

Gauge invariant averages for the cosmological backreaction

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Introduction

- The study of the possible dynamical influence of (small) inhomogeneities on the large-scale evolution of a cosmological background has recently attracted considerable interest, from both a theoretical and a phenomenological point of view
- One needs a well defined averaging procedure for smoothing-out the perturbed (non-homogeneous) geometric parameters.
- The computation of these averages is affected in principle by a well-known ambiguity due to the possible choice of different “gauges”.

Outline

- **Gauge (non)-invariance of space-time integrals**
- **Gauge-invariant averaging prescriptions at second order**
- **Examples**
- **Conclusions**

Based on: M. Gasperini, G. M. and G. Veneziano, e-Print: arXiv:0901.1303 [gr-qc].

Gauge (non)-invariance of space-time integrals, 1

General coordinate transformations (GCT) \iff Gauge transformations (GT).

Consider a (typically non-homogeneous) scalar field $S(x)$. Under a GCT:

$$x \rightarrow \tilde{x} = f(x), \quad x = f^{-1}(\tilde{x}), \quad S(x) \rightarrow \tilde{S}(\tilde{x}) = S(x)$$

Under the associated GT old and new fields are evaluated at the same space-time point x and

$$S(x) \rightarrow \tilde{S}(x) = S(f^{-1}(x)).$$

Consider now the space-time integral of S over a four-dimensional region Ω :

$$F(S, \Omega) = \int_{\Omega(x)} d^4x \sqrt{-g(x)} S(x).$$

Claim: $F(S, \Omega)$ is invariant under GT if the region Ω itself changes as a scalar under GT.

Gauge (non)-invariance of space-time integrals, 2

Indeed, let us define Ω in terms of a window function W_Ω

$$F(S, \Omega) = \int_{\Omega(x)} d^4x \sqrt{-g(x)} S(x) \equiv \int_{\mathcal{M}_4} d^4x \sqrt{-g(x)} S(x) W_\Omega(x).$$

The integral will be gauge invariant only if under a GT

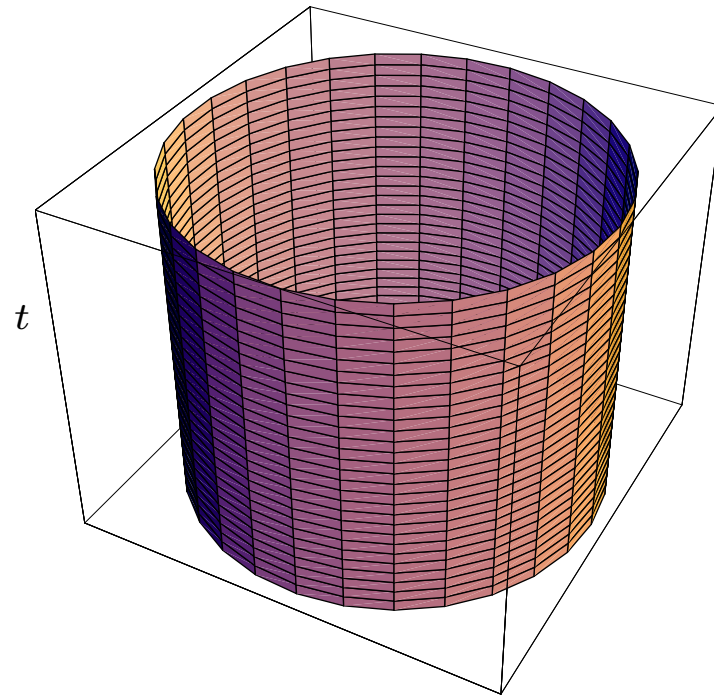
$$W_\Omega(x) \rightarrow \tilde{W}_\Omega(x) = W_\Omega(f^{-1}(x)),$$

In our (cosmological) case $W_\Omega(x)$ can be represented as a step-like window function, selecting a cylinder-like region (see picture for 2+1 dimensional spacetime) with temporal boundaries determined by the two space-like hypersurfaces on which a function $A(x)$ (with time-like gradient) takes the constant values A_1 and A_2 and by the coordinate condition $B(x) < r_0$, where $B(x)$ is a suitable function with space-like gradient. More explicitly:

$$W_\Omega(x) = \theta(A(x) - A_1)\theta(A_2 - A(x))\theta(r_0 - B(x))$$

In this case the integral will be GI only if the functions $A(x)$ and $B(x)$ are scalars.

Gauge (non)-invariance of space-time integrals, 3



Gauge (non)-invariance of space-time integrals, 4

For the cosmological backgrounds all fields are naturally of quasi-homogeneous type, and their gradients are typically time-like. In such a context we cannot covariantly define the spatial boundaries for lack of appropriate fields at our disposal and we have a non gauge invariant integral.

$$\tilde{F}(\tilde{S}, \Omega) - F(S, \Omega) = \int_{\mathcal{M}_4} d^4x \sqrt{-g(x)} S(x) \Delta W_\Omega(x)$$

where

$$\Delta W_\Omega(x) = \theta(A(x) - A_1) \theta(A_2 - A(x)) [\theta(r_0 - B(f(x))) - \theta(r_0 - B(x))].$$

However the breaking of gauge invariance comes from the region $r \sim r_0$ and goes away for large enough volumes.

A gauge-invariant averaging prescription, 1

Depending on the context in which the backreaction is considered, there are two types of averaging procedure: spatial (or *ensemble*) average of classical variables, and (vacuum) expectation values of quantized fields.

In both cases, one has to face the problem of the gauge dependence of the results. So the question is: **is it possible to define a gauge-invariant averaging prescription?**

Spatial volume averages can be covariantly obtained from the four-dimensional integrals discussed before simply by using a delta-like window function:

$$W_{\Omega}(x) = \delta(A(x) - A_0)\theta(r_0 - B(x))$$

Let us then define:

$$\langle S \rangle_{\{A_0, r_0\}} = \frac{F(S, \Omega)}{F(1, \Omega)} = \frac{\int d^4x \sqrt{-g} S \delta(A - A_0) \theta(r_0 - B)}{\int d^4x \sqrt{-g} \delta(A - A_0) \theta(r_0 - B)}$$

A gauge-invariant averaging prescription, 2

Considering the change of integration variable from t to \bar{t} , defined by $t = h(\bar{t}, x)$, such as

$$A(h(\bar{t}, x), x) = \bar{A}(\bar{t}, x) \equiv A^{(0)}(\bar{t}),$$

one obtains

$$\langle S \rangle_{\{A_0, r_0\}} = \frac{\int d^3x \sqrt{-\bar{g}(t_0, x)} \bar{S}(t_0, x) \theta(r_0 - B(h(t_0, x), x))}{\int d^3x \sqrt{-\bar{g}(t_0, x)} \theta(r_0 - B(h(t_0, x), x))}$$

where we have called t_0 the time \bar{t} when $A^{(0)}(\bar{t})$ takes the constant values A_0 and we are averaging on a section of the three-dimensional hypersurface Σ_{A_0} , where $A(x) = A_0$.

As said, the above integrals will be strictly gauge invariant only in the limit of an infinite spatial volume. In this limit the step-like boundary disappears, and we obtain:

$$\langle S \rangle_{A_0} = \frac{\int_{\Sigma_{A_0}} d^3x \sqrt{-\bar{g}(t_0, x)} \bar{S}(t_0, x)}{\int_{\Sigma_{A_0}} d^3x \sqrt{-\bar{g}(t_0, x)}}.$$

Note the presence, under the integral, not of S but of \bar{S} , i.e. of S transformed to the coordinate frame in which $A(x)$ is homogeneous.

A gauge-invariant averaging prescription, 3

This results can be generalized to the quantum case.

Expectation values of quantum operators can be extensively interpreted (and re-written) as spatial integrals weighted by the integration volume V , according to the general prescription

$$\langle \dots \rangle \rightarrow V^{-1} \int_V d^3x (\dots) ,$$

where the integration volume extends to all three-dimensional space. In this way the above gauge invariant prescription becomes

$$\langle S \rangle_{A_0} = \frac{\langle \sqrt{-\bar{g}}(t_0, x) \bar{S}(t_0, x) \rangle}{\langle \sqrt{-\bar{g}}(t_0, x) \rangle}$$

where it is important to note that the two entries of this ratio are not separately gauge invariant, but the ratio itself, equivalent to the above prescription, is indeed invariant.

A gauge-invariant averaging prescription, 4

Let us now present an explicit expansion (up to second order) of the generalized average $\langle S \rangle_{A_0}$ in terms of conventional averages defined in an arbitrary gauge. Expanding to second order the previous expression we obtain

$$\langle S \rangle_{A_0} = S^{(0)} + \langle \bar{S}^{(2)} \rangle + \frac{1}{(\sqrt{-g})^{(0)}} \langle \bar{S}^{(1)} (\sqrt{-g})^{(1)} \rangle$$

We can now express the transformed (barred) fields in terms of the original (unbarred) fields, in a general gauge. Considering the particular “infinitesimal” coordinate transformation (Bruni, Matarrese, Mollerach, Sonogo 1997) that connects t to \bar{t} , we can write the transformed quantities $\bar{S}^{(1)}$, $\bar{S}^{(2)}$ and $(\sqrt{-g})^{(1)}$ in terms of A and of the unbarred fields S and g and get:

$$\begin{aligned} \langle S \rangle_{A_0} &= S^{(0)} + \langle \Delta^{(2)} \rangle + \frac{1}{(\sqrt{-g})^{(0)}} \langle \Delta^{(1)} (\sqrt{-g})^{(1)} \rangle \\ &- \frac{1}{2} \frac{\Lambda^{(0)}}{(\dot{A}^{(0)})^2} \langle (A^{(1)})^2 \rangle - \frac{1}{(\sqrt{-g})^{(0)}} \partial_t \left\langle \frac{(\sqrt{-g})^{(0)} A^{(1)} \Delta^{(1)}}{\dot{A}^{(0)}} \right\rangle \end{aligned}$$

where:

$$\Delta^{(i)} = S^{(i)} - \frac{\dot{S}^{(0)}}{\dot{A}^{(0)}} A^{(i)}, \quad i = 1, 2, \quad ; \quad \Lambda^{(0)} = \ddot{S}^{(0)} - \frac{\dot{S}^{(0)}}{\dot{A}^{(0)}} \ddot{A}^{(0)}, \quad (1)$$

This is our basic result: it depends on the scalar observable A (chosen to specify the hypersurface the averaging is physically referred to) but, for any given choice of A , is fully gauge independent (up to second order): $\langle \tilde{S} \rangle_{A_0} = \langle S \rangle_{A_0}$,

A gauge-invariant averaging prescription, 5

Our result can be shown to pass several consistency checks. Suppose, for instance, that S and A are related by an arbitrary function $S = S(A)$. It is easy to check that in such case our formula simply gives $\langle S \rangle_{A_0} = S(A_0)$ as it should be.

As a second check one may replace the scalar A by $f(A)$, with f an arbitrary function, and check that $\langle S \rangle_{A_0}$ does not change.

The gauge invariance of our proposal can be very useful: it allows to compute the average in a gauge that has been conveniently chosen for other purposes; it also allows to evaluate and compare the average of a scalar $S(x)$ on different hypersurfaces, defined by different $A(x)$, while solving the dynamics of the problem in a single gauge.

We should note, instead, that the result of the conventional average procedure, i.e.

$\langle S \rangle = \langle S^{(0)} + S^{(1)} + S^{(2)} \rangle = S^{(0)} + \langle S^{(2)} \rangle$, is not gauge invariant, even if this expression is computed in the barred coordinates, because of the extra term proportional to $\langle \bar{S}^{(1)} (\sqrt{-\bar{g}})^{(1)} \rangle$.

Another gauge-invariant average prescription, 1

The gauge invariant prescription I presented contains, in the integration measure, the determinant of the full metric $\bar{g}_{\mu\nu}$, rather than the determinant $\bar{\gamma} \equiv \det(\bar{\gamma}_{ij})$ of the intrinsic metric $\bar{\gamma}_{ij}$ of the hypersurface Σ_A (as, for example, in Buchert 2001, and Buchert and Carfora 2002).

However, if we replace $\delta(A - A_0)$ by the following, more complicated but still covariant, window function:

$$\delta(A(x) - A_0) \sqrt{|g^{\mu\nu} \partial_\mu A \partial_\nu A|}.$$

and we repeat our procedure using the relation $\bar{g}^{00} \bar{g} = \bar{\gamma}$, we end up with a gauge invariant result which can be written exactly as before, but with $-\bar{g}$ replaced by $|\bar{\gamma}|$.

$$\langle S \rangle_{A_0} = \frac{\int_{\Sigma_{A_0}} d^3x \sqrt{|\bar{\gamma}(t_0, x)|} \bar{S}(t_0, x)}{\int_{\Sigma_{A_0}} d^3x \sqrt{|\bar{\gamma}(t_0, x)|}},$$

and similarly for the quantum case.

Another gauge-invariant average prescription, 2

This different averaging prescription can be written in arbitrary gauge as:

$$\begin{aligned}\langle S \rangle_{A_0} &= S^{(0)} + \langle \Delta^{(2)} \rangle + \frac{1}{(\sqrt{|\gamma|})^{(0)}} \langle \Delta^{(1)} (\sqrt{|\gamma|})^{(1)} \rangle - \frac{1}{\dot{A}^{(0)}} \langle A^{(1)} \Lambda^{(1)} \rangle \\ &- \frac{1}{\dot{A}^{(0)}} \partial_t \left(\ln (\sqrt{|\gamma|})^{(0)} \right) \langle A^{(1)} \Delta^{(1)} \rangle + \frac{1}{2} \frac{\Lambda^{(0)}}{(\dot{A}^{(0)})^2} \langle (A^{(1)})^2 \rangle\end{aligned}$$

where $\Lambda^{(1)} = \dot{S}^{(1)} - \frac{\dot{S}^{(0)}}{\dot{A}^{(0)}} \dot{A}^{(1)}$, and shares all the nice properties of the previous prescription.

The two averaging prescriptions are clearly inequivalent in all cases where $\sqrt{-\bar{g}}/\sqrt{|\gamma|}$ is non-homogeneous.

Examples of expectation values, 1

The computation of backreaction effects, due to the presence of small inhomogeneities perturbing the large-scale cosmological evolution, is plagued by substantial ambiguities arising from the gauge dependence of the perturbative approach and from the adopted averaging procedure.

In the quantum case for inflationary backgrounds, in particular, there is a controversial literature (see, for example, Abramo and Woodard 2002, Geshnizjani and Brandenberger 2002, and Finelli, Marozzi, Vacca and Venturi 2004) where different results are obtained on the grounds of computations performed with different methods in different gauges.

The generalized average prescription here presented, differently from the conventional procedures used in the literature, always gives gauge-invariant results (up to second perturbative order), thus avoiding interpretation problems and preventing ambiguous conclusions.

Let us show a particular example in which the ambiguities arising from the use of the standard approach are evident.

Examples of expectation values, 2

Let us consider a spatially flat, FRW background geometry, sourced by a single scalar field ϕ according to the Einstein equations, and we expand our background fields $\{\phi, g_{\mu\nu}\}$ up to second order in the non-homogeneous perturbations, without fixing any gauge, as follows:

$$\begin{aligned}\phi(t, \vec{x}) &= \phi^{(0)}(t) + \phi^{(1)}(t, \vec{x}) + \phi^{(2)}(t, \vec{x}), \\ g_{00} &= -1 - 2\alpha^{(1)} - 2\alpha^{(2)}, \quad g_{i0} = -\frac{a}{2} \left(\beta_{,i}^{(1)} + B_i^{(1)} \right) - \frac{a}{2} \left(\beta_{,i}^{(2)} + B_i^{(2)} \right), \\ g_{ij} &= a^2 \left[\delta_{ij} \left(1 - 2\psi^{(1)} - 2\psi^{(2)} \right) + D_{ij} (E^{(1)} + E^{(2)}) + \frac{1}{2} \left(\chi_{i,j}^{(1)} + \chi_{j,i}^{(1)} + h_{ij}^{(1)} \right) \right. \\ &\quad \left. + \frac{1}{2} \left(\chi_{i,j}^{(2)} + \chi_{j,i}^{(2)} + h_{ij}^{(2)} \right) \right],\end{aligned}$$

where $D_{ij} = \partial_i \partial_j - \delta_{ij} (\nabla^2 / 3)$.

To fix a gauge, to first and to second order, we can, in particular, set to zero two scalar variables among ϕ , α , β , ψ and E , and one vector variable between B_i and χ_i .

The so-called Uniform Field Gauge (UFG) is fixed by the condition $\phi^{(1)} = 0 = \phi^{(2)}$, i.e. is the gauge where the scalar field is homogeneous.

However, we are free to fix another condition on the scalar components of the metric perturbations: in particular, we can set to zero one scalar variable among β , ψ and E (but not α , or the background would become trivially homogeneous up to second order (Gasperini, Marozzi and Veneziano, in preparation)).

Examples of expectation values, 3

A scalar quantity playing a central role in the computation of the backreaction is the volume expansion Θ . For a scalar-field-dominated geometry, in particular, we have

$$\Theta = \nabla_{\mu} u^{\mu}, \quad u_{\mu} = \frac{\partial_{\mu} \phi}{(-g^{\alpha\beta} \partial_{\alpha} \phi \partial_{\beta} \phi)^{1/2}},$$

so that $\Theta = 3H$ for the unperturbed homogeneous geometry.

If we include perturbations, up to second order, and we compute the average of Θ according to the standard prescription, the result is notoriously gauge dependent.

Considering the choice $A = \phi$, for instance, we find that the covariant average $\langle S \rangle_{\phi_0}$ reduces in the UFG to the first three terms of our prescription. It differs from the conventional average $\langle S \rangle$ by the contribution of $\langle S^{(1)}(\sqrt{-g})^{(1)} \rangle$. Only when this term is zero the two results coincide.

Let us show the differences among the various UFG choice presenting the results of $\langle \Theta \rangle$ for a cosmological background perturbed around a power-law geometry with scale factor $a(t) \sim |t|^{1/3}$. (Gasperini, Marozzi and Veneziano, in preparation)

Examples of expectation values, 4

Working in the UFG fixed by $\beta^{(1)} = 0 = \beta^{(2)}$ one obtains:

$$\langle \Theta \rangle_{UFG\beta} = 3H \left[1 + \frac{45}{8} \frac{\langle Q^{(1)2} \rangle_{\text{REN}}}{M_{\text{P}}^2} \right],$$

while in the UFG with $E^{(1)} = 0 = E^{(2)}$ one obtains:

$$\langle \Theta \rangle_{UFG E} = 3H \left[1 - \frac{3}{4} \frac{\langle Q^{(1)2} \rangle_{\text{REN}}}{M_{\text{P}}^2} \right].$$

where Q denotes the gauge-invariant Mukhanov variable (Mukhanov 1988), and the suffix REN denotes that the v.e.v. has been regularized through a suitable adiabatic subtraction.

These two results not only differ between each other, but also differ from the one obtained with the gauge invariant prescription with $A = \phi$, which gives

$$\langle \Theta \rangle_{\phi_0} = 3H \left[1 - \frac{15}{8} \frac{\langle Q^{(1)2} \rangle_{\text{REN}}}{M_{\text{P}}^2} \right]$$

in any gauge.

It follows, in particular, that we cannot try to solve the problem of the gauge dependence of the backreaction considering the UFG as a privileged gauge, as often suggested in the literature.

Conclusions

- By studying the gauge-transformation properties of four-dimensional integrals we have been able to propose a general formula for the classical or quantum average of any scalar quantity, on hypersurfaces on which another given scalar quantity is constant. Our non-trivial proposal can be shown to be gauge-invariant in the quantum case and in the classical one for very large spatial-averaging volumes.
- Our proposal contains, in *any* gauge, important extra terms respect the standard proposal. Neglecting those terms induces, in general, gauge-invariance violations that are of the same order as the effect one wishes to estimate.
- Unfortunately, the requirement of gauge invariance alone does not fix uniquely the prescription. I have presented two different gauge invariant prescriptions. The choice between one or the other definition is unclear at the moment and should be possibly dictated by the physics of the problem.