Cosmology and astrophysics with exact inhomogeneous solutions of General Relativity

Marie-Noëlle Célérier

SIGRAV, Florence, 27 January 2009

l'Observatoire de Paris — LUTH

Laboratoire de l'Univers et de ses Théories

Outline

(1) Exact inhomogeneous solutions of GR used here

(2) Structure evolution with L–T models

(3) Structure evolution with Szekeres models

(4) Solving cosmological problems with inhomogeneous models

(5) Conclusion

Exact inhomogeneous solutions of GR

The Lemaître–Tolman solution (1)

Spherically symmetric non-static solution with a dust source.

Metric in comoving and synchronous coordinates

$$ds^2 = dt^2 - \frac{R'^2}{1+2E(r)} dr^2 - R^2(t,r)(d\vartheta^2 + \sin^2\vartheta \, d\varphi^2)$$

A first integral of the Einstein equations = dynamical equation for *R*:

$$\dot{R}^2 = 2E + \frac{2M}{R} + \frac{\Lambda}{3} R^2$$

The mass density in energy units

$$\frac{8\pi G}{c^4}\rho = \frac{2M'}{R^2 R'}$$

Integrating the dynamical equation for *R* gives

$$\int_0^R \frac{d\widetilde{R}}{\sqrt{2E + 2M/\widetilde{R} + \frac{1}{3}\Lambda \widetilde{R}^2}} = t - t_B(r)$$

Three integration functions:

- $E(r)$, **the energy per unit mass of the particles within the comoving shell of radius** r.
- $M(r)$, **the gravitational mass of the particles in that shell.**
- $t_B(r)$, **the "bang time": the singularity (bang or crunch) is not simultaneous as in Friedmann models, but occurs at different times at different distances from the origin.**

The Lemaître–Tolman solution (2)

For $\Lambda = 0$ the first integral can be solved explicitly.

$E < 0$ (elliptic evolution)

$$R(t, r) = \frac{M}{(-2E)}(1 - \cos\eta)$$

$$\eta - \sin\eta = \frac{(-2E)^{3/2}}{M}(t - t_B(r))$$

$E = 0$ (parabolic evolution)

$$R(t, r) = \left[9/2\, M\, (t - t_B(r))^2\right]^{1/3}$$

$E > 0$ (hyperbolic evolution)

$$R(t, r) = \frac{M}{2E}(\cosh\eta - 1)$$

$$\sinh\eta - \eta = \frac{(2E)^{3/2}}{M}(t - t_B(r))$$

All the above formulas are covariant under coordinate transformations: $\widetilde{r} = g(r)$. Thus, one of the functions $E(r)$, $M(r)$ and $t_B(r)$ can be fixed at will by the choice of g.

Two degrees of freedom remain.

The Friedmann solutions are contained in the L–T class as the limit:

$$t_B = \text{const}, \qquad |E|^{3/2}/M = \text{const}.$$

The Lemaître–Tolman solution (3)

- **Origin conditions**

  An origin, or centre of spherical symmetry, occurs at $r = r_c$ where $R(t, r_c) = 0$ for all t. The conditions for a regular centre follow from the requirement that there is no point-mass and no curvature singularity at $r = r_c$ at all times after the Big Bang and before the Big Crunch. Mustapha and Hellaby 2001.

- **Shell crossings**

  Regular extremum. The density and the Kretschmann scalar diverge where $R' = 0$ and $M' \neq 0$, but loci where M'/R' is finite when $R' = 0$ are also possible on constant r shells. These constitute regular extrema in the spatial section of constant t. Both M' and R' change sign across a regular extremum.

  Shell crossings (constant r shell colliding with its neighbour) are loci of $R' = 0$ that are not regular maxima or minima of R. They create undesirable singularities where the density diverges and changes sign. Hellaby and Lake 1985 have derived the conditions on the 3 arbitrary functions that ensure none be present anywhere in an L–T model.

The Szekeres solution

Solution of GR with no symmetry (no Killing vector) and dust source.

$$ds^2 = dt^2 - e^{2\alpha} dr^2 - e^{2\beta}\left(dx^2 + dy^2\right)$$

Two families of Sz solutions. Only the one with $\beta' \neq 0$ yields useful applications in astrophysics and cosmology.

Can be parametrised so that a set of independent functions of r determines each solution.

The sign of one of them fixes the geometry of the $\{t = \mathrm{const.}\}$ 3-surfaces and the type of evolution (elliptic, parabolic, hyperbolic).

The sign of another function determines the geometry of the $\{t = \mathrm{const.}, r = \mathrm{const.}\}$ 2-surfaces $\Rightarrow$ quasi-spherical, quasi-plane and quasi-hyperbolic models. Only the quasi-spherical model has been found useful in astrophysical cosmology.

A function $t_B(r)$ represents the Big-Bang (Crunch) time as in L–T models.

A shell-cross singularity can occur if $\left(e^{\beta}\right)' = 0$. Its intersection with a $\{t = \mathrm{const.}\}$ space is not a sphere, but a circle or a point.

Reparametrisation of the Szekeres solution

$$ds^2 = dt^2 - \frac{(R' - RE'/E)^2}{\varepsilon - k(r)} dr^2 - \frac{R^2}{E^2}(dx^2 + dy^2)$$

$$\mathbf{E} \equiv \frac{S}{2}\left[\left(\frac{x-P}{S}\right)^2 + \left(\frac{y-Q}{S}\right)^2 + \varepsilon\right]$$

$R(t, r), S(r), P(r), Q(r)$ **and for the quasi-spherical model** $\varepsilon = +1$**.**

$$\dot{R}^2 = -k(r) + \frac{2M}{R} + \frac{\Lambda}{3}R^2.$$

For $\Lambda = 0$ **the solutions** $R(t, r)$ **are the same as the L–T solutions.**

The density in this parametrisation is

$$\kappa\rho = \frac{2(M' - 3ME'/E)}{R^2(R' - RE'/E)}.$$

The number of arbitrary functions corresponding to physical degrees of freedom is 5 (6 functions: $k(r), M(r), t_B(r), S(r), P(r), Q(r)$ **- the choice of** r**).**

Properties of the quasi-spherical Szekeres solution

$R = 0$ **is an origin (Bang or Crunch).** $R < 0$ **impossible.**

$M(r) > 0 \Rightarrow$ **any vacuum exterior has positive Schwarzschild mass.**

$|S(r)| \neq 0$ **is needed so** $S > 0$ **is a reasonable choice.**

For a well-behaved metric signature $k < 1$ **except where** $(R' - RE'/E) = 0$

Either $M' - 3ME'/E \geq 0$ **and** $R' - RE'/E \geq 0$
or $M' - 3ME'/E \leq 0$ **and** $R' - RE'/E \leq 0$**.**

If $R' - RE'/E$ **passes through** 0 **anywhere other than at a regular extremum** $\Rightarrow$ **shell cross.**

Conditions of regularity at the origin and no shell-crossing have been derived by Hellaby and Krasiński 2002.

The distribution of mass over each single sphere $\{t = \text{const}, r = \text{const}\}$ **has the form of a mass-dipole superposed on a monopole.**

Structure evolution with L–T models

Initial conditions

Initial conditions are imposed at the CMB and at the observer. Then, the model is evolved in between.

Krasiński and Hellaby 2002 have estimated what angle would be subtended in the CMB sky by matter that today forms various objects.

At the current best resolution, the only structures whose trace has a chance of appearing on the CMB maps are large cluster concentrations like the Great Attractor (angle on the CMB sky: 0.2-0.3°).

No data exist on the shape of the density or velocity profiles at last scattering or on the spatial extent of the region that is perturbed. For the shape, simple functions are assumed.

It turns out that the final outcome of evolution is sensitive to the initial velocity profile ⇒ consider these models as preliminary.

Galaxy plus black hole formation

Aim: **model the formation of a galaxy with a central black hole (M87), starting from initial fluctuations at recombination and ending at present time** **Krasiński and Hellaby 2004.**

Model: **two parts matched across a comoving boundary** $M = M_{BH}$**, with** M_{BH} **the observationally estimated current mass inside the black hole.**

Exterior part = observed present-day density profile of M87 + initial fluctuation compatible with CMB observations.

Interior = no observational constraints $\Rightarrow$ **two descriptions proposed, both L–T models, 1) a collapsing body (the black hole is formed in the course of evolution), 2) a dense preexisting wormhole.**

Theorem: **the initial and final states uniquely define the L–T model evolving between them, i.e. the arbitrary functions** $E(M)$ **and** $t_B(M)$**.**

Conclusion. **The galaxy-black hole structure can be generated by both mechanisms, but the black hole appears in a much shorter time (almost instantly after the BB) by condensation around a wormhole.**

Rich galaxy cluster and void

Aim: check whether it is possible to obtain a rich galaxy cluster or a void from initial density and velocity perturbations.

Galaxy cluster

A pure velocity perturbation can nearly produce a galaxy cluster. A mere density perturbation fails to do it. Velocity perturbations generate structures much more efficiently than density perturbations.

Void

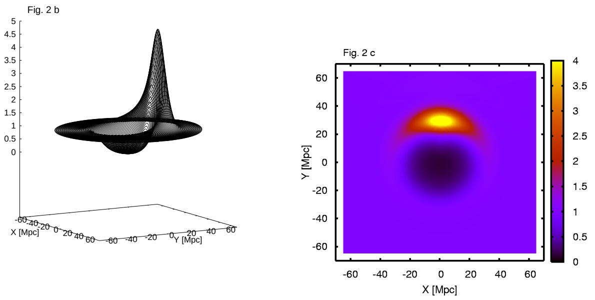
A void consistent with observational data (density contrast less than $\delta = -0.94$, smooth edges and high density in the surrounding regions) is very hard to obtain with L–T models without shell crossing Bolejko et al. 2005.

Adding a realistic distribution of radiation (using Tolman models, otherwise known as Misner-Sharp) helps forming observed voids Bolejko 2006.

Structure evolution with Szekeres models

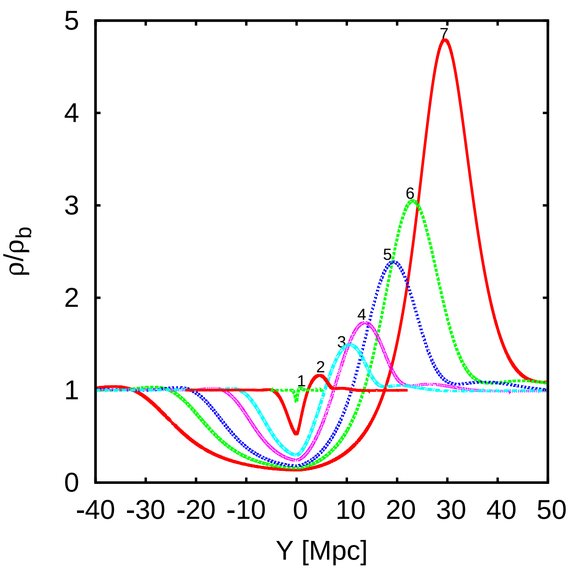
Double structure evolution

Model of a void with an adjourning supercluster evolved inside an homogeneous background Bolejko 2006.



Current density distribution in background units.

To estimate how two neighbouring structures influence each other, the evolution of a double structure in QSS models has been compared with that of single structures in L–T models and linear perturbation theory.

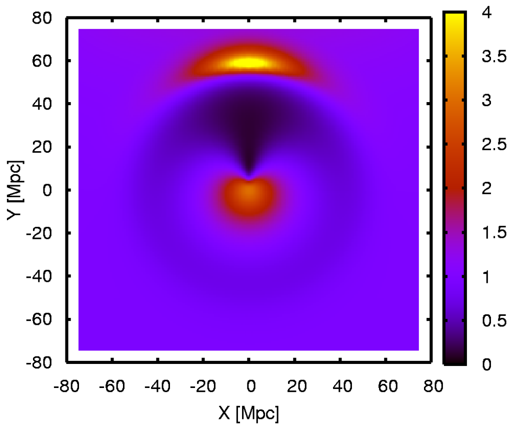


Evolution of the density profile from 100 Myr after the Big Bang (1) up to the present time (7).

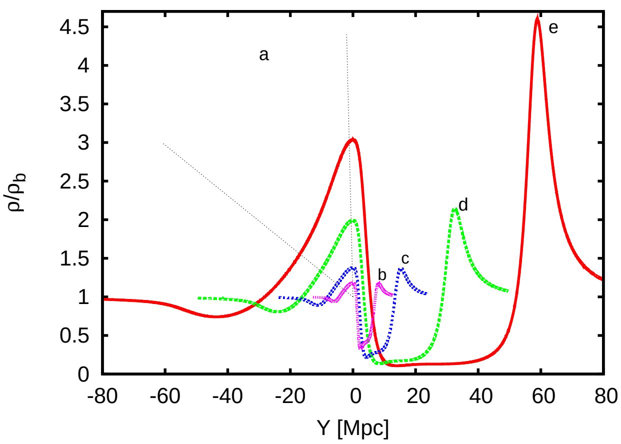
In the QSS models, the growth of the density contrast is 5 times faster than in L–T models and 8 times faster than in the linear approach.

Triple structure evolution

The model is composed of an overdense region at the origin, followed by a small void which spreads to a given r coordinate. At a larger distance from the origin, the void is huge and its larger side is adjacent to an overdense region Bolejko 2007.



Current density distribution.

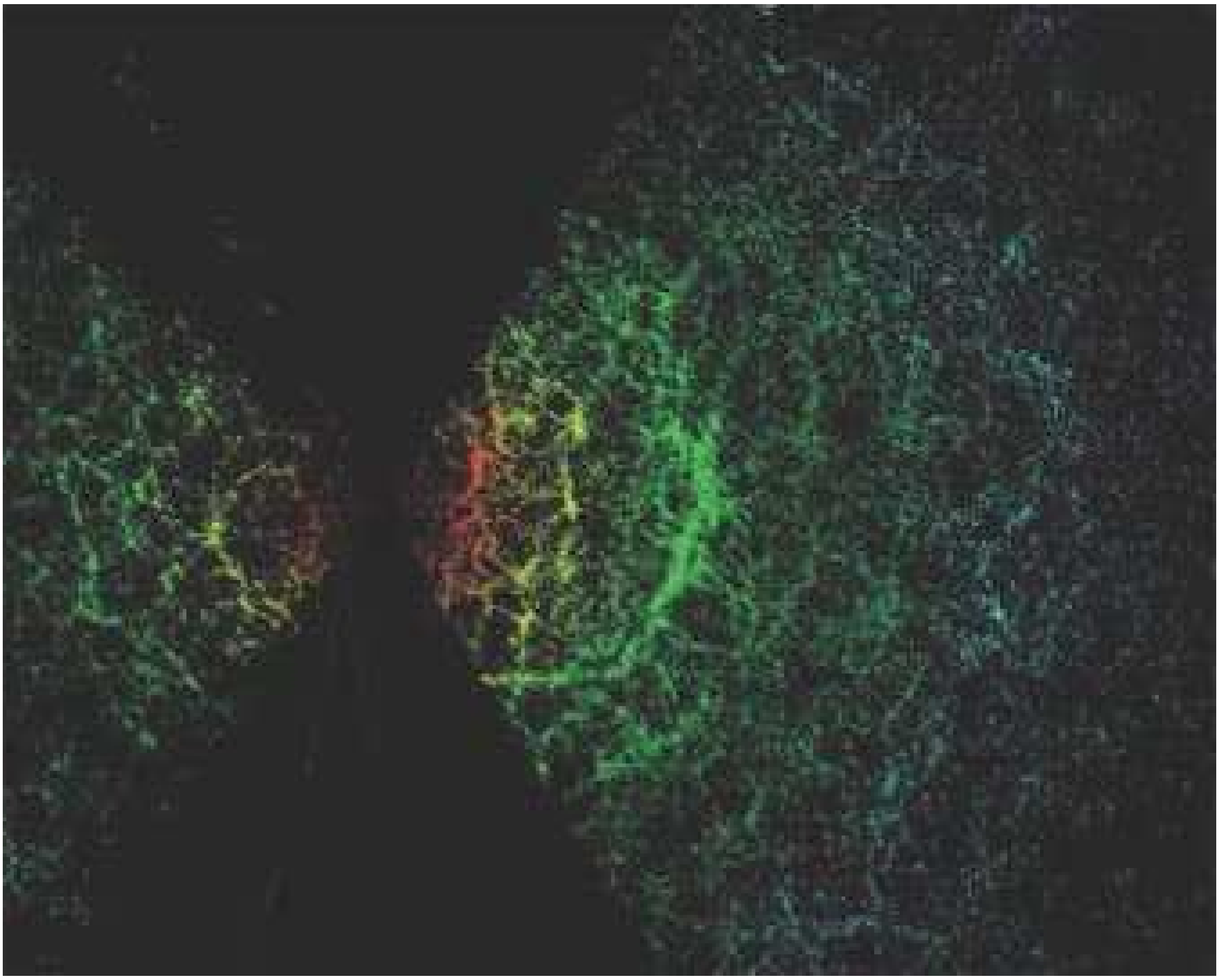


Evolution of the density profile from 0.5 Myr after the Big Bang (a) up o the present time (e).

Where the void is large, it evolves much faster than the underdense region closer to the "centered" cluster. The exterior overdense region close to the void along a large area evolves much faster than the more compact supercluster at the centre. This suggests that, in the Universe, small voids surrounded by large high densities evolve much more slowly than large isolated voids.

Solving cosmological problems with exact inhomogeneous models

The inhomogeneous Universe



Sloan Digital Sky Survey (false colors)

The Universe is not homogeneous. We see voids, groups of galaxies, clusters, superclusters, walls, filaments, etc. However, it is usually argued that it should be nearly homogeneous at large scales (thus the use of Friedmannian models); but how large are these scales?

SN Ia data and dark energy

Since its discovery (Riess et al. 1998, Perlmutter et al. 1999) the dimming of distant SNe Ia, as interpreted in a Friedmannian framework, has been mostly ascribed to the influence of a mysterious "dark energy" component.

This "dark energy" acts as a negative pressure fluid (possibly a mere cosmological constant) which tends to accelerate the Universe expansion.

Even if, from some theoretical point view , a geometrical cosmological constant could be a natural component of the Einstein equations (Nottale 1993, 1996), its actual value remains to be confirmed.

It is well-known that the inhomogeneities observed in our Universe can have an effect upon the values of the cosmological parameters derived in the framework of a smoothed out Friedmannian model Ellis and Stoeger 1987.

Moreover, the onset of "dark energy" domination in a previously matter dominated Universe appears at the epoch when structure formation enters the non-linear regime. Hence the failure of linear perturbation theory at these scales and the need of either averaging schemes or exact solutions.

Accelerated expansion: a mirage?

What we see is not an accelerated expansion (this is only the outcome of the Friedmannian assumption) but the "dimming" of the supernovae, or more exactly their luminosity distance-redshift relation.

However, the acceleration interpretation was sufficiently misleading such as to induce some authors to try to derive or rule out no-go theorems, i.e., theorems stating that a locally defined expansion cannot be accelerating in models satisfying the strong energy condition. But this is not the point.

Other stressed, more accurately, that the definition of a deceleration parameter in an inhomogeneous model is tricky Hirata and Seljak 2005, Apostopoulos et al. 2006.

To understand intuitively how inhomogeneities can mimic an accelerated expansion, consider a very simple toy model. A low-density inner homogeneous region is connected at some redshift to an outer homogeneous region of higher density. Both regions decelerate, but since the inner void expands faster than the outer region, an apparent acceleration is experienced by the observer located inside this void (Tomita, 2001).

Use of exact inhomogeneous models in cosmology

L–T models have been most widely used as exact inhomogeneous models in cosmology since they are the most tractable among the few available (but QSS models are currently slightly coming into play).

But caution with L–T is required since:

- An origin, or centre of spherical symmetry, occurs at $r = r_c$ where $R(t, r_c) = 0$ for all t. The conditions for a regular centre were derived by Mustapha and Hellaby 2001.
- Shell crossings, where a constant r shell collides with its neighbour, create undesirable singularities where the density diverges and changes sign. The conditions on the 3 arbitrary functions that ensure none be present anywhere in an L–T model were given by Hellaby and Lake 1985.
- The assumption of central observer, generally retained for simplicity, can be considered as grounded on the observed quasi-isotropy of the CMB temperature, and thus as a good working approximation at large scales. At smaller scales, it gives simplified models of the Universe averaged only over the angular coordinates around the observer, i. e., with the relax of only one degree of symmetry as regards the homogeneity assumption.

However, models assuming an off-centre observer and L–T Swiss-cheeses have also been studied to get rid of possible misleading features of spherical symmetry.

Degeneracy of L–T models

It is well-known, from the work of Mustapha, Hellaby and Ellis 1997, that an infinite class of L–T models can fit a given set of observations isotropic around the observer. This has been confirmed by MNC 2000 while studying the fitting of L–T models with a central observer to the supernova data.

The problem of finding THE spherically symmetric model able to solve the cosmological constant problem is therefore completely degenerate. It is the reason why many different centered L–T models have been proposed and shown to do the job rather well.

Thus, to constrain the model further on, it is mandatory to fit it to other cosmological data.

Direct and inverse problem

Two procedures for trying to solve the dark energy problem with inhomogeneous (L–T) models:

- **The direct way: first guess the form of the parameter functions defining a class of models supposed to represent our Universe with no cosmological constant, and write the dependence of these functions in terms of a limited number of constant parameters; then fit these constant parameters to the observed SN Ia data or to the luminosity distance-redshift relation of the standard ΛCDM model.**

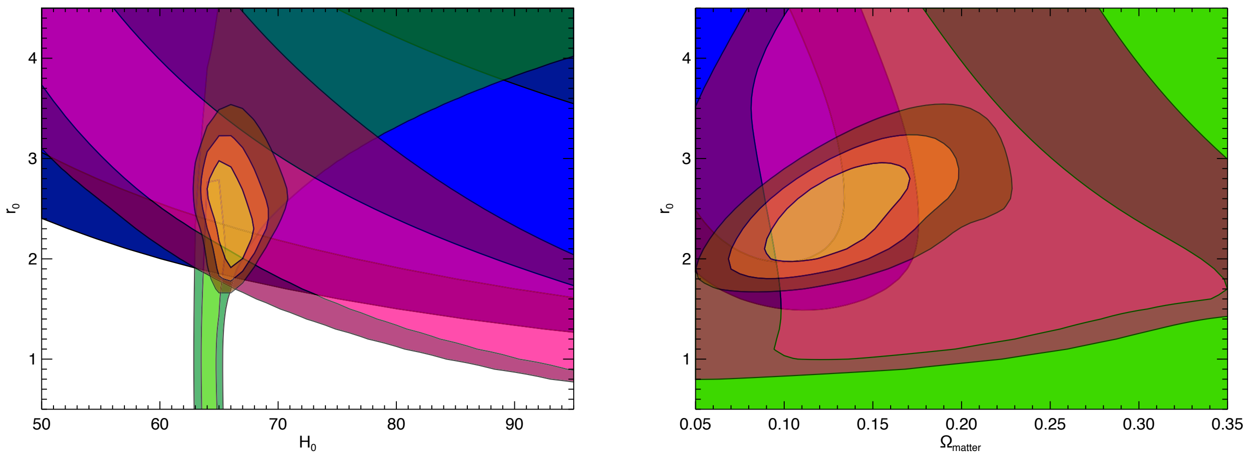
- **The inverse problem: consider the luminosity distance $D_L(z)$ as given by observations or by the ΛCDM model as an input and try to select a specific L–T model with zero cosmological constant best fitting this relation.**

Then, to avoid degeneracy, jump to a further step and try to reproduce more and possibly all the available observational data, Alnes et al. 2006, Bolejko 2008, Garcia-Bellido and Haugbolle 2008, Hellaby et al. 2007, 2008.

Example of centered L–T: the GBH model

The GBH void class of L–T model studied by Garcia-Bellido and Haugbolle 2008a,b is specified by its matter content $\Omega_M(r)$ and its expansion rate $H(r)$, governed by 5 free constant parameters and matching to an E-deS universe at large scales. A constrained model with an homogeneous Big-Bang is also considered.

These models are fitted to a series of observations (CMB, LSS, BAO, SN Ia, HST measure of H_0, age of the globular clusters, gas fraction in clusters, kinematic SZ effect for 9 distant galaxy clusters) to constrain their parameters.



Two of the six likelihoods for the GBH constrained model: in yellow with 1-, 2-, 3-σ contours = the likelihood for the combined data set; in blue, purple and green respectively with 1- and 2-σ contours = the likelihood for the individual SN Ia, BAO and CMB data sets.

Current data do put significant constraints on the models. Current observations do not exclude the possibility that we leave close to the centre of a large (around 2.5 Gpc) void within an Einstein-de Sitter Universe, i. e., no dark energy.

Example of L–T Swiss-cheese: Marra et al.'s model

The model: Lattice of L–T bubbles with radius ≥350 Mpc in E-de S background. Initially, the void at the center of each hole is dominated by negative curvature and a compensating overdensity matches smoothly the density and curvature EdS values at the border of the hole Marra, Kolb, Matarrese and Riotto 2007.

Since the voids expand faster than the cheese, the overdense regions contract and become thin shells at the borders of the bubbles while underdense regions turn into emptier voids, eventually occupying most of the volume.

Conclusion: Redshift effects are suppressed by a compensating effect acting on the size of a half bubble while evolution, i.e. the presence of voids, bends the photon paths and affects more photon physics than inhomogeneity geometry ⇒ ΛCDM best reproduced by large void models which have more time to evolve while photons are passing through.

Applying a fitting approach to the same Swiss-cheese, Marra, Kolb and Matarrese 2008 find no important effects due to global expansion but sensible ones with respect to density due to structure evolution.

Future prospects: consider QSS models which exhibit enhanced structure evolution.

Extracting the cosmic L–T metric from observations

The inverse problem of deriving the arbitrary functions of a L–T model from observations is very much involved. This is the reason why most of the authors who have tried to deal with this issue have added some a priori constraints to the model Vanderveld et al. 2006, Chung and Romano 2006, Tanimoto and Nambu 2007.

LU and Hellaby 2007 and McClure and Hellaby 2007 initiated a program to extract the metric from observations. This is the full inverse problem and is not degenerate. To date it has assumed the metric has the L–T form, as a relatively simple case to start from, though the long term intention is to remove the assumption of spherical symmetry.

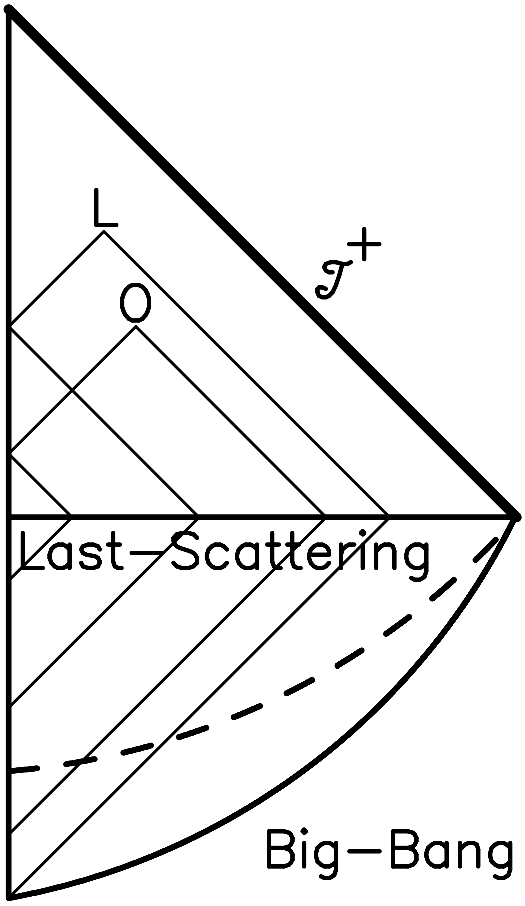
They developed and coded an algorithm that generates the L–T metric functions, given observational data on the redshifts, apparent luminosities or angular diameters, number counts of galaxies, estimates for the absolute luminosities or true diameters and source masses, as functions of z. This allows both of the physical functions of an L–T model to be determined without any a priori assumption on their form.

McClure and Hellaby 2008 improved several aspects of the numerical procedure so that it can handle input data that are subject to statistical fluctuations.

Horizon problem and inflation

Horizon problem. **The comoving region over which the CMB is observed as quasi-homogeneous at the last-scattering surface is much larger than the intersection of this surface with future light cones issuing from the BB surface. This problem develops sooner or later in any universe model with a spacelike BB singularity.**

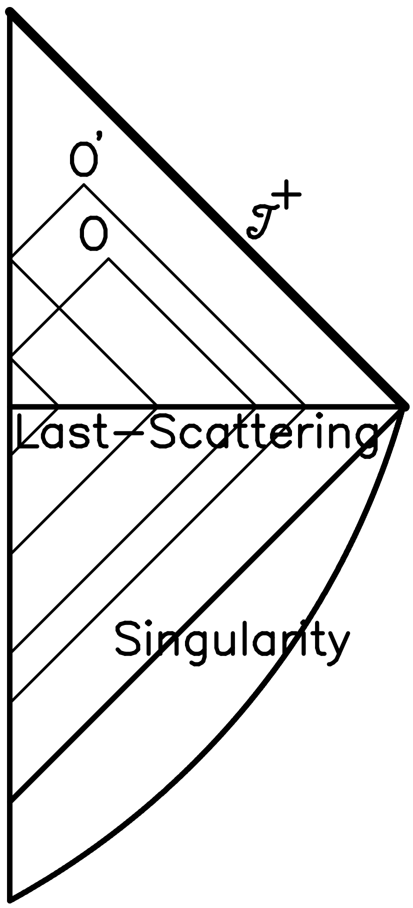
Inflation only postpones the occurrence of the problem.



Permanent solution: DBB L–T model

Theorem: a spherically symmetric model with $g_{00} = [t - b(r)]^a f(r, t)$ exhibits a timelike shell-crossing surface at $t = b(r)$ if $a > 1/2$ and $g'_{00} \neq 0$.

We have shown that a L–T model with $t'_B(r) > 0$ for all r (delayed Big-Bang) and some other properties exhibits the conditions of application of the above theorem allowing us to solve permanently the horizon problem with no inflation MNC and Schneider 1998, MNC 2000, MNC and Szekeres 2002.



Other applications of inhomogeneous solutions

- **Study of light propagation and temperature fluctuations of the CMB in QSS Swiss-cheese models.**

Clearly, underdense regions induce negative temperature fluctuations, overdense regions induce positive fluctuations (Rees-Sciama effect, less than one order of magnitude than the measured rms temperature fluctuations in the CMB). Although it is highly unlikely that the signal caused by local structures has a signature of acoustic oscillations, these local structures can have some visible impact on observations Bolejko 2009.

- **CMB low multipoles in L–T models with off-center observer.**

Large scale inhomogeneities can affect the amplitude of low-order multipoles and possibly reproduce the dipole, quadrupole and octopole anisotropies of the CMB Schneider and MNC 1999, Alnes and Amarzguioui 2006.

Conclusions

- **The concordance model is built on a spatially homogeneous background metric combined with first order perturbation theory to account for structure formation. Although this assumption yielded many important successes in cosmology it is not the whole story and we must not lose sight of the fact that the present-day Universe is actually very inhomogeneous.**

- **The increasing precision of observational data implies that the use of FLRW models, which has been appropriate up to now, must be considered just a zeroth order approximation, and linear perturbation theory a first order approximation whose domain of validity is an early, nearly homogeneous Universe.**

- **In the nonlinear regime, which was entered since structures formed, there is no escape from the use of exact methods (or of averaging schemes aiming at investigating this issue from the view-point of backreaction).**

- **In the era of "precision cosmology", inhomogeneity effects on the determination of cosmological models cannot be ignored.**

- **Exact inhomogeneous solutions can be employed not only for studying the geometry and dynamics of the Universe, but also to investigate the formation and evolution of structures.**