Matching of the Hagedorn Temperature in AdS/CFT

-How to see free strings in Yang-Mills theory

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Motivation:

Can we find strings in Yang-Mills theory?

't Hooft (1973):

At large N the diagrams of SU(N) Yang-Mills theory can be arranged into a topological expansion

Define $\lambda = g_{YM}^2 N \leftarrow$ The 't Hooft coupling

Then we can write the sum of vacuum diagrams as

$$\sum_g N^{2-2g} \sum_n c_{g,n} \lambda^n = \sum_g N^{2-2g} f_g(\lambda)$$

g: genus of the associated Riemann surface

For large λ: Loop corrections will fill out the holes in the diagrams and you have closed Riemann surface $\rightarrow$ The string world-sheet

The topological expansion is a string world-sheet expansion

This is provided we identify the string coupling to be $g_s = \dfrac{1}{N}$

The leading contribution for large N is given by g=0:

- ► Free string theory: the world-sheet is the two-sphere
- ► Corresponds to the planar diagrams for the Yang-Mills theory

$\rightarrow$ Planar Yang-Mills theory is dual to free string theory

Maldacena (1997):

First explicit conjecture: The AdS/CFT correspondence

$\rightarrow \mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$ dual to type IIB strings on $AdS_5 \times S^5$

Dictionary relating λ, N to g_s, l_s and R (the AdS_5, S^5 radius):

$$T_{\text{str}} = \frac{1}{2}\sqrt{\lambda} \qquad g_s = \frac{\lambda}{N} \qquad \text{with} \quad T_{\text{str}} = \frac{R^2}{4\pi l_s^2}$$

This is in accordance with 't Hooft's expectations

- ► g_s is inversely proportional with N
- ► Large λ corresponds to semi-classical limit for world-sheet theory

Planar $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$ a free string theory?

Sign of free strings: The Hagedorn temperature

For $\lambda \ll 1$ planar $\mathcal{N} = 4$ on $\mathbb{R} \times S^3$ has a Hagedorn density of states
$\rho(E) \sim E^{-1} \exp(T_H E)$ for high energies

Conjecture: The Hagedorn temperature of $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$ is dual to the Hagedorn temperature of string theory on $AdS_5 \times S^5$

If we can match the two $\rightarrow$ Evidence of free strings in Yang-Mills theory

Is it possible to match the Hagedorn temperature in AdS/CFT?

Gauge theory:

We can only compute Hagedorn temperature for $\lambda \ll 1$
Current status: Free part + one-loop part computed

String theory:

No known first quantization of strings on $AdS_5 \times S^5$

However, Hagedorn temperature computable for pp-wave background (strings on $AdS_5 \times S^5$ with large R-charge)

Problem:

Matching of spectra in Gauge-theory/pp-wave correspondence requires $\lambda \gg 1$

$\Rightarrow$ Seemingly no possibility of match of Hagedorn temperature

Why does matching of spectra in gauge-theory/pp-wave correspondence require $\lambda \gg 1$?

Consider gauge-theory/pp-wave correspondence of BMN

Z, X: two complex scalars

Consider the three single-trace operators:

$\mathcal{O}_1 = \mathrm{Tr}\left[\mathrm{sym}\left(X^2 Z^J\right)\right]$ ⟵ Chiral primary (BPS) ⇒ Survives the limit

$\mathcal{O}_2 = \mathrm{Tr}\left[X^2 Z^J\right]$ ⟵ Conjectured to decouple in the limit

$\mathcal{O}_3 = \sum_l e^{2\pi i \frac{ln}{J}} \mathrm{Tr}\left[X Z^l X Z^{J-l}\right]$ ⟵ Near-BPS ⇒ Survives the limit

For $\lambda = 0$: All quantum numbers of $\mathcal{O}_1$, $\mathcal{O}_2$, $\mathcal{O}_3$ the same
⇒ They contribute the same in the partition function

One-loop contribution just a perturbation of this result.

Gauge-theory/pp-wave correspondence needs $\lambda \gg 1$ since we are expanding around chiral primaries
Conjecture of BMN: The unwanted states for $\lambda \ll 1$ decouple for $\lambda \gg 1$

Matching of Hagedorn temperature in AdS/CFT seems impossible
→ We need a new way to match gauge theory and string theory…

New way: Consistent subsector from decoupling limit of AdS/CFT:

T : temperature

Ω_i : Chemical potentials corresponding to R-charges J_i of SU(4) R-symmetry

We consider what happens near the critical point $T = 0$, $\Omega_1 = \Omega_2 = 1$, $\Omega_3 = 0$

Take limit $$T \to 0,\ \Omega \to 1,\ \lambda \to 0,\ \tilde{T} \equiv \frac{T}{1-\Omega},\ \tilde{\lambda} \equiv \frac{\lambda}{1-\Omega},\ N \text{ fixed}$$

of planar $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$ with $\Omega_1 = \Omega_2 = \Omega$ and $\Omega_3 = 0$. We get:

Ferromagnetic Heisenberg spin chain	⟷	Limit of weakly coupled planar $\mathcal{N} = 4$ SYM	⟷	Limit of free strings on $AdS_5 \times S^5$

Gauge theory: Weakly coupled, reduction to the SU(2) sector, described exactly by Heisenberg chain $\to$ A solvable model

String theory: Free strings, decoupled part of the string spectrum, zero string tension limit

We match succesfully the spectra and Hagedorn temperature (for $\tilde{\lambda}$ large)

Plan for talk:

- Motivation

- Gauge theory side:

  Thermal $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$

  Free planar $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$

  Decoupling limit of interacting $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$

  Gauge theory spectrum from Heisenberg chain

  Hagedorn temperature from Heisenberg chain

- String theory side:

  Decoupling limit of string theory on $AdS_5 \times S^5$

  Penrose limit, matching of spectra

  Computation and matching of the Hagedorn temperature

- Conclusions, Implications for AdS/CFT, Future directions

Thermal $\mathcal{N}=4$ SYM on $\mathbb{R}\times S^3$:

$$Z(\beta,\Omega_i) = \mathrm{Tr}_M\left(e^{-\beta D+\beta(\Omega_1 J_1+\Omega_2 J_2+\Omega_3 J_3)}\right)$$

Partition function with chemical potentials

D : Dilatation operator

J_i : R-charges for SU(4) R-symmetry of $\mathcal{N}=4$ SYM

Ω_i : Chemical potentials

State/operator correspondence:

State, CFT on $\mathbb{R}\times S^3$	Operator, CFT on $\mathbb{R}^4$
Energy E	Scaling dimension D
Gauge singlet	Gauge invariant operator

Gauge singlets:
Because flux lines on S^3 cannot escape

We put $R(S^3) = 1$, hence E=D

M : The set of gauge invariant operators

Given by linear combinations of all possible multi-trace operators

$\mathrm{Tr}(\cdots)\mathrm{Tr}(\cdots)\cdots\mathrm{Tr}(\cdots)$

Planar limit N = ∞ of U(N) $\mathcal{N} = 4$ SYM

$\rightarrow$ Large N factorization, traces do not mix

$\rightarrow$ We can single out the single-trace sector

Single-trace partition function

$$Z_{\mathsf{ST}}(\beta, \Omega_i) = \mathrm{Tr}_S \left(e^{-\beta D + \beta(\Omega_1 J_1 + \Omega_2 J_2 + \Omega_3 J_3)} \right)$$

S : The set of single-trace operators

Introduce $x = e^{-\beta}$, $y_i = e^{\beta \Omega_i}$

Then we can write $Z_{\mathsf{ST}}(x, y_i) = \mathrm{Tr}_S \left(x^D \prod_{i=1}^{3} y_i^{J_i} \right)$

Multi-trace partition function is then

$$\log Z(x, y_i) = \sum_{n=1}^{\infty} \frac{1}{n} Z_{\mathsf{ST}} \left(\omega^{n+1} x^n, y_i^n \right)$$

$\omega = e^{2\pi i}$

Equals -1 when uplifted to half-integer

Free planar $\mathcal{N}$ = 4 SYM on $\mathbb{R} \times S^3$:

$\lambda = 0 : D = D_0 \leftarrow$ The bare scaling dimension

Computation of $Z_{\mathrm{ST}}(\beta, \Omega_i) = \mathrm{Tr}_S \left(e^{-\beta D_0 + \beta(\Omega_1 J_1 + \Omega_2 J_2 + \Omega_3 J_3)} \right)$

Single-trace operators $\mathrm{Tr}(A_1 A_2 \cdots A_L)$, $A_i \in \mathcal{A}$

$\mathcal{A}$: The set of letters of $\mathcal{N}$ = 4 SYM

	SU(4) rep
6 real scalars	[0,1,0]
1 gauge boson	[0,0,0]
8 fermions	[1,0,0] $\oplus$ [0,0,1]

plus descendants using the covariant derivative

Compute first the letter partition function:

$$x = e^{-\beta} \ , \ \ y_i = e^{\beta \Omega_i}$$

$$z(x, y_i) = \mathrm{Tr}_{\mathcal{A}} \left(x^{D_0} \prod_{i=1}^{3} y_i^{J_i} \right)$$

$$= \frac{6x^2 - 2x^3}{(1-x)^3} + \frac{x + x^2}{(1-x)^3} \sum_{i=1}^{3} \left(y_i + y_i^{-1} \right) + \frac{2x^{\frac{3}{2}}}{(1-x)^3} \prod_{i=1}^{3} \left(y_i^{\frac{1}{2}} + y_i^{-\frac{1}{2}} \right)$$

From the letter partition function $z(x,y_i)$ we obtain

$$Z_{\mathrm{ST}}(x,y_i) = -\sum_{k=1}^{\infty} \frac{\varphi(k)}{k} \log\left[1 - z(\omega^{k+1}x^k, y_i^k)\right]$$

Giving

$$\log Z(x,y_i) = -\sum_{k=1}^{\infty} \log\left[1 - z(\omega^{k+1}x^k, y_i^k)\right]$$

Partition function for free planar $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$

Sundborg. Polyakov. Aharony et al.
Yamada & Yaffe. TH & Orselli

Hagedorn temperature:

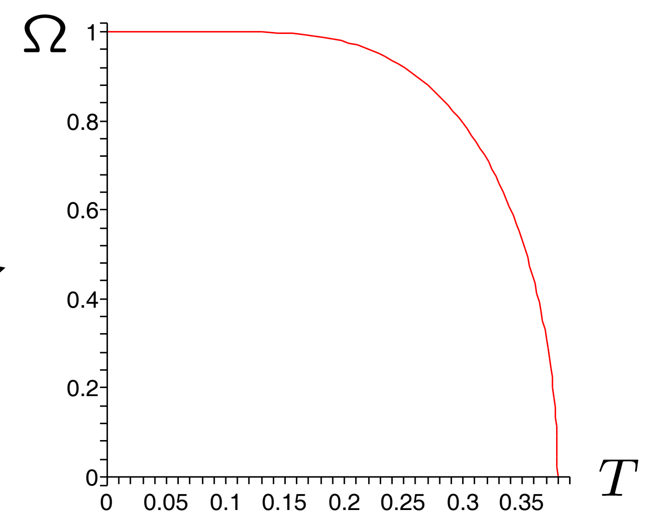
$Z(x,y_i)$ has a singularity when $z(x,y_i) = 1$ $\rightarrow$ The Hagedorn singularity

Given the chemical potentials Ω_i :
Defines Hagedorn temperature $T_H(\Omega_1,\Omega_2,\Omega_3)$

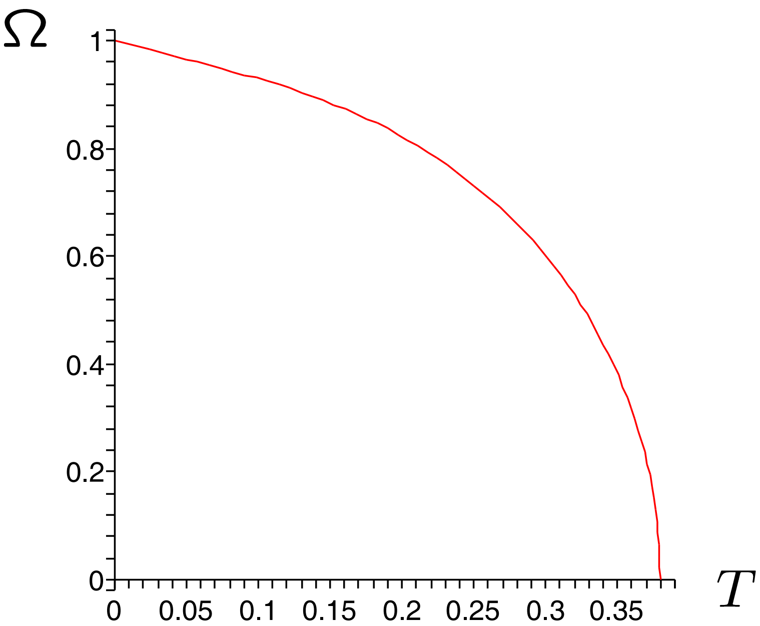
$$\Omega_i = 0 \;\; : \;\; T_H = \frac{1}{-\log(7-4\sqrt{3})}$$

Special cases:

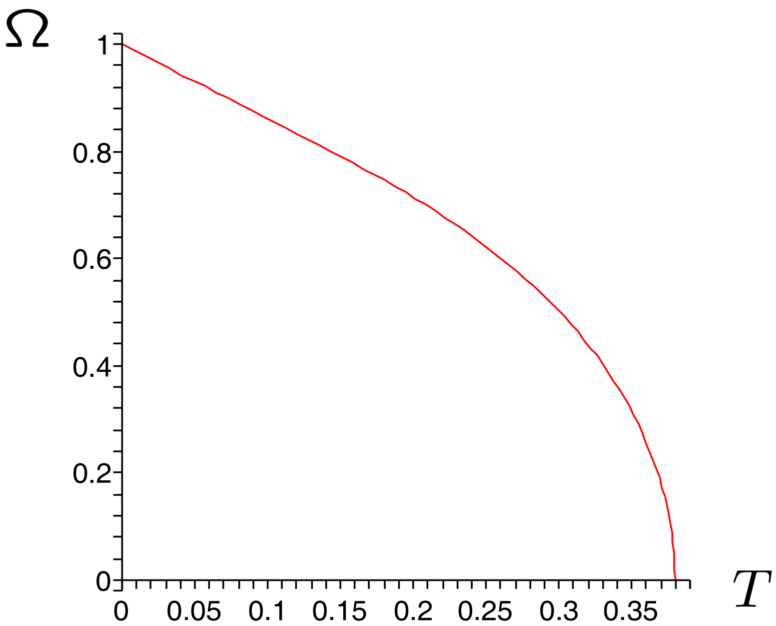
Case 1: $(\Omega_1,\Omega_2,\Omega_3)=(\Omega,0,0)$



Case 2: $(\Omega_1,\Omega_2,\Omega_3) = (\Omega,\Omega,0)$



Case 3: $(\Omega_1,\Omega_2,\Omega_3)=(\Omega,\Omega,\Omega)$



$$\Omega \to 1 : T_H(\Omega) \simeq \frac{1-\Omega}{\log 2}$$

$$\Omega \to 1 : T_H(\Omega) \simeq \frac{1-\Omega}{\log 4}$$

Consider case 2: $(\Omega_1,\Omega_2,\Omega_3)=(\Omega,\Omega,0)$

What happens for $\Omega \to 1$?

Should also take $T \to 0$

Try limit: $T \to 0,\ \Omega \to 1,\ \tilde{T} \equiv \dfrac{T}{1-\Omega}$ fixed

Gives finite Hagedorn temperature in the limit:

$$\tilde{T}_H = \frac{1}{\log 2}$$

The limit: $T \to 0, \ \Omega \to 1, \ \tilde{T} \equiv \frac{T}{1-\Omega}$ fixed

$(\Omega_1,\Omega_2,\Omega_3)=(\Omega,\Omega,0)$

Corresponds to $x \to 0, \ y \to \infty, \ \tilde{x} \equiv xy$ fixed

with $x = e^{-\beta}, \ y = e^{\beta\Omega}$

Take limit of letter partition function

$$z(x,y,y,1) = \frac{x + x^2}{(1-x)^3}\left(2 + 2y + 2y^{-1}\right) + \cdots$$

$$\lim_{x\to 0} z(x,\tilde{x}/x,\tilde{x}/x,1) = 2\tilde{x}$$

Corresponds to the two complex scalars:

Z : weight (1,0,0)
X : weight (0,1,0)

$\to$ In this limit only the two scalars Z, X survive and the possible operators are:

single-trace operators: $\mathrm{Tr}(A_1 A_2 \cdots A_L), \ A_i \in \{Z, X\}$

and multi-trace operators by combining these

Therefore: In the above limit we are precisely left with the SU(2) sector of $\mathcal{N} = 4$ SYM

Limit of partition function $$\log Z(\tilde{x}) = -\sum_{k=1}^{\infty} \log\left[1 - 2\tilde{x}^k\right]$$

Hagedorn singularity: $$\tilde{x}_H = \frac{1}{2} \Rightarrow \tilde{T}_H = \frac{1}{\log 2}$$

Partition function and Hagedorn temperature of the SU(2) sector

The two other cases:

Case 1: $(\Omega_1,\Omega_2,\Omega_3) = (\Omega,0,0)$ $$\lim_{x\to 0} z(x, \tilde{x}/x, 1, 1) = \tilde{x}$$

Single-trace operators: $\mathrm{Tr}(Z^L)$

half-BPS sector

Case 3: $(\Omega_1,\Omega_2,\Omega_3)=(\Omega,\Omega,\Omega)$ $$\lim_{x\to 0} z(x, \tilde{x}/x, \tilde{x}/x, \tilde{x}/x) = 3\tilde{x} + 2\tilde{x}^{3/2}$$

Single-trace operators: $\mathrm{Tr}(A_1 A_2 \cdots A_L),\ A_i \in \{Z, X, W, \chi_1, \chi_2\}$

Z,X,W : 3 complex scalars, weights (1,0,0), (0,1,0), (0,0,1)

χ_1, χ_2: 2 complex fermions, weight (1/2,1/2,1/2)

$\rightarrow$ The SU(2|3) sector of $\mathcal{N} = 4$ SYM

Decoupling limit of interacting $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$:

We consider weakly coupled U(N) $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$
near the critical point $(T,\Omega)=(0,1)$ and with $(\Omega_1,\Omega_2,\Omega_3) = (\Omega,\Omega,0)$

Full partition function:

$$Z(\beta,\Omega) = \mathrm{Tr}_M \left(e^{-\beta D + \beta \Omega J} \right) \longleftarrow \quad J = J_1 + J_2$$

Interacting $\mathcal{N} = 4$ SYM:

$$D = D_0 + \lambda D_2 + \lambda^{3/2} D_3 + \lambda^2 D_4 + \cdots$$

Convention here:

$$\lambda = \frac{g_{\mathrm{YM}}^2 N}{4\pi^2}$$

With this, we can rewrite the weight factor as:

$$e^{-\beta D + \beta \Omega J} = \exp\left(-\beta (D_0 - J) - \beta (1-\Omega) J - \beta \lambda D_2 - \beta \sum_{n=3}^{\infty} \lambda^{n/2} D_n \right)$$

Weight factor:

$$e^{-\beta D+\beta\Omega J} = \exp\left(-\beta(D_0 - J) - \beta(1-\Omega)J - \beta\lambda D_2 - \beta\sum_{n=3}^{\infty}\lambda^{n/2}D_n\right)$$

Consider the limit:

$$T\to 0,\ \Omega\to 1,\ \lambda\to 0,\ \tilde{T}\equiv\frac{T}{1-\Omega}\ \text{fixed},\ \tilde{\lambda}\equiv\frac{\lambda}{1-\Omega}\ \text{fixed},\ N\ \text{fixed}$$

$\beta\to\infty$ and $2(D_0 - J)$ is a non-negative integer

$\Rightarrow$ Effective truncation to states with $D_0 = J$ $\Rightarrow$ The SU(2) sector

The other terms:

$$\beta(1-\Omega)J\to\tilde{\beta}J \qquad \beta\lambda D_2\to\tilde{\beta}\tilde{\lambda}D_2 \qquad \beta\sum_{n=3}^{\infty}\lambda^{n/2}D_n\to 0$$

with $\tilde{\beta}\equiv\beta(1-\Omega)$

Partition function becomes

$$Z(\beta,\Omega) = Z(\tilde{\beta}) = \mathrm{Tr}_{\mathcal{H}}\left(e^{-\tilde{\beta}H}\right)$$

The SU(2) sector

Hilbert space: $\mathcal{H} = \{\alpha\in M|(D_0 - J)\alpha = 0\}$

Hamiltonian: $H = D_0 + \tilde{\lambda}D_2$

Result: For $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$ in the decoupling limit

$$T \to 0,\ \Omega \to 1,\ \lambda \to 0,\ \tilde{T} \equiv \frac{T}{1-\Omega} \text{ fixed},\ \tilde{\lambda} \equiv \frac{\lambda}{1-\Omega} \text{ fixed},\ N \text{ fixed}$$

The full partition function reduces to

$$Z(\tilde{\beta}) = \mathrm{Tr}_{\mathcal{H}}\left(e^{-\tilde{\beta} H}\right)$$

Hamiltonian: $H = D_0 + \tilde{\lambda} D_2$

Only states in the SU(2) sector contributes

The Hamiltonian truncate $\to$ has only the bare + one-loop term

Note also: $\tilde{\lambda}$ can be finite, i.e. it does not have to be small

The exact partition function can in principle be computed for finite $\tilde{\lambda}$ and finite N

$\mathcal{N} = 4$ SYM is weakly coupled in this limit

The result can be used to study $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$ near the critical point $(T,\Omega_1,\Omega_2,\Omega_3) = (0,1,1,0)$

Planar limit N = ∞

$\rightarrow$ we can focus on the single-trace sector

$$\mathrm{Tr}(A_1 A_2 \cdots A_L),\ \ A_i \in \{Z, X\}$$

$\rightarrow$ like a spin chain $\quad Z : \uparrow,\ \ X : \downarrow$

$$\mathrm{Tr}(XZZX \cdots Z)$$

$$\Downarrow$$

$$|\downarrow\uparrow\uparrow\downarrow \cdots \uparrow\rangle$$

Which spin chain?

$$D_2 = \frac{1}{2} \sum_{i=1}^{L} (I_{i,i+1} - P_{i,i+1})$$

L: Length of single-trace operator / spin chain

$\tilde{\lambda} D_2$: Hamiltonian of ferromagnetic $\mathrm{XXX}_{1/2}$ Heisenberg spin chain

Minahan & Zarembo

Total Hamiltonian: $H = L + \tilde{\lambda} D_2$

In the limit $T \to 0,\ \tilde{T} \equiv \frac{T}{1-\Omega}$ fixed, $\tilde{\lambda} \equiv \frac{\lambda}{1-\Omega}$ fixed, N fixed

planar $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$ has the partition function

$$\log Z(\tilde{\beta}) = \sum_{n=1}^{\infty} \sum_{L=1}^{\infty} \frac{1}{n} e^{-nL\tilde{\beta}} Z_L^{(\mathrm{XXX})}(n\tilde{\beta})$$

$$Z_L^{(XXX)}(\tilde{\beta}) = \mathrm{Tr}_L \left(e^{-\tilde{\beta}\tilde{\lambda}D_2} \right)$$

← Partition function for the ferromagnetic $XXX_{1/2}$ Heisenberg spin chain

↑ Chains of length L

→ The ferromagnetic Heisenberg model is obtained as a limit of weakly coupled planar $\mathcal{N} = 4$ SYM

Spectrum of gauge theory from Heisenberg chain:

We can now obtain the spectrum for large $\tilde{\lambda}, \ L$

Hamiltonian: $\tilde{\lambda} D_2$

Large $\tilde{\lambda} \leftrightarrow$ Low energy spectrum of D_2

Spectrum: Vacua (D_2 = 0) plus excitations (magnons)

Vacua are given by: $D_2 = 0$

Define the total spin: $S_z = \dfrac{J_1 - J_2}{2}$

Exists a vacuum for each value of S_z:

$$|S_z\rangle_L \sim \mathrm{Tr}\left(\mathrm{sym}(Z^{J_1} X^{J_2})\right) \longleftarrow \quad J_1 = \frac{1}{2}L + S_z, \quad J_2 = \frac{1}{2}L - S_z$$

These L+1 states are precisely all the possible states for which D_2 = 0, i.e. all the possible vacua

The vacua $|S_z\rangle_L$ are precisely the chiral primaries of $\mathcal{N}$ = 4 SYM obeying $D_0 = J_1 + J_2$ (=L)

$\rightarrow$ The low energy excitations are 'close' to BPS

Low energy excitations: Magnons

Assume thermodynamic limit, i.e. large L

Eigenvalue problem: $\tilde{\lambda} D_2 |\Psi\rangle = E|\Psi\rangle$

$$A_{\frac{1}{2}} = Z, \ A_{-\frac{1}{2}} = X$$

Ansatz for state with q impurities:

$$|\Psi\rangle = \sum_{l_1,\ldots,l_q} \Psi(l_1,\ldots,l_q) \prod_{i=1}^{q} S_{z,l_i} |S_z\rangle_L$$

$$= \sum_{l_1,\ldots,l_q} \sum_{s\in Q} \Psi(l_1,\ldots,l_q) \prod_{i=1}^{q} s(l_i) \mathrm{Tr}(A_{s(1)} \cdots A_{s(L)})$$

$$Q = \left\{ s = (s(1), s(2), \ldots, s(L)) \,\middle|\, \sum_{i=1}^{L} s(i) = S_z, \ s(i) = \pm\frac{1}{2} \right\}$$

Using Bethe ansatz techniques + integrability of the Heisenberg chain
we get the spectrum for large $\tilde{\lambda}, \ L$:

$$E = \frac{2\pi^2\tilde{\lambda}}{L^2} \sum_{n\neq 0} n^2 M_n \ , \quad \sum_{n\neq 0} n M_n = 0$$

A string-like spectrum
in weakly coupled gauge theory

Hagedorn temperature from Heisenberg chain:

Consider the partition function

$$\log Z(\tilde{\beta}) = \sum_{n=1}^{\infty} \sum_{L=1}^{\infty} \frac{1}{n} e^{-nL\tilde{\beta}} \mathrm{Tr}_L \left(e^{-n\tilde{\beta}\tilde{\lambda} D_2} \right)$$

Define

$$V(t) \equiv \lim_{L\to\infty} \frac{1}{L} \log \mathrm{Tr}_L \left(e^{-t^{-1} D_2} \right)$$

Notice: $f(t) = -tV(t)$ ⟵ f(t) is the thermodynamic limit of the free energy per site for the Heisenberg chain

We see then that

$$e^{-nL\tilde{\beta}} \mathrm{Tr}_L \left(e^{-n\tilde{\beta}\tilde{\lambda} D_2} \right) \simeq \exp\left(-nL\tilde{\beta} + LV((n\tilde{\beta}\tilde{\lambda})^{-1}) \right) \quad \text{for } L \to \infty$$

n=1 gives the first singularity

Therefore we have the Hagedorn singularity for temperature $\tilde{T} = \tilde{T}_H$ given by

$$\tilde{T}_H = \frac{1}{V\left(\tilde{\lambda}^{-1}\tilde{T}_H\right)}$$

A general relation between thermodynamics of Heisenberg chain and the Hagedorn temperature

$$\tilde{T}_H = \frac{1}{V\left(\tilde{\lambda}^{-1}\tilde{T}_H\right)}$$

← Defines $\tilde{T}_H$ as function of $\tilde{\lambda}$

Large $\tilde{\lambda}$ ⟷ Low temperatures $t \ll 1$

Small $\tilde{\lambda}$ ⟷ High temperatures $t \gg 1$

Small $\tilde{\lambda}$ /high temperatures:

$$V(t) = \log 2 - \frac{1}{4t} + \frac{3}{32t^2} - \frac{1}{64t^3} - \frac{5}{1024t^4} + \frac{3}{1024t^5} + \mathcal{O}(t^{-6})$$

Obtained from the integral equation:

$$u(x) = 2 + \oint_C \frac{dy}{2\pi i}\left\{\frac{1}{x-y-2i}\exp\left[-\frac{2t^{-1}}{y(y+2i)}\right] + \frac{1}{x-y+2i}\exp\left[-\frac{2t^{-1}}{y(y-2i)}\right]\right\}\frac{1}{u(y)}$$

$$V(t) = \log\left[u(0)\right]$$

Shiroishi & Takahashi

Using the general formula we get

$$\begin{aligned}\tilde{T}_H \; = \; & \frac{1}{\log 2} + \frac{1}{4\log 2}\tilde{\lambda} - \frac{3}{32}\tilde{\lambda}^2 + \left(\frac{3}{128} + \frac{\log 2}{64}\right)\tilde{\lambda}^3 + \left(-\frac{3}{512} - \frac{17\log 2}{1024} + \frac{5(\log 2)^2}{1024}\right)\tilde{\lambda}^4 \\ & + \left(\frac{3}{2048} + \frac{39\log 2}{4096} + \frac{3(\log 2)^2}{4096} - \frac{3(\log 2)^3}{1024}\right)\tilde{\lambda}^5 + \mathcal{O}(\tilde{\lambda}^6)\end{aligned}$$

Large $\tilde{\lambda}$ /low temperatures:

Using the low-energy spectrum $E = \frac{2\pi^2\tilde{\lambda}}{L^2}\sum_{n\neq 0} n^2 M_n \ , \quad \sum_{n\neq 0} n M_n = 0$

we find for t $\ll$ 1: $V(t) = \zeta\left(\frac{3}{2}\right)\sqrt{\frac{t}{2\pi}}$

This gives $\tilde{T}_H = (2\pi)^{1/3}\left[\zeta\left(\frac{3}{2}\right)\right]^{-2/3}\tilde{\lambda}^{1/3}$ for $\tilde{\lambda} \gg 1$

Sensible that $\tilde{T}_H \to \infty$ for $\tilde{\lambda} \to \infty$ since from the Hamiltonian $\tilde{\lambda} D_2$ we see that the vacua gives the dominant contribution

$\to$ Partition function becomes the trace over chiral primaries

Correction computed in the Heisenberg chain: $V(t) = \zeta\left(\frac{3}{2}\right)\sqrt{\frac{t}{2\pi}} - t$

Takahashi

Gives correction: $\tilde{T}_H = \frac{(2\pi)^{1/3}}{\zeta(\frac{3}{2})^{2/3}}\tilde{\lambda}^{1/3} + \frac{4\pi}{3\zeta(\frac{3}{2})^2} + \mathcal{O}(\tilde{\lambda}^{-1/3})$

Microcanonical version of the limit:

In the following we turn to the string side
$\to$ Important to formulate a microcanonical version of the limit

We consider U(N) $\mathcal{N}$ =4 SYM on $\mathbb{R} \times$ S^3 in the limit

$$\epsilon \to 0, \ \tilde{H} \equiv \frac{E-J}{\epsilon} \text{ fixed}, \ \tilde{\lambda} \equiv \frac{\lambda}{\epsilon} \text{ fixed}, \ J_i \text{ fixed}, \ N \text{ fixed}$$

In this limit planar $\mathcal{N}$ = 4 SYM becomes the ferromagnetic $XXX_{1/2}$ Heisenberg model (for the single-trace sector)

$\tilde{H}$: Hamiltonian for Heisenberg model

ε is a way to define $\tilde{\lambda}$ in the microcanonical ensemble

Alternatively, we can formulate the limit as

$$\lambda \to 0, \ \frac{E-J}{\lambda} \text{ fixed}, \ J_i \text{ fixed}, \ N \text{ fixed}$$

Limit very different from pp-wave limits
where E – J is fixed while J $\to \infty$ and N $\to \infty$

$\mathcal{N}$ = 4 SYM is weakly coupled

We are particularly interested in the regime with large $\tilde{\lambda}, \; J$

What does large $\tilde{\lambda}$ corresponds to?

One considers energies $\tilde{H} \sim 1 \longrightarrow \frac{E-J}{\lambda} \sim \frac{1}{\tilde{\lambda}}$

Thus large $\tilde{\lambda}$ corresponds to E-J $\ll \lambda$

Therefore, $\mathcal{N}$=4 SYM on $\mathbb{R} \times S^3$ has a string like spectrum in the regime

$$E - J \ll \lambda \ll 1 \quad , \qquad J \gg 1$$

$\leftarrow$ This defines the regime in terms of microcanonical variables

In this regime we can find free strings in Yang-Mills theory!

Decoupling limit of string theory:

$\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$ dual to type IIB string theory on $AdS_5 \times S^5$

$$T_{\text{str}} = \frac{1}{2}\sqrt{\lambda} \qquad g_s = \frac{\lambda}{N} \qquad \text{with} \quad T_{\text{str}} = \frac{R^2}{4\pi l_s^2}$$

Dual decoupling limit:

$$\epsilon \to 0, \ \tilde{H} \equiv \frac{E-J}{\epsilon} \text{ fixed}, \ \tilde{T}_{\text{str}} \equiv \frac{T_{\text{str}}}{\sqrt{\epsilon}} \text{ fixed}, \ \tilde{g}_s \equiv \frac{g_s}{\epsilon} \text{ fixed}, \ J_i \text{ fixed}$$

A zero string-tension, zero string-coupling limit

Consider planar limit $N = \infty$ / free strings $g_s = 0$:

Ferromagnetic Heisenberg spin chain ⟷ Limit of weakly coupled planar $\mathcal{N} = 4$ SYM ⟷ Limit of free strings on $AdS_5 \times S^5$

$$\log Z(\tilde{\beta}) = \sum_{n=1}^{\infty} \sum_{L=1}^{\infty} \frac{1}{n} e^{-nL\tilde{\beta}} Z_L^{(\text{XXX})}(n\tilde{\beta})$$

$$\tilde{T}_{\text{str}} = \frac{1}{2}\sqrt{\tilde{\lambda}}$$

Correspondence for large J / Penrose limit of $AdS_5 \times S^5$:

Want to find appropriate pp-wave background

Vacua in gauge theory are chiral primaries with $D = J_1 + J_2$

$\rightarrow$ String theory vacua: $E = J_1 + J_2$

$\Rightarrow$ We should consider string spectrum near $E = J_1 + J_2$

Leads to consider a Penrose limit resulting in the pp-wave background:

$$\frac{1}{\sqrt{\epsilon}} ds^2 = -4dx^+ dx^- - \mu^2 \sum_{I=3}^{8} x^I x^I (dx^+)^2 + \sum_{i=1}^{8} dx^i dx^i + 4\mu x^2 dx^1 dx^+$$

$\leftarrow$ Michelson

$$\frac{1}{\epsilon} F_{(5)} = 2\mu dx^+ (dx^1 dx^2 dx^3 dx^4 + dx^5 dx^6 dx^7 dx^8)$$

Penrose limit:
Bertolini, de Boer, TH, Imeroni & Obers

with currents

$$H_{lc} = \sqrt{\epsilon}\mu(E - J) \ , \quad p^+ = \frac{E + J}{2\mu R^2} \ , \quad p_1 = \frac{2S_z}{R}$$

x^1 a flat direction

New thing in Penrose limit: The ϵ factor $\leftarrow$ Wait one slide…

Light-cone string spectrum:

$$\frac{1}{\sqrt{\epsilon}} l_s^2 p^+ H_{\rm lc} = 2fN_0 + \sum_{n\neq 0} [(\omega_n + f)N_n + (\omega_n - f)M_n] + \sum_{n\in Z}\sum_{I=3}^{8} \omega_n N_n^{(I)} + \sum_{n\in Z}\left[\sum_{b=1}^{4}\left(\omega_n - \frac{1}{2}f\right)F_n^{(b)} + \sum_{b=5}^{8}\left(\omega_n + \frac{1}{2}f\right)F_n^{(b)}\right]$$

Level-matching condition:

$$\sum_{n\neq 0} n\left[N_n + M_n + \sum_{I=3}^{8} N_n^{(I)} + \sum_{b=1}^{8} F_n^{(b)}\right] = 0$$

with

$$f = \mu l_s^2 p^+ \ , \quad \omega_n = \sqrt{n^2 + f^2}$$

why the right pp-wave? $H_{lc} = 0 \leftrightarrow E = J$

A vacuum for each $p_1 \leftrightarrow$ a vacuum for each S_z

$\rightarrow$ The pp-wave has the right vacuum structure due to the flat direction

The ϵ factor:

We take the limit $\epsilon \to 0, \; \tilde{T}_{\rm str} \equiv \frac{T_{\rm str}}{\sqrt{\epsilon}}$ fixed with $T_{\rm str} = \frac{R^2}{4\pi l_s^2}$

Define therefore $\tilde{R}^4 = \frac{R^4}{\epsilon}$ ⟵ The rescaled AdS radius

Penrose limit: $\tilde{R} \to \infty \; , \quad J \to \infty \; , \quad p^+ = \frac{J}{\tilde{R}^2}$ fixed

Translates on the gauge theory side to: $\tilde{\lambda} \to \infty \; , \quad J \to \infty \; , \quad \frac{\tilde{\lambda}}{J^2}$ fixed

This is precisely the correct regime, as we shall see

<u>We can now implement the decoupling limit on the pp-wave background:</u>

$$\epsilon \to 0 \; , \quad \sqrt{\epsilon}\mu \text{ fixed} \; , \quad \tilde{H}_{\rm lc} \equiv \frac{1}{\epsilon} H_{\rm lc} \text{ fixed} \; , \quad \tilde{g}_s \equiv \frac{g_s}{\epsilon} \text{ fixed} \; , \quad l_s, p^+ \text{ fixed}$$

We see that $\mu \to \infty$ in the limit

Decoupling limit for pp-wave:

$$\epsilon \to 0\,, \quad \sqrt{\epsilon}\mu \text{ fixed}\,, \quad \tilde{H}_{\text{lc}} \equiv \frac{1}{\epsilon} H_{\text{lc}} \text{ fixed}\,, \quad \tilde{g}_s \equiv \frac{g_s}{\epsilon} \text{ fixed}\,, \quad l_s, p^+ \text{ fixed}$$

Limit of spectrum

- Only the modes with number operator M_n survives since $f \to \infty$

$$\sqrt{n^2 + f^2} - f = f\left(\sqrt{1 + f^{-2} n^2} - 1\right) \simeq \frac{n^2}{2f}$$

- Presence of flat direction gives non-trivial spectrum after limit, can be understood geometrically

Spectrum after decoupling limit

$$\tilde{H}_{\text{lc}} = \frac{1}{2(l_s^2 p^+)^2} \sum_{n \neq 0} n^2 M_n\,, \qquad \sum_{n \neq 0} n M_n = 0$$

Using $\tilde{T}_{\text{str}} = \frac{1}{2}\sqrt{\tilde{\lambda}}$ we can write this as

$$\tilde{H}_{\text{lc}} = \frac{2\pi^2 \tilde{\lambda}}{J^2} \sum_{n \neq 0} n^2 M_n\,, \qquad \sum_{n \neq 0} n M_n = 0$$

Valid for large $\tilde{\lambda}$, J

Matches spectrum of weakly coupled gauge theory!

Computation of Hagedorn temperature, I:

Computation using spectrum after decoupling limit

Multi-string partition function:

$$\log Z(\tilde{a},\tilde{b}) = \sum_{n=1}^{\infty} \frac{1}{n} \mathrm{Tr}(e^{-\tilde{a} n \tilde{H}_{\mathrm{lc}} - \tilde{b} n p^+})$$

Trace over single-string states

Z(a,b) has singularity for $\tilde{b}\sqrt{\tilde{a}} = l_s^2 \zeta(3/2)\sqrt{2\pi}$

From the Penrose limit one finds $\tilde{a} = \tilde{\beta}\ ,\quad \tilde{b} = \tilde{T}_{\mathrm{str}} l_s^2 \tilde{\beta}$

Using $\tilde{T}_{\mathrm{str}} = \frac{1}{2}\sqrt{\tilde{\lambda}}$

we get $\tilde{T}_H = (2\pi)^{1/3} \left[\zeta\left(\frac{3}{2}\right)\right]^{-2/3} \tilde{\lambda}^{1/3}$

Matches the Hagedorn temperature computed
in gauge theory/Heisenberg chain

Computation of Hagedorn temperature, II:

We can also consider the Hagedorn temperature as computed using the full pp-wave spectrum. Consider the partition function

$$\log Z(a,b) = \sum_{n=1}^{\infty} \frac{1}{n} \mathrm{Tr}\left((-1)^{(n+1)F} e^{-anH_{\mathrm{lc}} - bnp^+}\right)$$

This has a Hagedorn singularity for

$$b = 4 l_s^2 \mu \sum_{p=1}^{\infty} \frac{1}{p} \left[3 + \cosh(\mu a p) - 4(-1)^p \cosh\left(\frac{1}{2}\mu a p\right)\right] K_1(\mu a p)$$

Sugawara

Using now $a = \mu \tilde{\beta}$, $b = \tilde{T}_{\mathrm{str}} l_s^2 \tilde{\beta}$

we can take the $\varepsilon \to 0$ limit, obtaining again

$$\tilde{T}_H = (2\pi)^{1/3} \left[\zeta\left(\frac{3}{2}\right)\right]^{-2/3} \tilde{\lambda}^{1/3}$$

Check on the validity of the decoupling limit

$\to$ verifies commutativity of limits

We have matched spectrum and Hagedorn temperature of weakly coupled gauge theory and free string theory, in a sector of AdS/CFT

Why it worked?

- ► Because on the gauge theory side we could consider $\tilde{\lambda} \gg 1$

  Corresponds to looking at states near chiral primaries

  $\rightarrow$ We can ignore most states in the SU(2) sector, only the magnon states important for low energies

- ► Because we have a pp-wave with the same vacuum structure as for the gauge theory side

A non-trivial match between weakly coupled gauge theory and weakly coupled string theory

Can either be understood as matching of spectra (non-thermal) or matching of thermal partition function (thermodynamics)

Matching done for $\tilde{\lambda} \gg 1$ with $\tilde{T}_{\text{str}} = \frac{1}{2}\sqrt{\tilde{\lambda}}$

Meaning of large $\tilde{\lambda} = \frac{\lambda}{1-\Omega}$

Can be seen as strong coupling in the gauge theory even though $\lambda \to 0$

Why? Because at each order of $\tilde{\lambda}$ diagrams of the same order in λ contribute. For instance in the computation for the Hagedorn temperature:

$$\tilde{T}_H = \frac{1}{\log 2} + \frac{1}{4\log 2}\tilde{\lambda} - \frac{3}{32}\tilde{\lambda}^2 + \left(\frac{3}{128} + \frac{\log 2}{64}\right)\tilde{\lambda}^3 + \left(-\frac{3}{512} - \frac{17\log 2}{1024} + \frac{5(\log 2)^2}{1024}\right)\tilde{\lambda}^4 + \left(\frac{3}{2048} + \frac{39\log 2}{4096} + \frac{3(\log 2)^2}{4096} - \frac{3(\log 2)^3}{1024}\right)\tilde{\lambda}^5 + \mathcal{O}(\tilde{\lambda}^6)$$

Therefore: We have found a way to take the strong coupling limit of gauge theory in our subsector.

Works due to the truncation of the Hamiltonian: $H = D_0 + \tilde{\lambda} D_2$

Compare also to 't Hooft limit: $\lambda = g_{\text{YM}}^2 N$ fixed for $N \to \infty$

Means that $g_{\text{YM}} \to 0$. But $\lambda \gg 1$ is strong coupling for the planar limit.

Large $\tilde{\lambda}$

String theory on $AdS_5 \times S^5$
in pp-wave limit

$$\tilde{T}_H = (2\pi)^{1/3} \left[\zeta\left(\frac{3}{2}\right) \right]^{-2/3} \tilde{\lambda}^{1/3}$$

Heisenberg
spin chain

$$\tilde{T}_H = \frac{1}{V\left(\tilde{\lambda}^{-1} \tilde{T}_H\right)}$$

$\tilde{\lambda}$

Small $\tilde{\lambda}$

Perturbative SYM on $\mathbb{R} \times S^3$

$$\tilde{T}_H = \frac{1}{\log 2} + \frac{1}{4 \log 2} \tilde{\lambda} + \mathcal{O}(\tilde{\lambda}^2)$$

⇒ We can fully connect the Hagedorn temperature
from free SYM on $\mathbb{R} \times S^3$ to string theory on $AdS_5 \times S^5$

Conclusions:

We found the decoupling limit

$$T \to 0,\ \Omega \to 1,\ \lambda \to 0,\ \tilde{T} \equiv \frac{T}{1-\Omega},\ \tilde{\lambda} \equiv \frac{\lambda}{1-\Omega},\ N \text{ fixed}$$

of $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$ in which the partition function becomes

$$Z(\tilde{\beta}) = \mathrm{Tr}_{\mathcal{H}} \left(e^{-\tilde{\beta}(D_0 + \tilde{\lambda} D_2)} \right)$$

where $\mathcal{H}$ corresponds to SU(2) sector of $\mathcal{N} = 4$ SYM.

In the planar limit N = ∞:

Physics of Heisenberg model $\Longleftrightarrow$ Physics of decoupled planar $\mathcal{N} = 4$ SYM

- ▶ A manifestly integrable decoupled sector of planar $\mathcal{N} = 4$ SYM
- ▶ Describes planar $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$ near the critical point $(T,\Omega_1,\Omega_2,\Omega_3) = (0,1,1,0)$

Implications for AdS/CFT:

► Dual limit a zero string tension, zero string coupling limit of type IIB string theory on $AdS_5 \times S^5$

► Planar limit/zero string coupling: A solvable sector of AdS/CFT

Ferromagnetic Heisenberg spin chain	⟺	Limit of weakly coupled planar $\mathcal{N} = 4$ SYM	⟺	Limit of free strings on $AdS_5 \times S^5$

► Explicit matching for spectrum and Hagedorn temperature using pp-wave

$$E = \frac{2\pi^2 \tilde{\lambda}}{L^2} \sum_{n \neq 0} n^2 M_n, \quad \sum_{n \neq 0} n M_n = 0 \qquad \tilde{T}_H = (2\pi)^{1/3} \left[\zeta\left(\frac{3}{2}\right) \right]^{-2/3} \tilde{\lambda}^{1/3}$$

► planar $\mathcal{N} = 4$ SYM has string-like behavior in the regime

$$E - J \ll \lambda \ll 1 \quad , \qquad J \gg 1$$

In this regime can match the spectrum of gauge and string theory

Future directions:

- ► Turn on other chemical potentials
- ► Study Hagedorn transition on gauge theory side
- ► Finite size corrections on the string side
- ► 1/N corrections and pp-wave string interactions

Paper with K. R. Kristjansson & M. Orselli (hep-th/0611242):
Decoupling limit giving ferromagnetic Heisenberg chain with magnetic field

Modified decoupling limit with magnetic field:

Generalization of decoupling limit, same critical point $(T,\Omega_1,\Omega_2,\Omega_3) = (0,1,1,0)$ so $\Omega_1, \Omega_2 \to 1$ but now with $\Omega_1 \neq \Omega_2$

define $$\Omega \equiv \frac{\Omega_1 + \Omega_2}{2} \qquad h \equiv \frac{\Omega_1 - \Omega_2}{2}$$

Decoupling limit:

$$\Omega \to 1,\ \tilde{T} \equiv \frac{T}{1-\Omega} \text{ fixed},\ \tilde{h} \equiv \frac{h}{1-\Omega} \text{ fixed},\ \tilde{\lambda} \equiv \frac{\lambda}{1-\Omega} \text{ fixed},\ N \text{ fixed}$$

Full partition function of $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$ reduces to

$$Z(\tilde{\beta},\tilde{h}) = \mathrm{Tr}_{\mathcal{H}} \left(e^{-\tilde{\beta}(D_0 + \tilde{\lambda} D_2 - 2\tilde{h} S_z)} \right)$$

$\mathcal{H}$: The SU(2) sector of $\mathcal{N} = 4$ SYM

$$J \equiv J_1 + J_2$$

$$S_z \equiv \frac{J_1 - J_2}{2}$$

For planar $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$:

$\tilde{\lambda} D_2 - 2\tilde{h} S_z$ ⟵ Hamiltonian of ferromagnetic $XXX_{1/2}$ Heisenberg spin chain with magnetic field

Magnetic field $\longrightarrow$ Non-trivial effect on low energy spectrum

Degeneracy of vacuum sector broken, only one vacuum $\mathrm{Tr}\left(Z^L\right)$

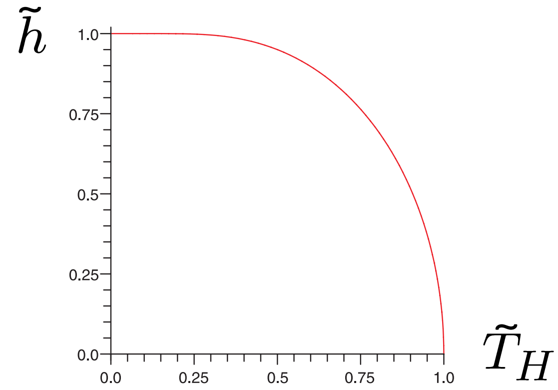
Spectrum of $\tilde{\lambda} D_2 - 2\tilde{h} S_z$ is:

$$E = -\tilde{h}L + \frac{2\pi^2\tilde{\lambda}}{L^2}\sum_n n^2 M_n + 2\tilde{h}\sum_n M_n \,, \quad \sum_n n M_n = 0$$

$\tilde{\lambda} \gg 1$

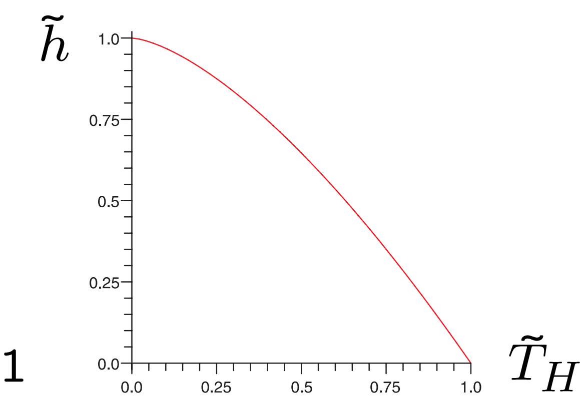
$L \gg 1$

Hagedorn temperature:



$$\tilde{T}_H = \frac{1}{\log(2\cosh(\tilde{h}/\tilde{T}_H))} \quad \text{for} \quad \tilde{\lambda} = 0$$

$$\tilde{T}_H = \frac{(2\pi)^{1/3}(1-\tilde{h})^{2/3}}{\zeta\left(\frac{3}{2}\right)^{2/3}}\tilde{\lambda}^{1/3} \quad \text{for } \tilde{\lambda} \gg 1$$



In general: Bound $0 \leq \tilde{h} \leq 1$ and $\tilde{T}_H \to 0$ for $\tilde{h} \to 1$

On the string side we find a new Penrose limit giving a geometric realization of the breaking of the symmetry by the magnetic field

We match the string spectrum with gauge theory for $\tilde{\lambda} \gg 1 \ , \ L \gg 1$

$$E = -\tilde{h}L + \frac{2\pi^2\tilde{\lambda}}{L^2}\sum_n n^2 M_n + 2\tilde{h}\sum_n M_n \ , \ \ \sum_n n M_n = 0$$

Using this, we match the Hagedorn temperature on the gauge theory and string theory sides for $\tilde{\lambda} \gg 1$

$$\tilde{T}_H = \frac{(2\pi)^{1/3}(1-\tilde{h})^{2/3}}{\zeta\left(\frac{3}{2}\right)^{2/3}}\tilde{\lambda}^{1/3} \text{ for } \tilde{\lambda} \gg 1$$