmetric** space, only generalized symmetric
- Need to take into account **gauge constraints**
- Non-compactness