Conifolds: Warped, Resolved, Deformed

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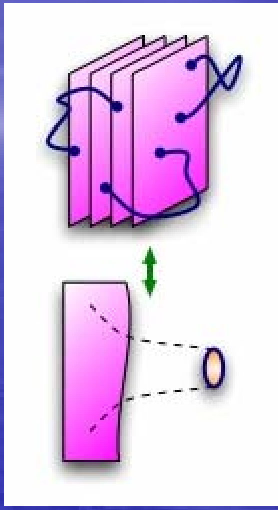
From D-branes to AdS/CFT



- A stack of N Dirichlet 3-branes realizes $\mathcal{N}$=4 supersymmetric SU(N) gauge theory in 4 dimensions. It also creates a curved background of 10-d theory of closed superstrings (artwork by E.Imeroni)

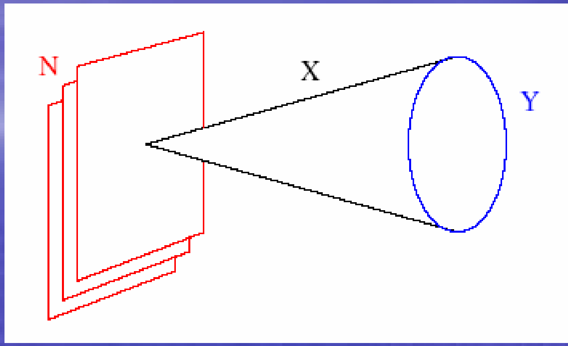
$$ds^2 = \left(1 + \frac{L^4}{r^4}\right)^{-1/2} \left(-(dx^0)^2 + (dx^i)^2\right) + \left(1 + \frac{L^4}{r^4}\right)^{1/2} \left(dr^2 + r^2 d\Omega_5^2\right)$$

which for small r approaches $AdS_5 \times \mathbf{S}^5$

Conebrane Dualities

- To reduce the number of supersymmetries in AdS/CFT, we may place the stack of N D3-branes at the tip of a 6-d Ricci-flat cone X whose base is a 5-d Einstein space Y:

$$ds_X^2 = dr^2 + r^2 ds_Y^2$$



- Taking the near-horizon limit of the background created by the N D3-branes, we find the space AdS_5 x Y, with N units of RR 5-form flux, whose radius is given by

$$L^4 = \frac{\sqrt{\pi}\kappa N}{2\,\mathrm{Vol}(Y)} = 4\pi g_s N \alpha'^2 \frac{\pi^3}{\mathrm{Vol}(Y)}$$

- This type IIB background is conjectured to be dual to the IR limit of the gauge theory on N D3-branes at the tip of the cone X.

Kachru, Silverstein; ...

D3-branes on the Conifold

- The conifold is a Calabi-Yau 3-fold cone X described by the constraint $\sum_{a=1}^{4} z_a^2 = 0$ on 4 complex variables.
- Its base Y is a coset $T^{1,1}$ which has symmetry $SU(2)_A x SU(2)_B$ that rotates the z's, and also $U(1)_R$: $z_a \to e^{i\theta} z_a$
- The Sasaki-Einstein metric on $T^{1,1}$ is

$$\mathrm{d}s^2_{T^{1,1}} = \frac{1}{9}\Big(\mathrm{d}\psi + \sum_{i=1}^{2} \cos\theta_i \mathrm{d}\phi_i\Big)^2 + \frac{1}{6}\sum_{i=1}^{2}\Big(\mathrm{d}\theta_i^2 + \sin^2\theta_i \mathrm{d}\phi_i^2\Big)$$

  where $\theta_i \in [0,\pi], \phi_i \in [0,2\pi], \psi \in [0,4\pi]$
- The topology of $T^{1,1}$ is S^2 x S^3.

- To `solve' the conifold constraint det Z = 0 we introduce another set of convenient coordinates:

$$Z = \begin{pmatrix} z^3 + iz^4 & z^1 - iz^2 \\ z^1 + iz^2 & -z^3 + iz^4 \end{pmatrix} = \begin{pmatrix} w_1 & w_3 \\ w_4 & w_2 \end{pmatrix} = \begin{pmatrix} a_1 b_1 & a_1 b_2 \\ a_2 b_1 & a_2 b_2 \end{pmatrix}$$

- The action of global symmetries is

$$\begin{array}{rcl} SU(2)\times SU(2)\ \text{symmetry} & : & \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \to L \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \qquad \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \to R \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \\ \text{R-symmetry} & : & (a_i, b_j) \to e^{i\frac{\alpha}{2}} (a_i, b_j)\ , \end{array}$$

- There is a redundancy under

$$a_i \to \lambda\, a_i \quad , \quad b_j \to \frac{1}{\lambda}\, b_j \quad (\lambda \in \mathbf{C})$$

which is partly fixed by imposing

$$|a_1|^2 + |a_2|^2 - |b_1|^2 - |b_2|^2 = 0$$

- It remains to quotient the space by the phase rotation $a \sim e^{i\alpha}a, b \sim e^{-i\alpha}b$ (in the gauge theory, this will have the meaning of the U(1) baryon number symmetry).
- In the IR gauge theory on D3-branes at the apex of the conifold, the coordinates a_1, a_2, b_1, b_2 are replaced by chiral superfields. For a single D3-brane it is necessary to introduce gauge group U(1) x U(1).

- The $\mathcal{N}=1$ SCFT on N D3-branes at the apex of the conifold has gauge group SU(N)xSU(N) coupled to bifundamental chiral superfields A_1, A_2, in $(\overline{\mathbf{N}}, \mathbf{N})$, and B_1, B_2 in $(\mathbf{N}, \overline{\mathbf{N}})$. IK, Witten
- The R-charge of each fields is ½. This insures $U(1)_R$ anomaly cancellation.
- The unique $SU(2)_A \times SU(2)_B$ invariant, exactly marginal quartic superpotential is added:

$$W = \epsilon^{ij}\epsilon^{kl}\,\mathrm{tr}\,A_i B_k A_j B_l$$

- This theory also has a baryonic U(1) symmetry under which $A_k \to e^{ia} A_k$; $B_l \to e^{-ia} B_l$, and a Z_2 symmetry which interchanges the A's with the B's and implements charge conjugation.

Resolution and Deformation

- There are two well-known Calabi-Yau blow-ups of the conifold singularity.
- The `deformation' replaces the constraint on the z-coordinates by

$$z_1^2+z_2^2+z_3^2+z_4^2=\text{♏}^2$$

- This replaces the singularity by a finite 3-sphere.
- In the `small resolution' the singularity is replaced by a finite 2-sphere. This is implemented by modifying the constraint on the a and b variables

$$|b_1|^2+|b_2|^2-|a_1|^2-|a_2|^2=u^2$$

- This suggests that in the gauge theory the resolution is achieved by giving VEV's to the chiral superfields. IK, Witten
- For example, we may give a VEV to only one of the four superfields: $B_2 = u 1_{N\times N}$
- The dual of such a gauge theory is a resolved conifold, which is warped by a stack of N D3-branes placed at the north pole of the blown up 2-sphere.

$$ds_{10}^2 = \sqrt{H^{-1}(y)}dx^\mu dx_\mu + \sqrt{H(y)}ds_6^2$$

- The explicit CY metric on the resolved conifold is Pando Zayas, Tseytlin

$$\kappa(r) = \frac{r^2 + 9u^2}{r^2 + 6u^2}$$

$$\begin{aligned} ds_6^2 &= \kappa^{-1}(r) dr^2 + \frac{1}{9}\kappa(r) r^2 \left(d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2\right)^2 \\ &\quad + \frac{1}{6} r^2 (d\theta_1^2 + \sin^2\theta_1 d\phi_1^2) + \frac{1}{6}(r^2 + 6u^2)(d\theta_2^2 + \sin^2\theta_2 d\phi_2^2) \end{aligned}$$

- The warp factor is the Green's function on this space IK, Murugan

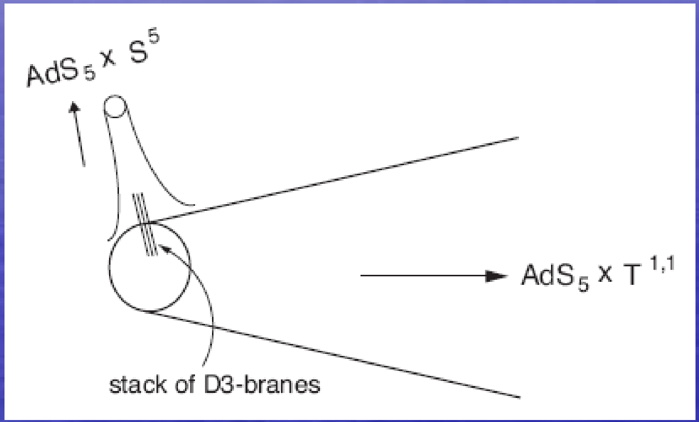
$$H(r, \theta_2) = L^4 \sum_{l=0}^{\infty} (2l+1) H_l^A(r) P_l(\cos\theta_2)$$

- The radial functions are hyper-geometric:

$$H_l^A(r) = \frac{2}{9u^2} \frac{C_\beta}{r^{2+2\beta}} \; {}_2F_1\left(\beta, 1+\beta; 1+2\beta; -\frac{9u^2}{r^2}\right)$$

$$C_\beta = \frac{(3u)^{2\beta}\Gamma(1+\beta)^2}{\Gamma(1+2\beta)}, \qquad \beta = \sqrt{1 + (3/2) l(l+1)}$$

- We get an explicit `localized' solution which describes SU(2)xU(1)xU(1) symmetric holographic RG flow to the $\mathcal{N}$=4 SU(N) SYM.



- A previously known `smeared' solution corresponds to taking just the l=0 harmonic. This solution is singular Pando Zayas, Tseytlin

$$\frac{2}{9u^2r^2} + \frac{4\beta^2}{81u^4}\ln r + \mathcal{O}(1) \quad \xleftarrow{0\leftarrow r} \quad H_l^A(r) \quad \xrightarrow{r\to\infty} \quad \frac{2C_\beta}{9u^2r^{2+2\beta}}$$

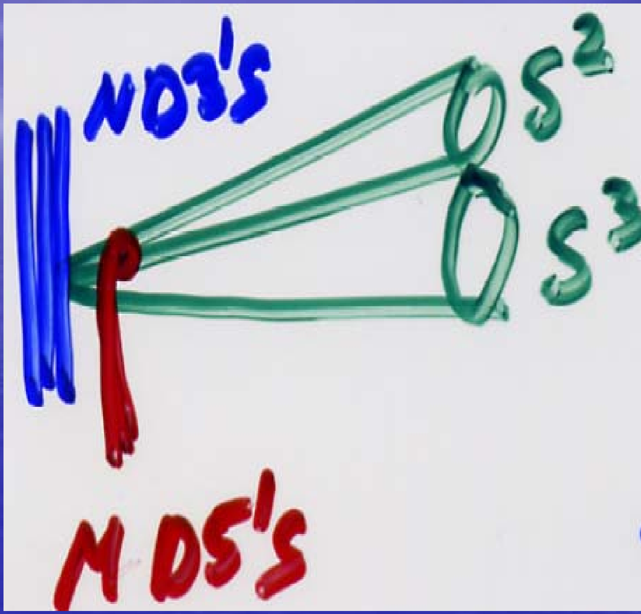
Baryonic Branch

- A baryonic operator det B_2 acquires a VEV. It can be calculated on the string side of the duality using a Euclidean D3-brane wrapping a 4-cycle inside the resolved conifold, located at fixed θ_2, ϕ_2 with a UV cut-off r_c:

$$e^{-S_{BI}} = \left(\frac{3e^{5/12}u}{r_c}\right)^{3N/4} \sin^N(\theta_2/2)$$

- No mesonic operators, e.g. Tr $(A_i B_j)$, have VEV's. This is a `baryonic branch' of the gauge theory.

Warped Deformed Conifold



- To achieve a deformation, change the gauge theory: add to the N D3-branes M D5-branes wrapped over the S^2 at the tip of the conifold.
- The 10-d geometry dual to the gauge theory on these branes is the warped deformed conifold (IK, Strassler)

$$ds_{10}^2 = h^{-1/2}(t)\big(-(dx^0)^2 + (dx^i)^2\big) + h^{1/2}(t)ds_6^2$$

- ds_6^2 is the metric of the deformed conifold, a simple Calabi-Yau space defined by the following constraint on 4 complex variables:

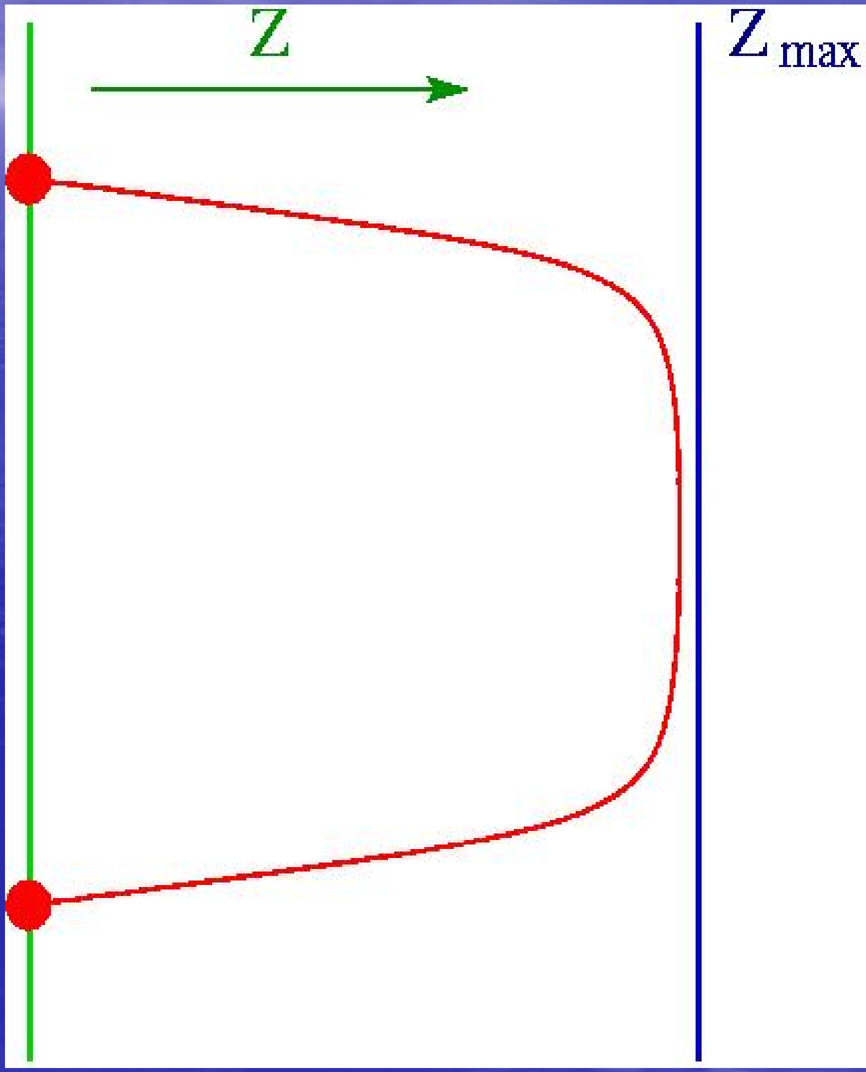
$$\sum_{i=1}^{4} z_i^2 = \varepsilon^2$$

String Theoretic Approaches to Confinement

- It is possible to generalize the AdS/CFT correspondence in such a way that the quark-antiquark potential is linear at large distance.
- A "cartoon" of the necessary metric is

$$ds^2 = \frac{dz^2}{z^2} + a^2(z)\left(-(dx^0)^2 + (dx^i)^2\right)$$

- The space ends at a maximum value of z where the warp factor is finite. Then the confining string tension is $\frac{a^2(z_{\max})}{2\pi\alpha'}$



- The warp factor is finite at the `end of space' t=0, as required for the confinement: h(t)= $2^{-8/3}\,\gamma\, I(t)$

$$I(t) = \int_t^\infty dx \frac{x\coth x - 1}{\sinh^2 x}(\sinh 2x - 2x)^{1/3}\ , \qquad \gamma = 2^{10/3}(g_s M \alpha')^2 \varepsilon^{-8/3}$$

- The standard warp factor a^2 , which measures the string tension, is identified with h(t) $^{-1/2}$ and is minimized at t=0. It blows up at large t (near the boundary).
- The dilaton is exactly constant due to the self-duality of the 3-form background

$$\star_6\, G_3 = iG_3\ , \qquad G_3 = F_3 - \frac{i}{g_s} H_3$$

- The radius-squared of the S^3 at $t=0$ is g_sM in string units.
- When g_sM is large, the curvatures are small everywhere, and the SUGRA solution is reliable in `solving’ this confining gauge theory.
- Even when g_sM is small, the curvature gets small at large t (in the UV). The SUGRA description of the duality cascade is robust.

Anomalously light glueballs

- The confining string tension is $$T_s = \frac{1}{2^{4/3} a_0^{1/2} \pi} \frac{\varepsilon^{4/3}}{(\alpha')^2 g_s M}$$
- The glueballs are the normalizable modes localized near at small t. In the supergravity limit (at large g_s M) their mass scales are

$$m_{glueball} \sim m_{KK} \sim \frac{\varepsilon^{2/3}}{g_s M \alpha'}$$

$$T_s \sim g_s M (m_{glueball})^2$$

- Curiously, the m^2 of the n-th radial excitation scales as $n^2\, T_s/(g_s\, M)$
- For $n^2 >> g_s$ M, we instead expect from semi-classical string quantization

  $m^2 \sim T_s\, n$

- In order to eliminate the anomalously light bound states, we need a small g_s M, which requires a departure from the SUGRA limit.
- Even for small g_s M, SUGRA becomes reliable in the UV (at large t).

Log running of couplings

- The large radius asymptotic solution is characterized by logarithmic deviations from $AdS_5 \times T^{1,1}$ IK, Tseytlin
- The near-AdS radial coordinate is $r \sim \varepsilon^{2/3} e^{t/3}$
- The NS-NS and R-R 2-form potentials:

$$F_3 = \frac{M\alpha'}{2}\omega_3 \ , \qquad B_2 = \frac{3 g_s M \alpha'}{2}\omega_2 \ln(r/r_0)$$

$$\omega_2 = \frac{1}{2}(g^1 \wedge g^2 + g^3 \wedge g^4) = \frac{1}{2}(\sin\theta_1 d\theta_1 \wedge d\phi_1 - \sin\theta_2 d\theta_2 \wedge d\phi_2)$$

$$\omega_3 = \frac{1}{2} g^5 \wedge (g^1 \wedge g^2 + g^3 \wedge g^4)$$

- This translates into log running of the gauge couplings through

$$\frac{4\pi^2}{g_1^2} + \frac{4\pi^2}{g_2^2} = \frac{\pi}{g_s e^{\Phi}} \,,$$

$$\left[\frac{4\pi^2}{g_1^2} - \frac{4\pi^2}{g_2^2}\right] g_s e^{\Phi} = \frac{1}{2\pi\alpha'}\left(\int_{S^2} B_2\right) - \pi$$

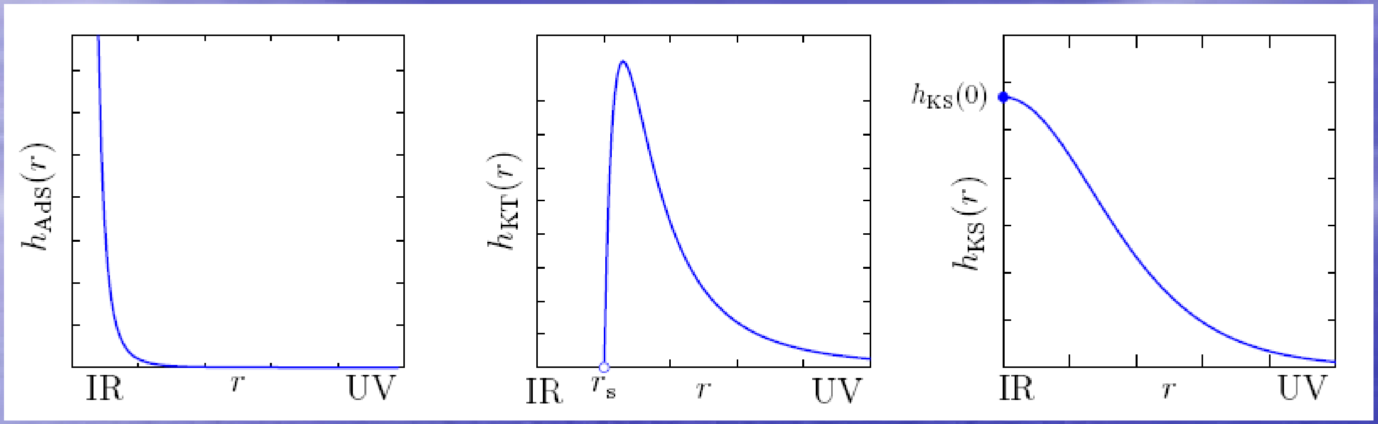
- The warp factor deviates from the M=0 solution logarithmically.

$$h(r) = \frac{27\pi(\alpha')^2[g_s N + a(g_s M)^2 \ln(r/r_0) + a(g_s M)^2/4]}{4r^4}$$

- Remarkably, the 5-form flux, dual to the number of colors, also changes logarithmically with the RG scale.

$$\tilde{F}_5 = \mathcal{F}_5 + \star\mathcal{F}_5 \,, \qquad \mathcal{F}_5 = 27\pi\alpha'^2 N_{eff}(r)\mathrm{vol}(\mathrm{T}^{1,1})$$

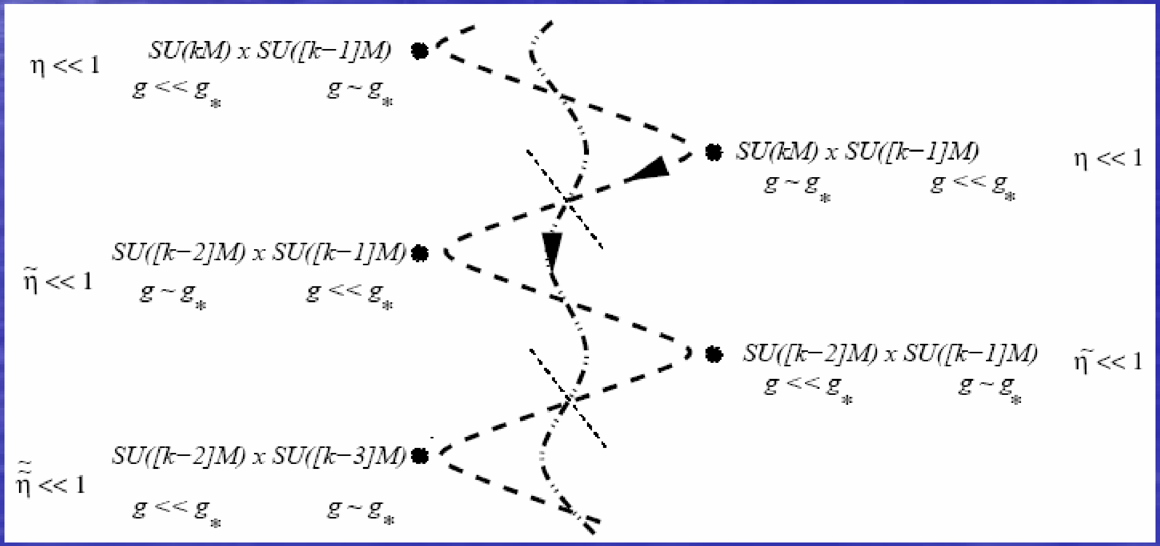
$$N_{eff}(r) = N + \frac{3}{2\pi} g_s M^2 \ln(r/r_0)$$



- Comparison of warp factors in the AdS, warped conifold, and warped deformed conifold cases. The warped conifold (KT) solution, which has a naked singularity, should be interpreted as asymptotic (UV) approximation to the correct solution.

- What is the explanation in the dual SU(kM)xSU((k-1)M) SYM theory coupled to bifundamental chiral superfields A_1, A_2, B_1, B_2 ? A novel phenomenon, called a **duality cascade**, takes place: k repeatedly changes by 1 as a result of the Seiberg duality IK, Strassler

(diagram of RG flows from a review by M. Strassler)



IR Behavior of the Conifold Cascade

- Here the dynamical deformation of the conifold renders the solution smooth, and explains the IR dynamics of the gauge theory.
- Dimensional transmutation in the IR. The dynamically generated confinement scale is
  $$\sim \varepsilon^{2/3}$$
- The pattern of R-symmetry breaking is the same as in the SU(M) SYM theory: $Z_{2M} \to Z_2$
- Yet, the IR gauge theory is somewhat more complicated.

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- In the IR the gauge theory cascades down to SU(2M) x SU(M). The SU(2M) gauge group effectively has $N_f=N_c$.
- The baryon and anti-baryon operators Seiberg

$$\mathcal{A} = \epsilon^{i_1 \ldots i_{N_c}} A^{a_1}_{\alpha_1 i_1} \ldots A^{a_{N_c}}_{\alpha_{N_c} i_{N_c}}$$

$$\mathcal{B} = \epsilon_{i_1 \ldots i_{N_c}} B^{i_1}_{\dot\alpha_1 a_1} \ldots B^{i_{N_c}}_{\dot\alpha_{N_c} a_{N_c}}$$

acquire expectation values and break the U(1) symmetry under which $A_k \to e^{ia} A_k$; $B_l \to e^{-ia} B_l$. Hence, we observe confinement without a mass gap: due to $U(1)_{baryon}$ chiral symmetry breaking there exist a Goldstone boson and its massless scalar superpartner. There exists a baryonic branch of the moduli space

$$\mathcal{A} = i\Lambda_1^{2M}\zeta \ , \qquad \mathcal{B} = i\Lambda_1^{2M}/\zeta$$

- The KS solution is part of a moduli space of confining SUGRA backgrounds, resolved warped deformed conifolds. Gubser, Herzog, IK; Butti, Grana, Minasian, Petrini, Zaffaroni
- To look for them we need to use the PT ansatz:

$$ds_{10}^2 = H^{-1/2} dx_m dx_m + e^x ds_6^2,$$

$$ds_6^2 = (e^g + a^2 e^{-g})(e_1^2 + e_2^2) + e^{-g} \sum_{i=1}^{2} \left(\epsilon_i^2 - 2a e_i \epsilon_i\right) + v^{-1}(\tilde{\epsilon}_3^2 + dt^2)$$

- H, x, g, a, v, and the dilaton are functions of the radial variable t.
- Additional radial functions enter into the ansatz for the 3-form field strengths. The PT ansatz preserves the SO(4) but breaks a Z_2 charge conjugation symetry, except at the KS point.

- BGMPZ used the method of SU(3) structures to derive the complete set of coupled first-order equations.
- A result of their integration is that the warp factor and the dilaton are related:

$$H(t) = \tilde{H}\left(e^{-2\phi(t)} - 1\right)$$

Dymarsky, IK, Seiberg

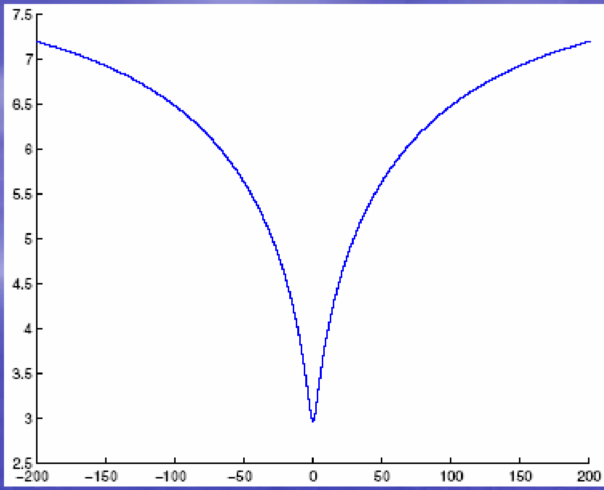
- The integration constant determines the \`modulus' U: $\tilde{H} = \gamma U^{-2}$ where $\gamma = 2^{10/3}(g_s M\alpha')^2\varepsilon^{-8/3}$
- At large t the solution approaches the KT \`cascade asymptotics': $a(t) = -2e^{-t} + Ue^{-5t/3}(-t+1) + \ldots$

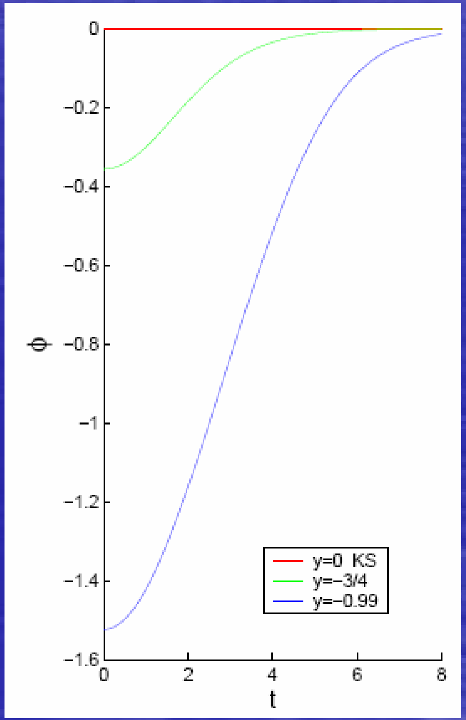
$$\gamma^{-1}H(t) = \frac{3}{32}e^{-4t/3}(4t-1) - \frac{3}{32\cdot 512}U^2(256t^3 - 864t^2 + 1752t - 847)e^{-8t/3} + O\left(e^{-10t/3}\right)$$

- The resolution parameter U is proportional to the VEV of the operator

$$\mathcal{U} = \mathrm{Tr}\left(\sum_{\alpha} A_{\alpha} A_{\alpha}^{\dagger} - \sum_{\dot{\alpha}} B_{\dot{\alpha}}^{\dagger} B_{\dot{\alpha}}\right)$$

- This family of resolved warped deformed conifolds is dual to the `baryonic branch' in the gauge theory (the quantum deformed moduli space).
- At large U the IR part of the solution approaches that of the Maldacena-Nunez solution. But we always have the `cascade' asymptotics at large t.
- Here are plots of the string tension (a fundamental string at the bottom of the throat is dual to an `emergent' chromo-electric flux tube) and of the dilaton profiles as a function of the modulus U=ln |ζ|. Dymarsky, IK, Seiberg





- In a very recent paper, Benna, Dymarsky and I confirmed the presence of the baryonic condensates on the string theory side of the duality. These 1-point functions are calculated using the Euclidean D5-brane wrapped over the 6 deformed conifold directions, with certain world volume gauge fields turned on (they are determined by the supersymmetry conditions). The behavior of the D5-brane action as a function of the large radius cut-off is in complete agreement with the expectations in the cascading gauge theory.

- All of this provides us with an exact solution of a class of 4-d large N confining supersymmetric gauge theories.
- This should be a good playground for testing various ideas about strongly coupled gauge theory.
- Some results on glueball spectra are already available, and further calculations are ongoing. Krasnitz; Caceres, Hernandez; Dymarsky, Melnikov; Berg, Haack, Muck
- High energy scattering of bound states in confining gauge/gravity models has also been studied successfully, e.g. the recent work on BFKL pomeron.
  Brower, Polchinski, Strassler, Tan

Conclusions

- Placing D3-branes at the tip of a CY cone, such as the conifold, leads to AdS/CFT dualities with $\mathcal{N}=1$ SUSY. Symmetry breaking in the gauge theory produces warped resolved conifolds.
- Adding wrapped D5-branes at the apex produces a cascading confining gauge theory whose dual is the warped deformed conifolds.
- This example of gauge/string duality gives a new geometrical view of such important phenomena as dimensional transmutation, chiral symmetry breaking, and quantum deformation of moduli space. We have also discussed motion along the baryonic branch of this theory, described by the resolved warped deformed conifolds.